MATH 2B: SAMPLE FINAL #2

- This exam consists of 13 questions and 100 total points.
- Read the directions for each problem carefully and answer all parts of each problem.
- Please show all work needed to arrive at your solutions (unless instructed otherwise). Label graphs and define any notation used. Cross out incorrect scratch-work.
- No calculators or other forms of assistance are allowed. Do not check your cell phones during the exam.
- Clearly indicate your final answer to each problem.

1. (8 points) Consider continuous functions f and f' (where f' denotes the derivative of f) with values given by the following table:

x	0	1	2	3	4	5
$f(x)$	3	4	6	9	13	18
$f'(x)$	1	2	4	6	7	5

a. Find $\int_0^4 f'(x)\,dx$ $= f(4)-f(0) = 13-3 = 10$

b. Estimate $\int_1^4 f(x)\,dx$ Using a left-hand Riemann sum with 3 equal subintervals.

$n=3 \Rightarrow \Delta x = \frac{4-1}{3} = 1.$ $\quad \int_1^4 f(x)\,dx \approx f(1)+f(2)+f(3) = 19$

c. Evaluate the following derivative at the point $x = 3$

$$\frac{d}{dx}\left(\int_2^x f(t)\,dt\right) = f(x)\Big|_{x=3} = f(3) = 9.$$

d. Suppose $f(x)$ gives the height of a rocket, measured in yards, x minutes after its launch. What are the units of $\int_0^4 f'(x)\,dx$ and what does this quantity represent.

units = 'yards'. $\int_0^4 f'(x) = f(4) - f(0)$ is increase in height over first 4 mins of flight.

2. (7 points) Evaluate $\displaystyle\int \frac{x}{1+x^4}\,dx$

let $u = x^2 \Rightarrow du = 2x\,dx$

So $\displaystyle\int \frac{x}{1+x^4}\,dx = \int \frac{1}{1+u^2}\cdot\frac{1}{2}\,du = \frac{1}{2}\tan^{-1}u + C$

$= \frac{1}{2}\tan^{-1}x^2 + C.$

3. (7 points) Evaluate $\int \frac{x^2}{e^{2x}}\,dx = \int x^2 e^{-2x}\,dx$

$u = x^2$, $dv = e^{-2x}\,dx$
$du = 2x\,dx$, $v = -\frac{1}{2}e^{-2x}$

$u = x$, $dv = e^{-2x}\,dx$
$du = dx$, $v = -\frac{1}{2}e^{-2x}$

$$= -\frac{1}{2}x^2 e^{-2x} + \int x e^{-2x}\,dx$$

$$= -\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}x e^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$$

$$= -\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x} + C$$

$$= -\frac{1}{4}e^{-2x}\left[1 + 2x + 2x^2\right] + C$$

4. (7 points) Evaluate $\int \sin^3(4t)\,dt$

$$= \int \sin(4t)\,(1 - \cos^2(4t))\,dt$$

$$= -\frac{1}{4}\cos(4t) + \frac{1}{12}\cos^3(4t) + C$$

5. (7 points) Evaluate the following integral by making an appropriate trigonometric substitution.

$$\int \frac{dx}{x^2\sqrt{x^2-9}}$$

$x = 3\sec\theta$

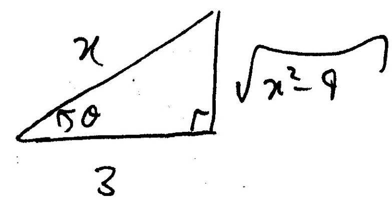
$dx = 3\sec\theta\tan\theta\, d\theta$

$$= \int \frac{3\sec\theta\tan\theta}{9\sec^2\theta \cdot 3\tan\theta}\, d\theta$$

$$= \frac{1}{9}\int \frac{1}{\sec\theta}\, d\theta = \frac{1}{9}\int \cos\theta\, d\theta$$

$$= \frac{1}{9}\sin\theta + C$$

$$= \frac{\sqrt{x^2-9}}{9x} + C.$$



6. (8 points) Determine whether the following integral is convergent or divergent. Evaluate the integral if it convergent. If it is divergent, explain why.

$$\int_0^\infty \frac{dz}{z^2+3z+2}$$

$$\frac{1}{z^2+3z+2} = \frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2}$$ (partial fraction)

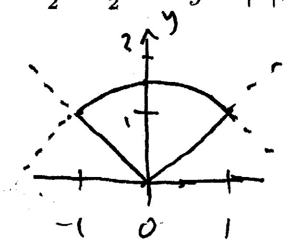
$$\Rightarrow \int_0^\infty \frac{dz}{z^2+3z+2} = \int_0^\infty \frac{1}{z+1} - \frac{1}{z+2}\, dz$$

$$= \lim_{s\to\infty} \ln|z+1| - \ln|z+2| \Big|_0^s$$

$$= \lim_{s\to\infty} \ln\left|\frac{z+1}{z+2}\right| - (\ln 1 - \ln 2)$$

$$= \ln 1 - (0 - \ln 2) = \ln 2$$ (convergent).

7. (10 points) Find the area of the region bounded by the curves $y = \frac{3}{2} - \frac{x^2}{2}$ and $y = |x|$.



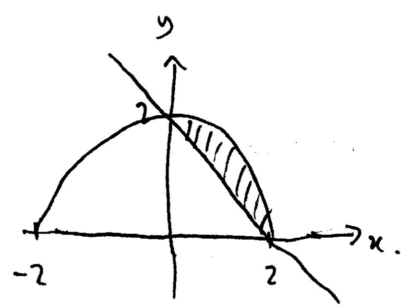
$$A = 2\int_0^1 \frac{3}{2} - \frac{x^2}{2} - x \, dx$$

$$= \int_0^1 3 - x^2 - 2x \, dx$$

$$= 3x - \frac{1}{3}x^3 - x^2 \Big|_0^1$$

$$= 3 - \frac{1}{3} - 1 = \frac{5}{3}$$

8. (10 points) Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $y = \sqrt{4 - x^2}$ and $y = 2 - x$.



$r_{out} = \sqrt{4-x^2}$, $r_{in} = 2 - x$.

$$\Rightarrow V = \pi \int_0^2 4 - x^2 - (2-x)^2 \, dx$$

$$= \pi \left[4x - \frac{1}{3}x^3 + \frac{1}{3}(2-x)^3 \right]_0^2$$

$$= \pi \left[8 - \frac{8}{3} - \frac{8}{3} \right] = 8\pi - \frac{16}{3}\pi$$

$$= \frac{8\pi}{3}.$$

9. (6 points) Determine whether each of the following sequences is convergent or divergent. Find the limit of the convergent sequences.

a. $a_n = \dfrac{e^{2n}}{\sqrt{n}}$ divergent $\Rightarrow +\infty$

$a_n = \sqrt{b_n}$ where $b_n = \dfrac{e^{4n}}{n} \longrightarrow 4e^{4n} \to \infty$

by L'Hôpital.

b. $a_n = \dfrac{(-1)^n}{n!}$ converges to zero

$-1 \leq (-1)^n \leq 1$

$\Rightarrow \underbrace{-\dfrac{1}{n!}}_{\downarrow\, 0} \leq \dfrac{(-1)^n}{n!} \leq \underbrace{\dfrac{1}{n!}}_{\downarrow\, 0}$

Squeeze thm.

c. $a_n = \tan^{-1}(n)$ converges to $\pi/2$

10. (6 points) Compute the arc length of the curve $y = \ln(\cos(x))$ over the interval $[0, \frac{\pi}{4}]$. (Hint. $\int \sec(x)\, dx = \ln|\sec(x) + \tan(x)| + C$.)

$$AL = \int_0^{\pi/4} \sqrt{1 + y'^2}\, dx = \int_0^{\pi/4} \sqrt{1 + \left(\frac{-\sin x}{\cos x}\right)^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx$$

$$= \ln|\sec x + \tan x| \Big|_0^{\pi/4} = \ln|\sqrt{2} + 1| - \ln|1 + 0|$$

$$= \ln(\sqrt{2} + 1).$$

11. (12 points) Use the indicated test to determine whether the given series is convergent or divergent.

a. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$ (integral test)

$\int_1^{\infty} \frac{1}{\sqrt{x+4}}\,dx = 2(x+4)^{+1/2}\Big|_1^{\to\infty} = +\infty$

divergent.

b. $\sum_{n=1}^{\infty} \frac{100^n}{n!}$ (ratio test)

$\left|\frac{100^{n+1}/(n+1)!}{10^n/n!}\right| = \frac{100}{n+1} \xrightarrow[n\to\infty]{} 0 < 1$

$\Rightarrow$ convergent.

c. $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{2n+5}$ (alternating series test)

For large n $(n \geq 3)$ have $\frac{\sqrt{n}}{2n+5}$ decreasing with limit 0.

$\Rightarrow \sum \frac{(-1)^n\sqrt{n}}{2n+5}$ converges.

d. $\sum_{n=2}^{\infty} \frac{n^2}{n^3-1}$ (comparison test or limit comparison test)

compare with divergent series $\sum \frac{1}{n}$.

$\frac{1/n}{n^2/n^3-1} = \frac{n^3-1}{n^3} \to 1$. Original series divergent

12. (6 points) Find the sum of the following convergent series.

a. $\sum_{n=1}^{\infty} \frac{3^n}{5^{n+1}}$ $= \frac{1}{5} \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n = \frac{1}{5} \cdot \frac{3/5}{1-3/5} = \frac{3}{10}$.

b. $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$ $= \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+3}$

$$S_N = \frac{1}{3} \left\{ \frac{1}{1} - \cancel{\frac{1}{4}} + \frac{1}{2} - \cancel{\frac{1}{5}} + \frac{1}{3} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{7}} + \cdots \right.$$

$$\left. + \cancel{\frac{1}{N-3}} - \cancel{\frac{1}{N}} + \cancel{\frac{1}{N-2}} - \frac{1}{N+1} + \cancel{\frac{1}{N-1}} - \frac{1}{N+2} + \cancel{\frac{1}{N}} - \frac{1}{N+3} \right\} = \frac{1}{3} \left\{ 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{N+1} - \frac{1}{N+2} - \frac{1}{N+3} \right\}$$

$$\xrightarrow[N \to \infty]{} \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{18} = \sum_{n=1}^{\infty} \frac{1}{n(n+3)}$$

13. (6 points) Find a power series representation for the function $f(x) = \frac{2}{3-x}$ and determine the interval of convergence.

$$f(x) = \frac{2}{3-x} = \frac{2}{3} \cdot \frac{1}{1 - x/3} = \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

Converges $\iff$ $-1 < \frac{x}{3} < 1$

$\iff$ $-3 < x < 3$

so $(-3, 3)$ is interval of convergence