Math 2B : Midterm # 2 Sample

This exam consists of 5 questions and 90 total points. The point value of each problem is indicated. Read directions for each problem carefully. Please show all work needed to arrive at your solutions. Clearly indicate your final answers.

1.) Evaluate each of the following integrals. [10 pts. each]

a) $\int \frac{\ln t}{t^5}\,dt$ $\quad u = \ln t \quad dv = \frac{1}{t^5}\,dt$

$du = \frac{1}{t}\,dt \quad v = \frac{-1}{4}t^{-4}$

$$= (\ln t)\cdot\left(\frac{-1}{4}t^{-4}\right) - \int \frac{-1}{4}t^{-4}\cdot\frac{1}{t}\,dt$$

$$= (\ln t)\cdot\left(\frac{-1}{4}t^{-4}\right) - \frac{1}{16}t^{-4} + C$$

b) $\int e^x \sin(3x)\,dx$ $\quad u = \sin(3x) \quad dv = e^x\,dx$

$du = 3\cos(3x)\,dx \quad v = e^x$

$$\int e^x \sin(3x)\,dx = e^x\sin(3x) - \int e^x\, 3\cos(3x)\,dx$$

$u = 3\cos(3x) \quad dv = e^x dx$

$du = -9\sin(3x)dx \quad v = e^x$

$$\int e^x \sin(3x)\,dx = e^x\sin(3x) - \left(e^x\cdot 3\cos(3x) + \int 9\sin(3x)\,e^x\,dx\right)$$

$$10\int e^x \sin(3x)\,dx = e^x\sin(3x) - 3e^x\cos(3x)$$

$$\int e^x \sin(3x)\,dx = \frac{1}{10}\left(e^x\sin(3x) - 3e^x\cos(3x)\right) + C$$

c) $\int \sin^5\theta \, d\theta$ $= \int (1-\cos^2\theta)^2 \sin\theta \, d\theta$

$u = \cos\theta \quad du = -\sin\theta \, d\theta$

$= \int (1-u^2)^2 \cdot -du$

$= \int (1-2u^2+u^4) \cdot -du$

$= \frac{-u^5}{5} + \frac{2}{3}u^3 - u + C$

$= \frac{-\cos^5\theta}{5} + \frac{2}{3}\cos^3\theta - \cos\theta + C$

d) $\int_{2\sqrt{2}}^{4} \frac{1}{x\sqrt{x^2-4}} \, dx$ $\quad x = 2\sec\theta \quad dx = 2\sec\theta\tan\theta \, d\theta$

If $x = 2\sqrt{2} = 2\sec\theta$, $\theta = \frac{\pi}{4}$.

If $x = 4 = 2\sec\theta$, $\theta = \frac{\pi}{3}$

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{2\sec\theta\sqrt{4\sec^2\theta - 4}} \cdot 2\sec\theta\tan\theta \, d\theta$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\sec\theta\tan\theta}{2\sec\theta \, 2\tan\theta} \, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{d\theta}{2}$

$= \frac{\theta}{2}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{\pi}{6} - \frac{\pi}{8} = \frac{\pi}{24}$

2.) Determine whether each improper integral below is convergent or divergent. Evaluate those that are convergent. [15 pts.]

$$\int_2^\infty \frac{dx}{\sqrt{x}} = \lim_{t\to\infty} \int_2^t \frac{dx}{\sqrt{x}} = \lim_{t\to\infty} \frac{1}{\left(-\frac{1}{2}+1\right)} x^{-\frac{1}{2}+1}\Big|_2^t$$

$$= \lim_{t\to\infty} 2\sqrt{x}\Big|_2^t$$

$$= \lim_{t\to\infty} 2\sqrt{t} - 2\sqrt{2} = \infty.$$

Integral diverges.

$$\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$$

$$= \int_0^3 \frac{1}{3\sqrt{1-\frac{x^2}{9}}}\,dx = \int_0^3 \frac{1}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}\,dx \qquad u = \frac{x}{3} \quad du = \frac{dx}{3}$$

$$= \int_{\frac{0}{3}}^{\frac{3}{3}} \frac{1}{\sqrt{1-u^2}}\,du = \int_0^1 \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \sin^{-1}(u)\Big|_0^1$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$ Converges.

$$\int_0^\infty \frac{dz}{z^2+3z+2}$$

Partial fractions: $\frac{a}{z+2} + \frac{b}{z+1} = \frac{1}{(z+2)(z+1)}$

$a(z+1) + b(z+2) = 1$

$(a+b)z + (a+2b) = 1 = 0z + 1$

$a+b=0, \quad a+2b=1$

$b=1, \; a=-1$

$$\int_0^\infty \frac{1}{z+1} - \frac{1}{z+2}\,dz$$

$$= \lim_{t\to\infty} \ln(z+1) - \ln(z+2)\Big|_0^t$$

$$= \lim_{t\to\infty} \ln\left(\frac{z+1}{z+2}\right)\Big|_0^t = \ln 1 - \ln\frac{1}{2} = \ln 2.$$ Converges.

3.) Find the length of the curve $f(x) = x^3 + \frac{1}{12x}$ on the interval $\left[\frac{1}{2}, 2\right]$. [10 pts.]

$$\int_{\frac{1}{2}}^{2} \sqrt{1 + \left(3x^2 - \frac{1}{12x^2}\right)^2}\, dx = \int_{\frac{1}{2}}^{2} \sqrt{1 + 9x^4 - \frac{1}{2} + \frac{1}{144x^4}}\, dx$$

$$= \int_{\frac{1}{2}}^{2} \sqrt{\left(3x^2 + \frac{1}{12x^2}\right)^2}\, dx$$

$$= \int_{\frac{1}{2}}^{2} 3x^2 + \frac{1}{12x^2}\, dx$$

$$= x^3 - \frac{1}{12x} \Big|_{\frac{1}{2}}^{2}$$

$$= \left(8 - \frac{1}{24}\right) - \left(\frac{1}{8} - \frac{1}{6}\right)$$

4.) Determine whether each of the following statements is true or false. [10 pts.]

i. If $\{a_n\}$ is decreasing and $a_n > 0$ for all n, then $\{a_n\}$ is convergent.

True *False*

By the monotonic sequence theorem

ii. If $f(x) \le g(x)$ and $\int_0^\infty g(x)dx$ diverges, then $\int_0^\infty f(x)dx$ also diverges.

True *False*

Not necessarily. Say $g(x) = \frac{1}{x+1}$ and $f(x) = \frac{1}{(x+1)^2}$.

iii. The integral $\int \frac{3}{x^2+7} dx$ can be solved by partial fractions.

True *False*

x^2+7 can't be factored.

iv. The integral $\int_1^\infty \frac{1}{x^\pi} dx$ converges.

True *False*

$\pi > 1$.

v. $\int_0^3 e^{x^2} dx = \int_0^5 e^{x^2} dx + \int_5^3 e^{x^2} dx$.

True *False*

$$\int_5^3 e^{x^2} dx = -\int_3^5 e^{x^2} dx$$

and $\int_0^3 e^{x^2} dx + \int_3^5 e^{x^2} dx = \int_0^5 e^{x^2} dx$

5.) Determine whether each of the following sequences is convergent or divergent. If a sequence is convergent, find its limit. [15 pts.]

a.) $a_n = n \sin\left(\frac{1}{n}\right) = \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}$ l'Hospital

$\lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{-\frac{1}{x^2} \cancel{\sin} \cos\left(\frac{1}{x}\right)}{\frac{-1}{x^2}} = \lim_{x\to\infty} \cos\left(\frac{1}{x}\right)$

$= \cos(0) = 1$

Convergent

b.) $a_n = \arcsin\left(\frac{3n}{3n+8}\right)$

$\lim_{n\to\infty} \left(\frac{3n}{3n+8}\right) = 1$, so

$\lim_{n\to\infty} \arcsin\left(\frac{3n}{3n+8}\right) = \arcsin(1) = \frac{\pi}{2}$. Convergent.

c.) $a_n = -5 + (0.9)^n$

$\lim_{n\to\infty} (.9)^n = 0$

So $\lim_{n\to\infty} a_n = -5 + 0 = -5$. Convergent.

d.) $a_n = 4 + (-1)^n$

$a_1 = 3,\ a_2 = 5,\ a_3 = 3,\ a_4 = 5, \ldots$

Divergent.

e.) $a_n = \frac{n^2+2n-12}{n+2} \longrightarrow \infty$

Divergent