SOLUTIONS

MOCK MIDTERM 1

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Instructions: This is a mock midterm, designed to give you some practice for the actual midterm. It should be similar in length (and in spirit) as the actual midterm.

1		25
2		15
3		30
4		30
Total		100

ANSWERS

1. 21

2. (a) $F(x) = \tan(x) + \sec(x) + 1$

   (b) $\frac{9}{2}\pi$

   (c) $\frac{1}{x^2}\sqrt{1+e^{\frac{1}{x}}}$

3. (a) $\tan^{-1}(x) + \frac{1}{2}\ln|1+x^2| + C$

   (b) $\frac{1}{5}(1+x^2)^{\frac{5}{2}} - \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C$

   (c) $-\ln(4)$

4. (a) $1/6$

   (b) (i) $\int_0^1 \pi(x-x^2)\,dx$

   (ii) $\int_0^1 \pi(y^2-y^4)\,dy$

   (iii) $\int_0^1 \pi\left((y+1)^2-(y^2+1)^2\right)dy$

   (iv) $\int_0^1 \pi\left((2-x)^2-(2-\sqrt{x})^2\right)dx$

Date: Friday, October 20, 2017.

1. *(25 points)* Use the **definition** of the integral (in terms of Riemann sums) to evaluate

$$\int_1^4 x^2 dx$$

You are allowed to use the following facts:

$$\sum_{i=1}^n 1 = n, \quad \sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

PREP WORK

$$\Delta x = \frac{4-1}{N} = \frac{3}{N}$$

$$x_i = 1 + (\Delta x) i = 1 + \frac{3i}{N}$$

$$\int_1^4 x^2 dx = \lim_{N \to \infty} (\Delta x) \sum_{i=1}^N f(x_i), \quad f(x_i) = (x_i)^2 = \left(1 + \frac{3i}{N}\right)^2$$

$$= \lim_{N \to \infty} \frac{3}{N} \sum_{i=1}^N \left(1 + \frac{3i}{N}\right)^2, \quad (a+b)^2 = a^2 + 2ab + b^2$$

$$= \lim_{N \to \infty} \frac{3}{N} \sum_{i=1}^N 1 + \frac{6i}{N} + \frac{9i^2}{N^2}$$

$$= \lim_{N \to \infty} \frac{3}{N}\left(\sum_{i=1}^N 1\right) + \left(\frac{3}{N}\right) \sum_{i=1}^N \frac{6i}{N} + \left(\frac{3}{N}\right) \sum_{i=1}^N \frac{9i^2}{N^2}$$

$$= \lim_{N \to \infty} \frac{3}{N}\left(\sum_{i=1}^N 1\right) + \left(\frac{3}{N}\right)\left(\frac{6}{N}\right)\left(\sum_{i=1}^N i\right) + \left(\frac{3}{N}\right)\left(\frac{9}{N^2}\right)\left(\sum_{i=1}^N i^2\right) \quad \text{FACTS}$$

$$= \lim_{N \to \infty} \left(\frac{3}{\cancel{N}}\right)(\cancel{N}) + \frac{18}{N^{\cancel{2}}} \frac{\cancel{N}(N+1)}{2} + \frac{27}{N^{\cancel{3}2}} \frac{\cancel{N}(N+1)(2N+1)}{6}$$

USE L'HÔPITAL, FOR EXAMPLE

$$= \lim_{N \to \infty} 3 + 9\left(\frac{N+1}{N}\right) + \frac{27}{6} \frac{(N+1)(2N+1)}{N^2} = 3 + 9 + \left(\frac{27}{6}\right)(2)$$

$$= 3 + 9 + 9 = 21$$

2. *(15 points, 5 points each)* Find the following:

(a) The antiderivative F of $f(x) = \sec(x)(\sec(x) + \tan(x))$ that satisfies [redacted] $F(0) = 2$

↳ TYPO

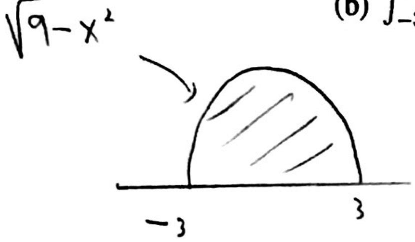
$f(x) = \sec^2(x) + \sec(x)\tan(x)$

$F(x) = \tan(x) + \sec(x) + C$

$F(0) = \tan(0) + \sec(0) + C = 1 + C = 2 \Rightarrow C = 1$

$\boxed{F(x) = \tan(x) + \sec(x) + 1}$

(b) $\int_{-3}^{3} \sqrt{9 - x^2}dx$



$= \text{AREA OF SEMICIRCLE OF RADIUS } 3$

$= \frac{1}{2}\pi(3^2)$

$= \frac{9}{2}\pi$

F IS AN AD OF $f(x) = \sqrt{1+e^x}$

(c) The derivative of $\int_{\frac{1}{x}}^{2} \sqrt{1 + e^x}dx$

$$\left(\int_{1/x}^{2} \sqrt{1+e^x}\,dx\right)' = \left(F(2) - F\left(\frac{1}{x}\right)\right)'$$

$$= -F'\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right)$$

$$= -f\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right)$$

$$= \frac{1}{x^2} f\left(\frac{1}{x}\right)$$

$$= \boxed{\frac{1}{x^2}\sqrt{1+e^{\frac{1}{x}}}}$$

3. *(30 points, 10 points each)* Find the following integrals

(a) $\int \frac{1+x}{1+x^2} dx$

Hint: Split the fraction up into two parts

$$\int \frac{1+x}{1+x^2} dx = \int \frac{1}{1+x^2} + \int \frac{x}{1+x^2} dx$$

$U = 1+x^2$

$dU = 2x\,dx$

$x\,dx = \frac{dU}{2}$

$$= \tan^{-1}(x) + \int \frac{\frac{1}{2}dU}{U}$$

$$= \tan^{-1}(x) + \frac{1}{2}\int \frac{1}{U} dU = \tan^{-1}(x) + \frac{1}{2}\ln|U| + C$$

$$= \boxed{\tan^{-1}(x) + \frac{1}{2}\ln|1+x^2| + C}$$

(b) $\int x^3\sqrt{1+x^2}dx$

$U = 1+x^2$, $dU = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}dU$

$$\int x^3\sqrt{1+x^2}\,dx = \int x^2\sqrt{1+x^2}\,(x\,dx) = \int x^2\sqrt{U}\left(\frac{1}{2}dU\right)$$

$U = 1+x^2$

$x^2 = U-1$

$$= \int (U-1)\sqrt{U}\cdot\left(\frac{1}{2}dU\right) = \frac{1}{2}\int U^{\frac{3}{2}} - U^{\frac{1}{2}}\,dU = \frac{1}{2}\left(\frac{2}{5}U^{\frac{5}{2}} - \frac{2}{3}U^{\frac{3}{2}}\right) + C$$

$$= \frac{1}{5}U^{5/2} - \frac{1}{3}U^{\frac{3}{2}} + C$$

$$= \boxed{\frac{1}{5}(1+x^2)^{\frac{5}{2}} - \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C}$$

(c) $\int_{e^{-4}}^{e^{-1}} \frac{1}{x\ln(x)}dx$

$U = \ln(x)$, $dU = \frac{1}{x}dx$,

$U(e^{-1}) = -1$, $U(e^{-4}) = -4$

$$\int_{e^{-4}}^{e^{-1}} \frac{1}{x\ln(x)}dx = \int_{-4}^{-1} \frac{1}{U}dU = \left[\ln|U|\right]_{-4}^{-1}$$

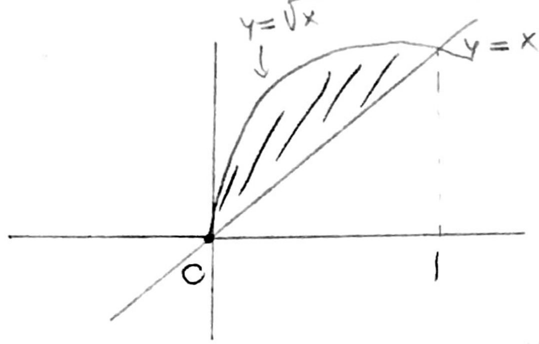
$$= \ln|-1| - \ln|-4|$$

$$= \ln(1) - \ln(4)$$

$$= -\ln(4)$$

4. *(30 points)*

(a) *(10 points)* Find the area of the region enclosed by the curves $y = x$ and $y = \sqrt{x}$. Illustrate with a picture.



POINTS OF INTERSECTION

$x = \sqrt{x} \Rightarrow x^2 = x \Rightarrow x^2 - x = 0$

$\Rightarrow (x)(x-1) = 0 \Rightarrow x = 0$ or $x = 1$

$$\text{AREA} = \int_0^1 \sqrt{x} - x \, dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2\right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{4}{6} - \frac{3}{6} = \frac{1}{6}$$

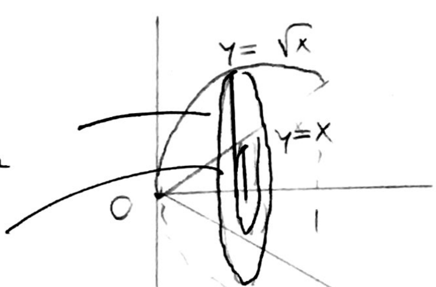
(b) *(20 points, 5 points each)* Find an expression of **but do not evaluate** the volume obtained by rotating the region in (a) about:

(i) The $x-$axis
(ii) The $y-$axis
(iii) The line $x = -1$
(iv) The line $y = 2$

(i)

$\sqrt{x}$ OUTER

x INNER



WASHER METHOD

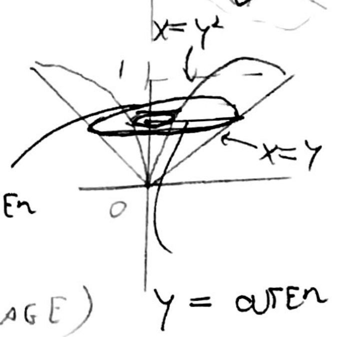
OUTER $= \sqrt{x}$

INNER $= x$

$$\int_0^1 \pi\left((\sqrt{x})^2 - x^2\right) dx = \int_0^1 \pi(x - x^2)\, dx$$

(ii)

y^2 = INNER



y = OUTER

WASHER METHOD

OUTER $= y$

INNER $= y^2$ $(y = \sqrt{x} \Rightarrow x = y^2)$

$$\int_0^1 \pi\left(y^2 - (y^2)^2\right) dy = \int_0^1 \pi(y^2 - y^4)\, dy$$

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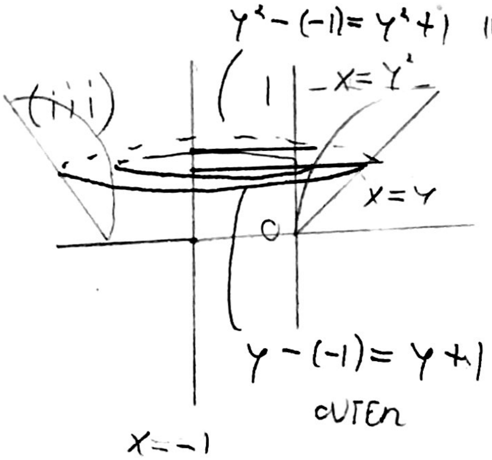


WASHER METHOD

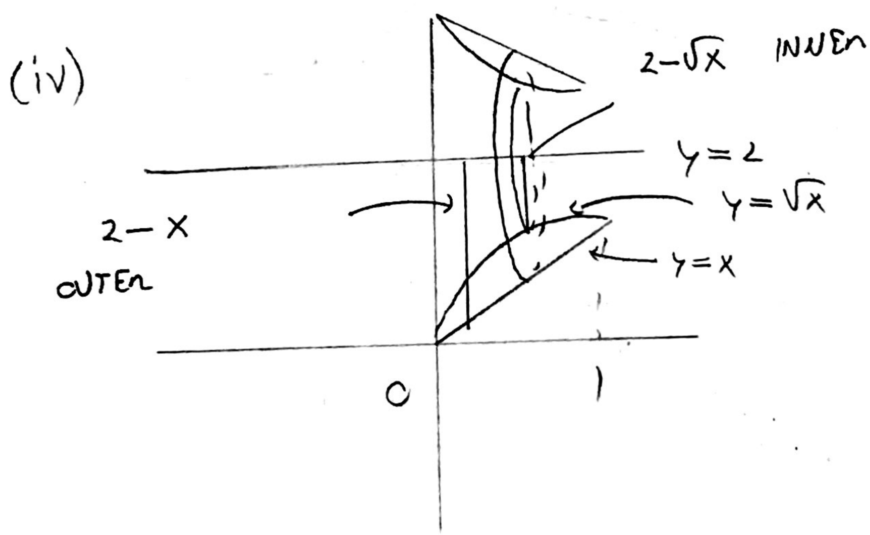
OUTER : $y+1$

INNER : y^2+1

$(y=\sqrt{x} \Rightarrow x=y^2)$

$$\int_0^1 \pi\left((y+1)^2-(y^2+1)^2\right)dy$$

(iv)



WASHER METHOD

OUTER : $2-x$

INNER : $2-\sqrt{x}$

$$\int_0^1 \pi\left((2-x)^2-(2-\sqrt{x})^2\right)dx$$