$x_{n+1} = x_n - \dfrac{x_n^5 - (2+r)x_n^4 + (1+2r)x_n^3 - (1+r)x_n^2 + 2(1-r)x_n + r - 1}{5x_n^4 - 4(2+r)x_n^3 + 3(1+2r)x_n^2 - 2(1+r)x_n + 2(1-r)}$. Again, we substitute $r \approx 3.04042 \times 10^{-6}$. $L_2$ is slightly more than 1 AU from the sun and, judging from the result of part (a), probably less than 0.02 AU from earth. So we take $x_1 = 1.02$ and get $x_2 \approx 1.01422$, $x_3 \approx 1.01118$, $x_4 \approx 1.01018$, $x_5 \approx 1.01008 \approx x_6$. So, to five decimal places, $L_2$ is located 1.01008 AU from the sun (or 0.01008 AU from the earth).

## 4.9 Antiderivatives

**1.** $f(x) = 4x + 7 = 4x^1 + 7 \quad \Rightarrow \quad F(x) = 4\dfrac{x^{1+1}}{1+1} + 7x + C = 2x^2 + 7x + C$

*Check:* $F'(x) = 2(2x) + 7 + 0 = 4x + 7 = f(x)$

**2.** $f(x) = x^2 - 3x + 2 \quad \Rightarrow \quad F(x) = \dfrac{x^3}{3} - 3\dfrac{x^2}{2} + 2x + C = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + C$

*Check:* $F'(x) = \frac{1}{3}(3x^2) - \frac{3}{2}(2x) + 2 + 0 = x^2 - 3x + 2 = f(x)$

**3.** $f(x) = 2x^3 - \frac{2}{3}x^2 + 5x \quad \Rightarrow \quad F(x) = 2\,\dfrac{x^{3+1}}{3+1} - \dfrac{2}{3}\dfrac{x^{2+1}}{2+1} + 5\,\dfrac{x^{1+1}}{1+1} = \frac{1}{2}x^4 - \frac{2}{9}x^3 + \frac{5}{2}x^2 + C$

*Check:* $F'(x) = \frac{1}{2}(4x^3) - \frac{2}{9}(3x^2) + \frac{5}{2}(2x) + 0 = 2x^3 - \frac{2}{3}x^2 + 5x = f(x)$

**4.** $f(x) = 6x^5 - 8x^4 - 9x^2 \quad \Rightarrow \quad F(x) = 6\,\dfrac{x^6}{6} - 8\,\dfrac{x^5}{5} - 9\,\dfrac{x^3}{3} + C = x^6 - \frac{8}{5}x^5 - 3x^3 + C$

**5.** $f(x) = x(12x + 8) = 12x^2 + 8x \quad \Rightarrow \quad F(x) = 12\,\dfrac{x^3}{3} + 8\,\dfrac{x^2}{2} + C = 4x^3 + 4x^2 + C$

**6.** $f(x) = (x-5)^2 = x^2 - 10x + 25 \quad \Rightarrow \quad F(x) = \dfrac{x^3}{3} - 10\dfrac{x^2}{2} + 25x + C = \frac{1}{3}x^3 - 5x^2 + 25x + C$

**7.** $f(x) = 7x^{2/5} + 8x^{-4/5} \quad \Rightarrow \quad F(x) = 7\left(\frac{5}{7}x^{7/5}\right) + 8(5x^{1/5}) + C = 5x^{7/5} + 40x^{1/5} + C$

**8.** $f(x) = x^{3.4} - 2x^{\sqrt{2}-1} \quad \Rightarrow \quad F(x) = \dfrac{x^{4.4}}{4.4} - 2\left(\dfrac{x^{\sqrt{2}}}{\sqrt{2}}\right) + C = \dfrac{5}{22}x^{4.4} - \sqrt{2}\,x^{\sqrt{2}} + C$

**9.** $f(x) = \sqrt{2}$ is a constant function, so $F(x) = \sqrt{2}\,x + C$.

**10.** $f(x) = e^2$ is a constant function, so $F(x) = e^2 x + C$.

**11.** $f(x) = 3\sqrt{x} - 2\sqrt[3]{x} = 3x^{1/2} - 2x^{1/3} \quad \Rightarrow \quad F(x) = 3\left(\frac{2}{3}x^{3/2}\right) - 2\left(\frac{3}{4}x^{4/3}\right) + C = 2x^{3/2} - \frac{3}{2}x^{4/3} + C$

**12.** $f(x) = \sqrt[3]{x^2} + x\sqrt{x} = x^{2/3} + x^{3/2} \quad \Rightarrow \quad F(x) = \frac{3}{5}x^{5/3} + \frac{2}{5}x^{5/2} + C$

**13.** $f(x) = \dfrac{1}{5} - \dfrac{2}{x} = \dfrac{1}{5} - 2\left(\dfrac{1}{x}\right)$ has domain $(-\infty, 0) \cup (0, \infty)$, so $F(x) = \begin{cases} \frac{1}{5}x - 2\ln|x| + C_1 & \text{if } x < 0 \\ \frac{1}{5}x - 2\ln|x| + C_2 & \text{if } x > 0 \end{cases}$

See Example 1(b) for a similar problem.

**14.** $f(t) = \dfrac{3t^4 - t^3 + 6t^2}{t^4} = 3 - \dfrac{1}{t} + \dfrac{6}{t^2}$ has domain $(-\infty, 0) \cup (0, \infty)$, so $F(t) = \begin{cases} 3t - \ln|t| - \dfrac{6}{t} + C_1 & \text{if } t < 0 \\ 3t - \ln|t| - \dfrac{6}{t} + C_2 & \text{if } t > 0 \end{cases}$

See Example 1(b) for a similar problem.

**15.** $g(t) = \dfrac{1 + t + t^2}{\sqrt{t}} = t^{-1/2} + t^{1/2} + t^{3/2} \quad \Rightarrow \quad G(t) = 2t^{1/2} + \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + C$

**16.** $r(\theta) = \sec\theta\,\tan\theta - 2e^\theta \quad \Rightarrow \quad R(\theta) = \sec\theta - 2e^\theta + C_n$ on the interval $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$.

**17.** $h(\theta) = 2\sin\theta - \sec^2\theta \quad \Rightarrow \quad H(\theta) = -2\cos\theta - \tan\theta + C_n$ on the interval $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$.

**18.** $g(v) = 2\cos v - \dfrac{3}{\sqrt{1 - v^2}} \quad \Rightarrow \quad G(v) = 2\sin v - 3\sin^{-1} v + C$

**19.** $f(x) = 2^x + 4\sinh x \quad \Rightarrow \quad F(x) = \dfrac{2^x}{\ln 2} + 4\cosh x + C$

**20.** $f(x) = 1 + 2\sin x + 3/\sqrt{x} = 1 + 2\sin x + 3x^{-1/2} \quad \Rightarrow \quad F(x) = x - 2\cos x + 3\,\dfrac{x^{1/2}}{1/2} + C = x - 2\cos x + 6\sqrt{x} + C$

**21.** $f(x) = \dfrac{2x^4 + 4x^3 - x}{x^3}$, $x > 0$; $f(x) = 2x + 4 - x^{-2} \quad \Rightarrow$

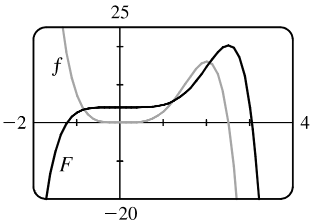
$F(x) = 2\,\dfrac{x^2}{2} + 4x - \dfrac{x^{-2+1}}{-2+1} + C = x^2 + 4x + \dfrac{1}{x} + C$, $x > 0$

**22.** $f(x) = \dfrac{2x^2 + 5}{x^2 + 1} = \dfrac{2(x^2 + 1) + 3}{x^2 + 1} = 2 + \dfrac{3}{x^2 + 1} \quad \Rightarrow \quad F(x) = 2x + 3\tan^{-1} x + C$

**23.** $f(x) = 5x^4 - 2x^5 \quad \Rightarrow \quad F(x) = 5 \cdot \dfrac{x^5}{5} - 2 \cdot \dfrac{x^6}{6} + C = x^5 - \frac{1}{3}x^6 + C$.

$F(0) = 4 \quad \Rightarrow \quad 0^5 - \frac{1}{3} \cdot 0^6 + C = 4 \quad \Rightarrow \quad C = 4$, so $F(x) = x^5 - \frac{1}{3}x^6 + 4$.

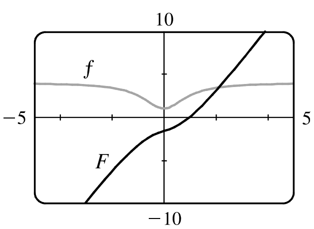
The graph confirms our answer since $f(x) = 0$ when $F$ has a local maximum, $f$ is positive when $F$ is increasing, and $f$ is negative when $F$ is decreasing.




**24.** $f(x) = 4 - 3(1 + x^2)^{-1} = 4 - \dfrac{3}{1 + x^2} \quad \Rightarrow \quad F(x) = 4x - 3\tan^{-1} x + C$.

$F(1) = 0 \quad \Rightarrow \quad 4 - 3\left(\frac{\pi}{4}\right) + C = 0 \quad \Rightarrow \quad C = \frac{3\pi}{4} - 4$, so

$F(x) = 4x - 3\tan^{-1} x + \frac{3\pi}{4} - 4$. Note that $f$ is positive and $F$ is increasing on $\mathbb{R}$. Also, $f$ has smaller values where the slopes of the tangent lines of $F$ are smaller.




**25.** $f''(x) = 20x^3 - 12x^2 + 6x \quad \Rightarrow \quad f'(x) = 20\left(\dfrac{x^4}{4}\right) - 12\left(\dfrac{x^3}{3}\right) + 6\left(\dfrac{x^2}{2}\right) + C = 5x^4 - 4x^3 + 3x^2 + C \quad \Rightarrow$

$f(x) = 5\left(\dfrac{x^5}{5}\right) - 4\left(\dfrac{x^4}{4}\right) + 3\left(\dfrac{x^3}{3}\right) + Cx + D = x^5 - x^4 + x^3 + Cx + D$

**26.** $f''(x) = x^6 - 4x^4 + x + 1 \quad \Rightarrow \quad f'(x) = \frac{1}{7}x^7 - \frac{4}{5}x^5 + \frac{1}{2}x^2 + x + C \quad \Rightarrow$

$f(x) = \frac{1}{56}x^8 - \frac{2}{15}x^6 + \frac{1}{6}x^3 + \frac{1}{2}x^2 + Cx + D$

**27.** $f''(x) = 2x + 3e^x \quad \Rightarrow \quad f'(x) = x^2 + 3e^x + C \quad \Rightarrow \quad f(x) = \frac{1}{3}x^3 + 3e^x + Cx + D$

**28.** $f''(x) = 1/x^2 = x^{-2} \quad \Rightarrow \quad f'(x) = \begin{cases} -1/x + C_1 & \text{if } x < 0 \\ -1/x + C_2 & \text{if } x > 0 \end{cases} \quad \Rightarrow \quad f(x) = \begin{cases} -\ln(-x) + C_1x + D_1 & \text{if } x < 0 \\ -\ln x + C_2x + D_2 & \text{if } x > 0 \end{cases}$

**29.** $f'''(t) = 12 + \sin t \quad \Rightarrow \quad f''(t) = 12t - \cos t + C_1 \quad \Rightarrow \quad f'(t) = 6t^2 - \sin t + C_1t + D \quad \Rightarrow$

$f(t) = 2t^3 + \cos t + Ct^2 + Dt + E$, where $C = \frac{1}{2}C_1$.

**30.** $f'''(t) = \sqrt{t} - 2\cos t = t^{1/2} - 2\cos t \quad \Rightarrow \quad f''(t) = \frac{2}{3}t^{3/2} - 2\sin t + C_1 \quad \Rightarrow \quad f'(t) = \frac{4}{15}t^{5/2} + 2\cos t + C_1t + D \quad \Rightarrow$

$f(t) = \frac{8}{105}t^{7/2} + 2\sin t + Ct^2 + Dt + E$, where $C = \frac{1}{2}C_1$.

**31.** $f'(x) = 1 + 3\sqrt{x} \quad \Rightarrow \quad f(x) = x + 3\left(\frac{2}{3}x^{3/2}\right) + C = x + 2x^{3/2} + C. \quad f(4) = 4 + 2(8) + C$ and $f(4) = 25 \quad \Rightarrow$

$20 + C = 25 \quad \Rightarrow \quad C = 5$, so $f(x) = x + 2x^{3/2} + 5$.

**32.** $f'(x) = 5x^4 - 3x^2 + 4 \quad \Rightarrow \quad f(x) = x^5 - x^3 + 4x + C. \quad f(-1) = -1 + 1 - 4 + C$ and $f(-1) = 2 \quad \Rightarrow$

$-4 + C = 2 \quad \Rightarrow \quad C = 6$, so $f(x) = x^5 - x^3 + 4x + 6$.

**33.** $f'(t) = \dfrac{4}{1 + t^2} \quad \Rightarrow \quad f(t) = 4\arctan t + C. \quad f(1) = 4\left(\dfrac{\pi}{4}\right) + C$ and $f(1) = 0 \quad \Rightarrow \quad \pi + C = 0 \quad \Rightarrow \quad C = -\pi$,

so $f(t) = 4\arctan t - \pi$.

**34.** $f'(t) = t + \dfrac{1}{t^3},\ t > 0 \quad \Rightarrow \quad f(t) = \dfrac{1}{2}t^2 - \dfrac{1}{2t^2} + C. \quad f(1) = \dfrac{1}{2} - \dfrac{1}{2} + C$ and $f(1) = 6 \quad \Rightarrow \quad C = 6$, so

$f(t) = \dfrac{1}{2}t^2 - \dfrac{1}{2t^2} + 6$.

**35.** $f'(x) = 5x^{2/3} \quad \Rightarrow \quad f(x) = 5\left(\frac{3}{5}x^{5/3}\right) + C = 3x^{5/3} + C$.

$f(8) = 3 \cdot 32 + C$ and $f(8) = 21 \quad \Rightarrow \quad 96 + C = 21 \quad \Rightarrow \quad C = -75$, so $f(x) = 3x^{5/3} - 75$.

**36.** $f'(x) = \dfrac{x + 1}{\sqrt{x}} = x^{1/2} + x^{-1/2} \quad \Rightarrow \quad f(x) = \frac{2}{3}x^{3/2} + 2x^{1/2} + C. \quad f(1) = \frac{2}{3} + 2 + C = \frac{8}{3} + C$ and $f(1) = 5 \quad \Rightarrow$

$C = 5 - \frac{8}{3} = \frac{7}{3}$, so $f(x) = \frac{2}{3}x^{3/2} + 2\sqrt{x} + \frac{7}{3}$.

**37.** $f'(t) = \sec t(\sec t + \tan t) = \sec^2 t + \sec t\,\tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2} \quad \Rightarrow \quad f(t) = \tan t + \sec t + C. \quad f\left(\frac{\pi}{4}\right) = 1 + \sqrt{2} + C$

and $f\left(\frac{\pi}{4}\right) = -1 \quad \Rightarrow \quad 1 + \sqrt{2} + C = -1 \quad \Rightarrow \quad C = -2 - \sqrt{2}$, so $f(t) = \tan t + \sec t - 2 - \sqrt{2}$.

*Note:* The fact that $f$ is defined and continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ means that we have only one constant of integration.

**38.** $f'(t) = 3^t - \dfrac{3}{t} \quad \Rightarrow \quad f(t) = \begin{cases} 3^t/\ln 3 - 3\ln(-t) + C & \text{if} \quad t < 0 \\ 3^t/\ln 3 - 3\ln t + D & \text{if} \quad t > 0 \end{cases}$

$f(-1) = \dfrac{1}{3\ln 3} - 3\ln 1 + C$ and $f(-1) = 1 \quad \Rightarrow \quad C = 1 - \dfrac{1}{3\ln 3}$.

$f(1) = \dfrac{3}{\ln 3} - 3\ln 1 + D$ and $f(1) = 2 \quad \Rightarrow \quad D = 2 - \dfrac{3}{\ln 3}$.

Thus, $f(t) = \begin{cases} 3^t/\ln 3 - 3\ln(-t) + 1 - 1/(3\ln 3) & \text{if} \quad t < 0 \\ 3^t/\ln 3 - 3\ln t + 2 - 3/\ln 3 & \text{if} \quad t > 0 \end{cases}$

**39.** $f''(x) = -2 + 12x - 12x^2 \quad \Rightarrow \quad f'(x) = -2x + 6x^2 - 4x^3 + C$. $f'(0) = C$ and $f'(0) = 12 \quad \Rightarrow \quad C = 12$, so $f'(x) = -2x + 6x^2 - 4x^3 + 12$ and hence, $f(x) = -x^2 + 2x^3 - x^4 + 12x + D$. $f(0) = D$ and $f(0) = 4 \quad \Rightarrow \quad D = 4$, so $f(x) = -x^2 + 2x^3 - x^4 + 12x + 4$.

**40.** $f''(x) = 8x^3 + 5 \quad \Rightarrow \quad f'(x) = 2x^4 + 5x + C$. $f'(1) = 2 + 5 + C$ and $f'(1) = 8 \quad \Rightarrow \quad C = 1$, so $f'(x) = 2x^4 + 5x + 1$. $f(x) = \frac{2}{5}x^5 + \frac{5}{2}x^2 + x + D$. $f(1) = \frac{2}{5} + \frac{5}{2} + 1 + D = D + \frac{39}{10}$ and $f(1) = 0 \quad \Rightarrow \quad D = -\frac{39}{10}$, so $f(x) = \frac{2}{5}x^5 + \frac{5}{2}x^2 + x - \frac{39}{10}$.

**41.** $f''(\theta) = \sin\theta + \cos\theta \quad \Rightarrow \quad f'(\theta) = -\cos\theta + \sin\theta + C$. $f'(0) = -1 + C$ and $f'(0) = 4 \quad \Rightarrow \quad C = 5$, so $f'(\theta) = -\cos\theta + \sin\theta + 5$ and hence, $f(\theta) = -\sin\theta - \cos\theta + 5\theta + D$. $f(0) = -1 + D$ and $f(0) = 3 \quad \Rightarrow \quad D = 4$, so $f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$.

**42.** $f''(t) = t^2 + \dfrac{1}{t^2} = t^2 + t^{-2},\ t > 0 \quad \Rightarrow \quad f'(t) = \frac{1}{3}t^3 - \dfrac{1}{t} + C$. $f'(1) = \frac{1}{3} - 1 + C$ and $f'(1) = 2 \quad \Rightarrow$ $C - \frac{2}{3} = 2 \quad \Rightarrow \quad C = \frac{8}{3}$, so $f'(t) = \frac{1}{3}t^3 - \dfrac{1}{t} + \frac{8}{3}$ and hence, $f(t) = \frac{1}{12}t^4 - \ln t + \frac{8}{3}t + D$. $f(2) = \frac{4}{3} - \ln 2 + \frac{16}{3} + D$ and $f(2) = 3 \quad \Rightarrow \quad \frac{20}{3} - \ln 2 + D = 3 \quad \Rightarrow \quad D = \ln 2 - \frac{11}{3}$, so $f(t) = \frac{1}{12}t^4 - \ln t + \frac{8}{3}t + \ln 2 - \frac{11}{3}$.

**43.** $f''(x) = 4 + 6x + 24x^2 \quad \Rightarrow \quad f'(x) = 4x + 3x^2 + 8x^3 + C \quad \Rightarrow \quad f(x) = 2x^2 + x^3 + 2x^4 + Cx + D$. $f(0) = D$ and $f(0) = 3 \quad \Rightarrow \quad D = 3$, so $f(x) = 2x^2 + x^3 + 2x^4 + Cx + 3$. $f(1) = 8 + C$ and $f(1) = 10 \quad \Rightarrow \quad C = 2$, so $f(x) = 2x^2 + x^3 + 2x^4 + 2x + 3$.

**44.** $f''(x) = x^3 + \sinh x \quad \Rightarrow \quad f'(x) = \frac{1}{4}x^4 + \cosh x + C \quad \Rightarrow \quad f(x) = \frac{1}{20}x^5 + \sinh x + Cx + D$. $f(0) = D$ and $f(0) = 1 \quad \Rightarrow \quad D = 1$, so $f(x) = \frac{1}{20}x^5 + \sinh x + Cx + 1$. $f(2) = \frac{32}{20} + \sinh 2 + 2C + 1$ and $f(2) = 2.6 \quad \Rightarrow$ $\sinh 2 + 2C = 0 \quad \Rightarrow \quad C = -\frac{1}{2}\sinh 2$, so $f(x) = \frac{1}{20}x^5 + \sinh x - \frac{1}{2}(\sinh 2)x + 1$.

**45.** $f''(x) = e^x - 2\sin x \quad \Rightarrow \quad f'(x) = e^x + 2\cos x + C \quad \Rightarrow \quad f(x) = e^x + 2\sin x + Cx + D$.
$f(0) = 1 + 0 + D$ and $f(0) = 3 \quad \Rightarrow \quad D = 2$, so $f(x) = e^x + 2\sin x + Cx + 2$. $f\left(\frac{\pi}{2}\right) = e^{\pi/2} + 2 + \frac{\pi}{2}C + 2$ and $f\left(\frac{\pi}{2}\right) = 0 \quad \Rightarrow \quad e^{\pi/2} + 4 + \frac{\pi}{2}C = 0 \quad \Rightarrow \quad \frac{\pi}{2}C = -e^{\pi/2} - 4 \quad \Rightarrow \quad C = -\frac{2}{\pi}(e^{\pi/2} + 4)$, so
$f(x) = e^x + 2\sin x + -\frac{2}{\pi}(e^{\pi/2} + 4)x + 2$.

**46.** $f''(t) = \sqrt[3]{t} - \cos t = t^{1/3} - \cos t \quad \Rightarrow \quad f'(t) = \frac{3}{4}t^{4/3} - \sin t + C \quad \Rightarrow \quad f(t) = \frac{9}{28}t^{7/3} + \cos t + Ct + D.$

$f(0) = 0 + 1 + 0 + D$ and $f(0) = 2 \quad \Rightarrow \quad D = 1$, so $f(t) = \frac{9}{28}t^{7/3} + \cos t + Ct + 1$. $\quad f(1) = \frac{9}{28} + \cos 1 + C + 1$ and

$f(1) = 2 \quad \Rightarrow \quad C = 2 - \frac{9}{28} - \cos 1 - 1 = \frac{19}{28} - \cos 1$, so $f(t) = \frac{9}{28}t^{7/3} + \cos t + \left(\frac{19}{28} - \cos 1\right)t + 1$.

**47.** $f''(x) = x^{-2}, x > 0 \quad \Rightarrow \quad f'(x) = -1/x + C \quad \Rightarrow \quad f(x) = -\ln|x| + Cx + D = -\ln x + Cx + D$ [since $x > 0$].

$f(1) = 0 \quad \Rightarrow \quad C + D = 0$ and $f(2) = 0 \quad \Rightarrow \quad -\ln 2 + 2C + D = 0 \quad \Rightarrow \quad -\ln 2 + 2C - C = 0$ [since $D = -C$] $\quad \Rightarrow$

$-\ln 2 + C = 0 \quad \Rightarrow \quad C = \ln 2$ and $D = -\ln 2$. So $f(x) = -\ln x + (\ln 2)x - \ln 2$.

**48.** $f'''(x) = \cos x \quad \Rightarrow \quad f''(x) = \sin x + C$. $f''(0) = C$ and $f''(0) = 3 \quad \Rightarrow \quad C = 3$. $f''(x) = \sin x + 3 \quad \Rightarrow$

$f'(x) = -\cos x + 3x + D$. $f'(0) = -1 + D$ and $f'(0) = 2 \quad \Rightarrow \quad D = 3$. $f'(x) = -\cos x + 3x + 3 \quad \Rightarrow$

$f(x) = -\sin x + \frac{3}{2}x^2 + 3x + E$. $f(0) = E$ and $f(0) = 1 \quad \Rightarrow \quad E = 1$. Thus, $f(x) = -\sin x + \frac{3}{2}x^2 + 3x + 1$.

**49.** "The slope of its tangent line at $(x, f(x))$ is $3 - 4x$" means that $f'(x) = 3 - 4x$, so $f(x) = 3x - 2x^2 + C$.

"The graph of $f$ passes through the point $(2, 5)$" means that $f(2) = 5$, but $f(2) = 3(2) - 2(2)^2 + C$, so $5 = 6 - 8 + C \quad \Rightarrow$

$C = 7$. Thus, $f(x) = 3x - 2x^2 + 7$ and $f(1) = 3 - 2 + 7 = 8$.

**50.** $f'(x) = x^3 \quad \Rightarrow \quad f(x) = \frac{1}{4}x^4 + C$. $\quad x + y = 0 \quad \Rightarrow \quad y = -x \quad \Rightarrow \quad m = -1$. Now $m = f'(x) \quad \Rightarrow \quad -1 = x^3 \quad \Rightarrow$

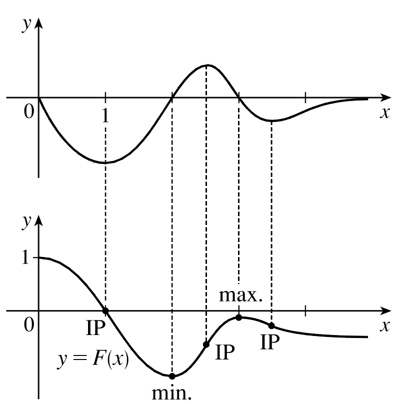
$x = -1 \quad \Rightarrow \quad y = 1$ (from the equation of the tangent line), so $(-1, 1)$ is a point on the graph of $f$. From $f$,

$1 = \frac{1}{4}(-1)^4 + C \quad \Rightarrow \quad C = \frac{3}{4}$. Therefore, the function is $f(x) = \frac{1}{4}x^4 + \frac{3}{4}$.

**51.** $b$ is the antiderivative of $f$. For small $x$, $f$ is negative, so the graph of its antiderivative must be decreasing. But both $a$ and $c$ are increasing for small $x$, so only $b$ can be $f$'s antiderivative. Also, $f$ is positive where $b$ is increasing, which supports our conclusion.

**52.** We know right away that $c$ cannot be $f$'s antiderivative, since the slope of $c$ is not zero at the $x$-value where $f = 0$. Now $f$ is positive when $a$ is increasing and negative when $a$ is decreasing, so $a$ is the antiderivative of $f$.

**53.**




The graph of $F$ must start at $(0, 1)$. Where the given graph, $y = f(x)$, has a local minimum or maximum, the graph of $F$ will have an inflection point.

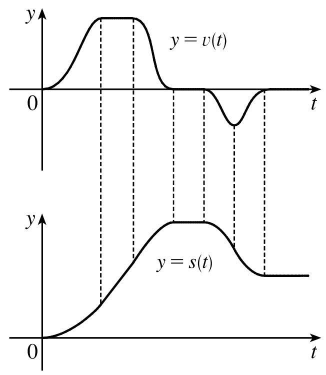
Where $f$ is negative (positive), $F$ is decreasing (increasing).

Where $f$ changes from negative to positive, $F$ will have a minimum.

Where $f$ changes from positive to negative, $F$ will have a maximum.

Where $f$ is decreasing (increasing), $F$ is concave downward (upward).

**54.**

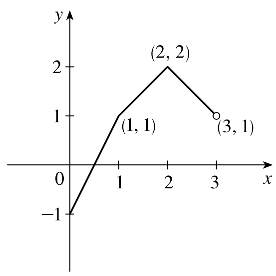



Where $v$ is positive (negative), $s$ is increasing (decreasing).

Where $v$ is increasing (decreasing), $s$ is concave upward (downward).

Where $v$ is horizontal (a steady velocity), $s$ is linear.

**55.**




$$f'(x) = \begin{cases} 2 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 < x < 2 \\ -1 & \text{if } 2 < x < 3 \end{cases} \quad \Rightarrow \quad f(x) = \begin{cases} 2x + C & \text{if } 0 \le x < 1 \\ x + D & \text{if } 1 < x < 2 \\ -x + E & \text{if } 2 < x < 3 \end{cases}$$

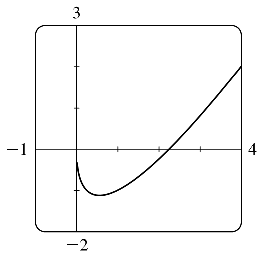
$f(0) = -1 \;\Rightarrow\; 2(0) + C = -1 \;\Rightarrow\; C = -1$. Starting at the point $(0, -1)$ and moving to the right on a line with slope 2 gets us to the point $(1, 1)$.

The slope for $1 < x < 2$ is 1, so we get to the point $(2, 2)$. Here we have used the fact that $f$ is continuous. We can include the point $x = 1$ on either the first or the second part of $f$. The line connecting $(1, 1)$ to $(2, 2)$ is $y = x$, so $D = 0$. The slope for $2 < x < 3$ is $-1$, so we get to $(3, 1)$. $\quad f(2) = 2 \;\Rightarrow\; -2 + E = 2 \;\Rightarrow\; E = 4$. Thus,

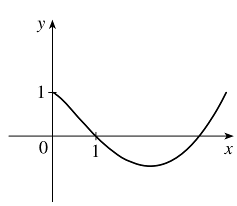
$$f(x) = \begin{cases} 2x - 1 & \text{if } 0 \le x \le 1 \\ x & \text{if } 1 < x < 2 \\ -x + 4 & \text{if } 2 \le x < 3 \end{cases}$$

Note that $f'(x)$ does not exist at $x = 1$, 2, or 3.

**56.** (a)



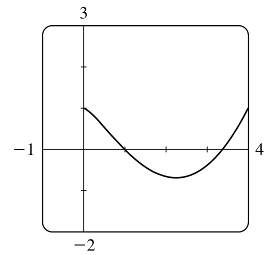

(b) Since $F(0) = 1$, we can start our graph at $(0, 1)$. $f$ has a minimum at about $x = 0.5$, so its derivative is zero there. $f$ is decreasing on $(0, 0.5)$, so its derivative is negative and hence, $F$ is CD on $(0, 0.5)$ and has an IP at $x \approx 0.5$. On $(0.5, 2.2)$, $f$ is negative and increasing ($f'$ is positive), so $F$ is decreasing and CU. On $(2.2, \infty)$, $f$ is positive and increasing, so $F$ is increasing and CU.



(c) $f(x) = 2x - 3\sqrt{x} \quad \Rightarrow \quad F(x) = x^2 - 3 \cdot \frac{2}{3}x^{3/2} + C.$

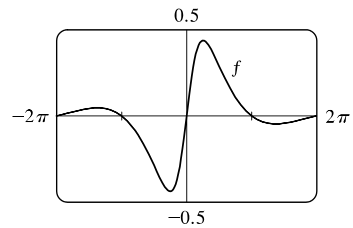
$F(0) = C$ and $F(0) = 1 \quad \Rightarrow \quad C = 1$, so $F(x) = x^2 - 2x^{3/2} + 1.$

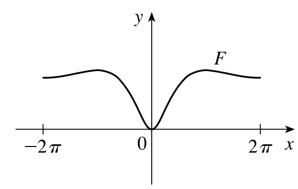
(d)




**57.** $f(x) = \dfrac{\sin x}{1 + x^2}, \ -2\pi \le x \le 2\pi$

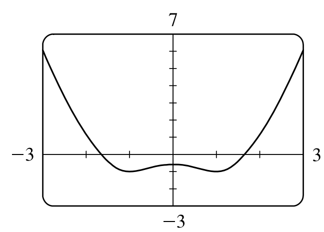
Note that the graph of $f$ is one of an odd function, so the graph of $F$ will be one of an even function.

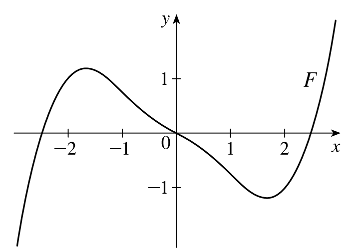



**58.** $f(x) = \sqrt{x^4 - 2x^2 + 2} - 2, \ -3 \le x \le 3$

Note that the graph of $f$ is one of an even function, so the graph of $F$ will be one of an odd function.




**59.** $v(t) = s'(t) = \sin t - \cos t \quad \Rightarrow \quad s(t) = -\cos t - \sin t + C.$ $s(0) = -1 + C$ and $s(0) = 0 \quad \Rightarrow \quad C = 1$, so $s(t) = -\cos t - \sin t + 1.$

**60.** $v(t) = s'(t) = t^2 - 3\sqrt{t} = t^2 - 3t^{1/2} \quad \Rightarrow \quad s(t) = \frac{1}{3}t^3 - 2t^{3/2} + C.$ $s(4) = \frac{64}{3} - 16 + C$ and $s(4) = 8 \quad \Rightarrow$ $C = 8 - \frac{64}{3} + 16 = \frac{8}{3}$, so $s(t) = \frac{1}{3}t^3 - 2t^{3/2} + \frac{8}{3}.$

**61.** $a(t) = v'(t) = 2t + 1 \quad \Rightarrow \quad v(t) = t^2 + t + C.$ $v(0) = C$ and $v(0) = -2 \quad \Rightarrow \quad C = -2$, so $v(t) = t^2 + t - 2$ and $s(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + D.$ $s(0) = D$ and $s(0) = 3 \quad \Rightarrow \quad D = 3$, so $s(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + 3.$

**62.** $a(t) = v'(t) = 3\cos t - 2\sin t \quad \Rightarrow \quad v(t) = 3\sin t + 2\cos t + C.$ $v(0) = 2 + C$ and $v(0) = 4 \quad \Rightarrow \quad C = 2$, so $v(t) = 3\sin t + 2\cos t + 2$ and $s(t) = -3\cos t + 2\sin t + 2t + D.$ $s(0) = -3 + D$ and $s(0) = 0 \quad \Rightarrow \quad D = 3$, so $s(t) = -3\cos t + 2\sin t + 2t + 3.$

**63.** $a(t) = v'(t) = 10\sin t + 3\cos t \quad \Rightarrow \quad v(t) = -10\cos t + 3\sin t + C \quad \Rightarrow \quad s(t) = -10\sin t - 3\cos t + Ct + D.$ $s(0) = -3 + D = 0$ and $s(2\pi) = -3 + 2\pi C + D = 12 \quad \Rightarrow \quad D = 3$ and $C = \frac{6}{\pi}$. Thus, $s(t) = -10\sin t - 3\cos t + \frac{6}{\pi}t + 3.$

**64.** $a(t) = t^2 - 4t + 6 \quad \Rightarrow \quad v(t) = \frac{1}{3}t^3 - 2t^2 + 6t + C \quad \Rightarrow \quad s(t) = \frac{1}{12}t^4 - \frac{2}{3}t^3 + 3t^2 + Ct + D$. $s(0) = D$ and $s(0) = 0 \quad \Rightarrow \quad D = 0$. $s(1) = \frac{29}{12} + C$ and $s(1) = 20 \quad \Rightarrow \quad C = \frac{211}{12}$. Thus, $s(t) = \frac{1}{12}t^4 - \frac{2}{3}t^3 + 3t^2 + \frac{211}{12}t$.

**65.** (a) We first observe that since the stone is dropped 450 m above the ground, $v(0) = 0$ and $s(0) = 450$.
$v'(t) = a(t) = -9.8 \quad \Rightarrow \quad v(t) = -9.8t + C$. Now $v(0) = 0 \quad \Rightarrow \quad C = 0$, so $v(t) = -9.8t \quad \Rightarrow$
$s(t) = -4.9t^2 + D$. Last, $s(0) = 450 \quad \Rightarrow \quad D = 450 \quad \Rightarrow \quad s(t) = 450 - 4.9t^2$.

(b) The stone reaches the ground when $s(t) = 0$. $450 - 4.9t^2 = 0 \quad \Rightarrow \quad t^2 = 450/4.9 \quad \Rightarrow \quad t_1 = \sqrt{450/4.9} \approx 9.58$ s.

(c) The velocity with which the stone strikes the ground is $v(t_1) = -9.8\sqrt{450/4.9} \approx -93.9$ m/s.

(d) This is just reworking parts (a) and (b) with $v(0) = -5$. Using $v(t) = -9.8t + C$, $v(0) = -5 \quad \Rightarrow \quad 0 + C = -5 \quad \Rightarrow$
$v(t) = -9.8t - 5$. So $s(t) = -4.9t^2 - 5t + D$ and $s(0) = 450 \quad \Rightarrow \quad D = 450 \quad \Rightarrow \quad s(t) = -4.9t^2 - 5t + 450$.
Solving $s(t) = 0$ by using the quadratic formula gives us $t = \left(5 \pm \sqrt{8845}\right)/(-9.8) \quad \Rightarrow \quad t_1 \approx 9.09$ s.

**66.** $v'(t) = a(t) = a \quad \Rightarrow \quad v(t) = at + C$ and $v_0 = v(0) = C \quad \Rightarrow \quad v(t) = at + v_0 \quad \Rightarrow$
$s(t) = \frac{1}{2}at^2 + v_0t + D \quad \Rightarrow \quad s_0 = s(0) = D \quad \Rightarrow \quad s(t) = \frac{1}{2}at^2 + v_0t + s_0$

**67.** By Exercise 66 with $a = -9.8$, $s(t) = -4.9t^2 + v_0t + s_0$ and $v(t) = s'(t) = -9.8t + v_0$. So
$[v(t)]^2 = (-9.8t + v_0)^2 = (9.8)^2 t^2 - 19.6v_0t + v_0^2 = v_0^2 + 96.04t^2 - 19.6v_0t = v_0^2 - 19.6\left(-4.9t^2 + v_0t\right)$.
But $-4.9t^2 + v_0t$ is just $s(t)$ without the $s_0$ term; that is, $s(t) - s_0$. Thus, $[v(t)]^2 = v_0^2 - 19.6\,[s(t) - s_0]$.

**68.** For the first ball, $s_1(t) = -16t^2 + 48t + 432$ from Example 7. For the second ball, $a(t) = -32 \quad \Rightarrow \quad v(t) = -32t + C$, but
$v(1) = -32(1) + C = 24 \quad \Rightarrow \quad C = 56$, so $v(t) = -32t + 56 \quad \Rightarrow \quad s(t) = -16t^2 + 56t + D$, but
$s(1) = -16(1)^2 + 56(1) + D = 432 \quad \Rightarrow \quad D = 392$, and $s_2(t) = -16t^2 + 56t + 392$. The balls pass each other
when $s_1(t) = s_2(t) \quad \Rightarrow \quad -16t^2 + 48t + 432 = -16t^2 + 56t + 392 \quad \Leftrightarrow \quad 8t = 40 \quad \Leftrightarrow \quad t = 5$ s.

*Another solution:* From Exercise 66, we have $s_1(t) = -16t^2 + 48t + 432$ and $s_2(t) = -16t^2 + 24t + 432$.
We now want to solve $s_1(t) = s_2(t-1) \quad \Rightarrow \quad -16t^2 + 48t + 432 = -16(t-1)^2 + 24(t-1) + 432 \quad \Rightarrow$
$48t = 32t - 16 + 24t - 24 \quad \Rightarrow \quad 40 = 8t \quad \Rightarrow \quad t = 5$ s.

**69.** Using Exercise 66 with $a = -32$, $v_0 = 0$, and $s_0 = h$ (the height of the cliff), we know that the height at time $t$ is
$s(t) = -16t^2 + h$. $v(t) = s'(t) = -32t$ and $v(t) = -120 \quad \Rightarrow \quad -32t = -120 \quad \Rightarrow \quad t = 3.75$, so
$0 = s(3.75) = -16(3.75)^2 + h \quad \Rightarrow \quad h = 16(3.75)^2 = 225$ ft.

**70.** (a) $EIy'' = mg(L - x) + \frac{1}{2}\rho g(L - x)^2 \quad \Rightarrow \quad EIy' = -\frac{1}{2}mg(L - x)^2 - \frac{1}{6}\rho g(L - x)^3 + C \quad \Rightarrow$
$EIy = \frac{1}{6}mg(L - x)^3 + \frac{1}{24}\rho g(L - x)^4 + Cx + D$. Since the left end of the board is fixed, we must have $y = y' = 0$
when $x = 0$. Thus, $0 = -\frac{1}{2}mgL^2 - \frac{1}{6}\rho gL^3 + C$ and $0 = \frac{1}{6}mgL^3 + \frac{1}{24}\rho gL^4 + D$. It follows that
$EIy = \frac{1}{6}mg(L - x)^3 + \frac{1}{24}\rho g(L - x)^4 + \left(\frac{1}{2}mgL^2 + \frac{1}{6}\rho gL^3\right)x - \left(\frac{1}{6}mgL^3 + \frac{1}{24}\rho gL^4\right)$ and

$$f(x) = y = \frac{1}{EI}\left[\tfrac{1}{6}mg(L - x)^3 + \tfrac{1}{24}\rho g(L - x)^4 + \left(\tfrac{1}{2}mgL^2 + \tfrac{1}{6}\rho gL^3\right)x - \left(\tfrac{1}{6}mgL^3 + \tfrac{1}{24}\rho gL^4\right)\right]$$

(b) $f(L) < 0$, so the end of the board is a *distance* approximately $-f(L)$ below the horizontal. From our result in (a), we calculate

$$-f(L) = \frac{-1}{EI}\left[\tfrac{1}{2}mgL^3 + \tfrac{1}{6}\rho gL^4 - \tfrac{1}{6}mgL^3 - \tfrac{1}{24}\rho gL^4\right] = \frac{-1}{EI}\left(\tfrac{1}{3}mgL^3 + \tfrac{1}{8}\rho gL^4\right) = -\frac{gL^3}{EI}\left(\frac{m}{3} + \frac{\rho L}{8}\right)$$

*Note:* This is positive because $g$ is negative.

**71.** Marginal cost $= 1.92 - 0.002x = C'(x) \quad\Rightarrow\quad C(x) = 1.92x - 0.001x^2 + K$. But $C(1) = 1.92 - 0.001 + K = 562 \quad\Rightarrow$ $K = 560.081$. Therefore, $C(x) = 1.92x - 0.001x^2 + 560.081 \quad\Rightarrow\quad C(100) = 742.081$, so the cost of producing 100 items is \$742.08.

**72.** Let the mass, measured from one end, be $m(x)$. Then $m(0) = 0$ and $\rho = \dfrac{dm}{dx} = x^{-1/2} \quad\Rightarrow\quad m(x) = 2x^{1/2} + C$ and $m(0) = C = 0$, so $m(x) = 2\sqrt{x}$. Thus, the mass of the 100-centimeter rod is $m(100) = 2\sqrt{100} = 20$ g.

**73.** Taking the upward direction to be positive we have that for $0 \le t \le 10$ (using the subscript 1 to refer to $0 \le t \le 10$),

$a_1(t) = -(9 - 0.9t) = v_1'(t) \quad\Rightarrow\quad v_1(t) = -9t + 0.45t^2 + v_0$, but $v_1(0) = v_0 = -10 \quad\Rightarrow$

$v_1(t) = -9t + 0.45t^2 - 10 = s_1'(t) \quad\Rightarrow\quad s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + s_0$. But $s_1(0) = 500 = s_0 \quad\Rightarrow$

$s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + 500$. $\;s_1(10) = -450 + 150 - 100 + 500 = 100$, so it takes

more than 10 seconds for the raindrop to fall. Now for $t > 10$, $a(t) = 0 = v'(t) \quad\Rightarrow$

$v(t) = \text{constant} = v_1(10) = -9(10) + 0.45(10)^2 - 10 = -55 \quad\Rightarrow\quad v(t) = -55$.

At 55 m/s, it will take $100/55 \approx 1.8$ s to fall the last 100 m. Hence, the total time is $10 + \frac{100}{55} = \frac{130}{11} \approx 11.8$ s.

**74.** $v'(t) = a(t) = -22$. The initial velocity is 50 mi/h $= \frac{50 \cdot 5280}{3600} = \frac{220}{3}$ ft/s, so $v(t) = -22t + \frac{220}{3}$.

The car stops when $v(t) = 0 \quad\Leftrightarrow\quad t = \frac{220}{3 \cdot 22} = \frac{10}{3}$. Since $s(t) = -11t^2 + \frac{220}{3}t$, the distance covered is

$s\left(\frac{10}{3}\right) = -11\left(\frac{10}{3}\right)^2 + \frac{220}{3} \cdot \frac{10}{3} = \frac{1100}{9} = 122.\overline{2}$ ft.

**75.** $a(t) = k$, the initial velocity is 30 mi/h $= 30 \cdot \frac{5280}{3600} = 44$ ft/s, and the final velocity (after 5 seconds) is

50 mi/h $= 50 \cdot \frac{5280}{3600} = \frac{220}{3}$ ft/s. So $v(t) = kt + C$ and $v(0) = 44 \quad\Rightarrow\quad C = 44$. Thus, $v(t) = kt + 44 \quad\Rightarrow$

$v(5) = 5k + 44$. But $v(5) = \frac{220}{3}$, so $5k + 44 = \frac{220}{3} \quad\Rightarrow\quad 5k = \frac{88}{3} \quad\Rightarrow\quad k = \frac{88}{15} \approx 5.87$ ft/s$^2$.

**76.** $a(t) = -16 \quad\Rightarrow\quad v(t) = -16t + v_0$ where $v_0$ is the car's speed (in ft/s) when the brakes were applied. The car stops when $-16t + v_0 = 0 \quad\Leftrightarrow\quad t = \frac{1}{16}v_0$. Now $s(t) = \frac{1}{2}(-16)t^2 + v_0t = -8t^2 + v_0t$. The car travels 200 ft in the time that it takes to stop, so $s\left(\frac{1}{16}v_0\right) = 200 \quad\Rightarrow\quad 200 = -8\left(\frac{1}{16}v_0\right)^2 + v_0\left(\frac{1}{16}v_0\right) = \frac{1}{32}v_0^2 \quad\Rightarrow\quad v_0^2 = 32 \cdot 200 = 6400 \quad\Rightarrow$

$v_0 = 80$ ft/s $[54.\overline{54}$ mi/h$]$.

**77.** Let the acceleration be $a(t) = k$ km/h$^2$. We have $v(0) = 100$ km/h and we can take the initial position $s(0)$ to be 0. We want the time $t_f$ for which $v(t) = 0$ to satisfy $s(t) < 0.08$ km. In general, $v'(t) = a(t) = k$, so $v(t) = kt + C$, where $C = v(0) = 100$. Now $s'(t) = v(t) = kt + 100$, so $s(t) = \frac{1}{2}kt^2 + 100t + D$, where $D = s(0) = 0$.

Thus, $s(t) = \frac{1}{2}kt^2 + 100t$. Since $v(t_f) = 0$, we have $kt_f + 100 = 0$ or $t_f = -100/k$, so

$$s(t_f) = \frac{1}{2}k\left(-\frac{100}{k}\right)^2 + 100\left(-\frac{100}{k}\right) = 10{,}000\left(\frac{1}{2k} - \frac{1}{k}\right) = -\frac{5{,}000}{k}.$$ The condition $s(t_f)$ must satisfy is

$$-\frac{5{,}000}{k} < 0.08 \quad\Rightarrow\quad -\frac{5{,}000}{0.08} > k \quad [k \text{ is negative}] \quad\Rightarrow\quad k < -62{,}500 \text{ km/h}^2,$$ or equivalently,

$k < -\frac{3125}{648} \approx -4.82\text{ m/s}^2$.

**78.** (a) For $0 \le t \le 3$ we have $a(t) = 60t \quad\Rightarrow\quad v(t) = 30t^2 + C \quad\Rightarrow\quad v(0) = 0 = C \quad\Rightarrow\quad v(t) = 30t^2$, so

$s(t) = 10t^3 + C \quad\Rightarrow\quad s(0) = 0 = C \quad\Rightarrow\quad s(t) = 10t^3$. Note that $v(3) = 270$ and $s(3) = 270$.

For $3 < t \le 17$: $a(t) = -g = -32$ ft/s $\quad\Rightarrow\quad v(t) = -32(t-3) + C \quad\Rightarrow\quad v(3) = 270 = C \quad\Rightarrow$

$v(t) = -32(t-3) + 270 \quad\Rightarrow\quad s(t) = -16(t-3)^2 + 270(t-3) + C \quad\Rightarrow\quad s(3) = 270 = C \quad\Rightarrow$

$s(t) = -16(t-3)^2 + 270(t-3) + 270$. Note that $v(17) = -178$ and $s(17) = 914$.

For $17 < t \le 22$: The velocity increases linearly from $-178$ ft/s to $-18$ ft/s during this period, so

$\dfrac{\Delta v}{\Delta t} = \dfrac{-18-(-178)}{22-17} = \dfrac{160}{5} = 32$. Thus, $v(t) = 32(t-17) - 178 \quad\Rightarrow$

$s(t) = 16(t-17)^2 - 178(t-17) + 914$ and $s(22) = 424$ ft.

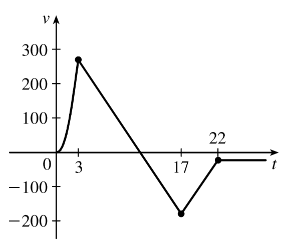
For $t > 22$: $v(t) = -18 \quad\Rightarrow\quad s(t) = -18(t-22) + C$. But $s(22) = 424 = C \quad\Rightarrow\quad s(t) = -18(t-22) + 424$.

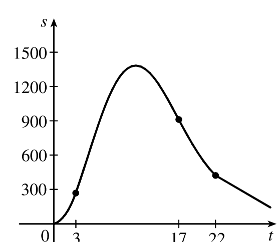
Therefore, until the rocket lands, we have

$$v(t) = \begin{cases} 30t^2 & \text{if } 0 \le t \le 3 \\ -32\,(t-3) + 270 & \text{if } 3 < t \le 17 \\ 32(t-17) - 178 & \text{if } 17 < t \le 22 \\ -18 & \text{if } t > 22 \end{cases}$$

and

$$s(t) = \begin{cases} 10t^3 & \text{if } 0 \le t \le 3 \\ -16(t-3)^2 + 270(t-3) + 270 & \text{if } 3 < t \le 17 \\ 16(t-17)^2 - 178\,(t-17) + 914 & \text{if } 17 < t \le 22 \\ -18(t-22) + 424 & \text{if } t > 22 \end{cases}$$




(b) To find the maximum height, set $v(t)$ on $3 < t \le 17$ equal to 0. $\;-32(t-3) + 270 = 0 \quad\Rightarrow\quad t_1 = 11.4375$ s and the maximum height is $s(t_1) = -16(t_1-3)^2 + 270(t_1-3) + 270 = 1409.0625$ ft.

(c) To find the time to land, set $s(t) = -18(t-22) + 424 = 0$. Then $t - 22 = \frac{424}{18} = 23.\overline{5}$, so $t \approx 45.6$ s.

**79.** (a) First note that $90 \text{ mi/h} = 90 \times \frac{5280}{3600} \text{ ft/s} = 132 \text{ ft/s}$. Then $a(t) = 4 \text{ ft/s}^2 \quad \Rightarrow \quad v(t) = 4t + C$, but $v(0) = 0 \quad \Rightarrow$ $C = 0$. Now $4t = 132$ when $t = \frac{132}{4} = 33$ s, so it takes 33 s to reach 132 ft/s. Therefore, taking $s(0) = 0$, we have $s(t) = 2t^2$, $0 \leq t \leq 33$. So $s(33) = 2178$ ft. 15 minutes $= 15(60) = 900$ s, so for $33 < t \leq 933$ we have $v(t) = 132 \text{ ft/s} \quad \Rightarrow \quad s(933) = 132(900) + 2178 = 120{,}978 \text{ ft} = 22.9125 \text{ mi}$.

(b) As in part (a), the train accelerates for 33 s and travels 2178 ft while doing so. Similarly, it decelerates for 33 s and travels 2178 ft at the end of its trip. During the remaining $900 - 66 = 834$ s it travels at 132 ft/s, so the distance traveled is $132 \cdot 834 = 110{,}088$ ft. Thus, the total distance is $2178 + 110{,}088 + 2178 = 114{,}444 \text{ ft} = 21.675 \text{ mi}$.

(c) $45 \text{ mi} = 45(5280) = 237{,}600$ ft. Subtract $2(2178)$ to take care of the speeding up and slowing down, and we have 233,244 ft at 132 ft/s for a trip of $233{,}244/132 = 1767$ s at 90 mi/h. The total time is $1767 + 2(33) = 1833 \text{ s} = 30 \text{ min } 33 \text{ s} = 30.55 \text{ min}$.

(d) $37.5(60) = 2250$ s. $2250 - 2(33) = 2184$ s at maximum speed. $2184(132) + 2(2178) = 292{,}644$ total feet or $292{,}644/5280 = 55.425$ mi.

## 4 Review

### TRUE-FALSE QUIZ

**1.** False. For example, take $f(x) = x^3$, then $f'(x) = 3x^2$ and $f'(0) = 0$, but $f(0) = 0$ is not a maximum or minimum; $(0, 0)$ is an inflection point.

**2.** False. For example, $f(x) = |x|$ has an absolute minimum at 0, but $f'(0)$ does not exist.

**3.** False. For example, $f(x) = x$ is continuous on $(0, 1)$ but attains neither a maximum nor a minimum value on $(0, 1)$. Don't confuse this with $f$ being continuous on the *closed* interval $[a, b]$, which would make the statement true.

**4.** True. By the Mean Value Theorem, $f'(c) = \dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{0}{2} = 0$. Note that $|c| < 1 \quad \Leftrightarrow \quad c \in (-1, 1)$.

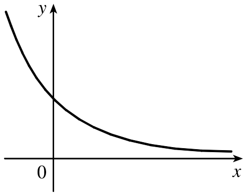
**5.** True. This is an example of part (b) of the I/D Test.

**6.** False. For example, the curve $y = f(x) = 1$ has no inflection points but $f''(c) = 0$ for all $c$.

**7.** False. $f'(x) = g'(x) \quad \Rightarrow \quad f(x) = g(x) + C$. For example, if $f(x) = x + 2$ and $g(x) = x + 1$, then $f'(x) = g'(x) = 1$, but $f(x) \neq g(x)$.

**8.** False. Assume there is a function $f$ such that $f(1) = -2$ and $f(3) = 0$. Then by the Mean Value Theorem there exists a number $c \in (1, 3)$ such that $f'(c) = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{0 - (-2)}{2} = 1$. But $f'(x) > 1$ for all $x$, a contradiction.

**9.** True. The graph of one such function is sketched.



**10.** False. At any point $(a, f(a))$, we know that $f'(a) < 0$. So since the tangent line at $(a, f(a))$ is not horizontal, it must cross the $x$-axis—at $x = b$, say. But since $f''(x) > 0$ for all $x$, the graph of $f$ must lie above all of its tangents; in particular, $f(b) > 0$. But this is a contradiction, since we are given that $f(x) < 0$ for all $x$.

**11.** True. Let $x_1 < x_2$ where $x_1, x_2 \in I$. Then $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$ [since $f$ and $g$ are increasing on $I$], so $(f+g)(x_1) = f(x_1) + g(x_1) < f(x_2) + g(x_2) = (f+g)(x_2)$.

**12.** False. $f(x) = x$ and $g(x) = 2x$ are both increasing on $(0, 1)$, but $f(x) - g(x) = -x$ is not increasing on $(0, 1)$.

**13.** False. Take $f(x) = x$ and $g(x) = x - 1$. Then both $f$ and $g$ are increasing on $(0, 1)$. But $f(x)\,g(x) = x(x-1)$ is not increasing on $(0, 1)$.

**14.** True. Let $x_1 < x_2$ where $x_1, x_2 \in I$. Then $0 < f(x_1) < f(x_2)$ and $0 < g(x_1) < g(x_2)$ [since $f$ and $g$ are both positive and increasing]. Hence, $f(x_1)\,g(x_1) < f(x_2)\,g(x_1) < f(x_2)\,g(x_2)$. So $fg$ is increasing on $I$.

**15.** True. Let $x_1, x_2 \in I$ and $x_1 < x_2$. Then $f(x_1) < f(x_2)$ [$f$ is increasing] $\Rightarrow$ $\dfrac{1}{f(x_1)} > \dfrac{1}{f(x_2)}$ [$f$ is positive] $\Rightarrow$ $g(x_1) > g(x_2)$ $\Rightarrow$ $g(x) = 1/f(x)$ is decreasing on $I$.

**16.** False. If $f$ is even, then $f(x) = f(-x)$. Using the Chain Rule to differentiate this equation, we get $f'(x) = f'(-x)\dfrac{d}{dx}(-x) = -f'(-x)$. Thus, $f'(-x) = -f'(x)$, so $f'$ is odd.

**17.** True. If $f$ is periodic, then there is a number $p$ such that $f(x+p) = f(p)$ for all $x$. Differentiating gives $f'(x) = f'(x+p)\cdot(x+p)' = f'(x+p)\cdot 1 = f'(x+p)$, so $f'$ is periodic.

**18.** False. The most general antiderivative of $f(x) = x^{-2}$ is $F(x) = -1/x + C_1$ for $x < 0$ and $F(x) = -1/x + C_2$ for $x > 0$ [see Example 4.9.1(b)].

**19.** True. By the Mean Value Theorem, there exists a number $c$ in $(0, 1)$ such that $f(1) - f(0) = f'(c)(1-0) = f'(c)$. Since $f'(c)$ is nonzero, $f(1) - f(0) \neq 0$, so $f(1) \neq f(0)$.

**20.** False. Let $f(x) = 1 + \dfrac{1}{x}$ and $g(x) = x$. Then $\lim\limits_{x\to\infty} f(x) = 1$ and $\lim\limits_{x\to\infty} g(x) = \infty$, but

$$\lim_{x\to\infty}[f(x)]^{g(x)} = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e, \text{ not } 1.$$

**21.** False. $\displaystyle\lim_{x\to 0}\frac{x}{e^x} = \frac{\lim_{x\to 0} x}{\lim_{x\to 0} e^x} = \frac{0}{1} = 0$, not 1.

## EXERCISES

**1.** $f(x) = x^3 - 9x^2 + 24x - 2$, $[0, 5]$. $f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x-2)(x-4)$. $f'(x) = 0$ $\Leftrightarrow$ $x = 2$ or $x = 4$. $f'(x) > 0$ for $0 < x < 2$, $f'(x) < 0$ for $2 < x < 4$, and $f'(x) > 0$ for $4 < x < 5$, so $f(2) = 18$ is a local maximum value and $f(4) = 14$ is a local minimum value. Checking the endpoints, we find $f(0) = -2$ and $f(5) = 18$. Thus, $f(0) = -2$ is the absolute minimum value and $f(2) = f(5) = 18$ is the absolute maximum value.

**2.** $f(x) = x\sqrt{1-x}$, $[-1, 1]$. $f'(x) = x \cdot \frac{1}{2}(1-x)^{-1/2}(-1) + (1-x)^{1/2}(1) = (1-x)^{-1/2}\left[-\frac{1}{2}x + (1-x)\right] = \dfrac{1-\frac{3}{2}x}{\sqrt{1-x}}$.

$f'(x) = 0 \;\Rightarrow\; x = \frac{2}{3}$. $f'(x)$ does not exist $\;\Leftrightarrow\; x = 1$. $f'(x) > 0$ for $-1 < x < \frac{2}{3}$ and $f'(x) < 0$ for $\frac{2}{3} < x < 1$, so $f\left(\frac{2}{3}\right) = \frac{2}{3}\sqrt{\frac{1}{3}} = \frac{2}{9}\sqrt{3}$ [$\approx 0.38$] is a local maximum value. Checking the endpoints, we find $f(-1) = -\sqrt{2}$ and $f(1) = 0$. Thus, $f(-1) = -\sqrt{2}$ is the absolute minimum value and $f\left(\frac{2}{3}\right) = \frac{2}{9}\sqrt{3}$ is the absolute maximum value.

**3.** $f(x) = \dfrac{3x-4}{x^2+1}$, $[-2, 2]$. $f'(x) = \dfrac{(x^2+1)(3) - (3x-4)(2x)}{(x^2+1)^2} = \dfrac{-(3x^2-8x-3)}{(x^2+1)^2} = \dfrac{-(3x+1)(x-3)}{(x^2+1)^2}$.

$f'(x) = 0 \;\Rightarrow\; x = -\frac{1}{3}$ or $x = 3$, but 3 is not in the interval. $f'(x) > 0$ for $-\frac{1}{3} < x < 2$ and $f'(x) < 0$ for $-2 < x < -\frac{1}{3}$, so $f\left(-\frac{1}{3}\right) = \frac{-5}{10/9} = -\frac{9}{2}$ is a local minimum value. Checking the endpoints, we find $f(-2) = -2$ and $f(2) = \frac{2}{5}$. Thus, $f\left(-\frac{1}{3}\right) = -\frac{9}{2}$ is the absolute minimum value and $f(2) = \frac{2}{5}$ is the absolute maximum value.

**4.** $f(x) = \sqrt{x^2+x+1}$, $[-2, 1]$. $f'(x) = \frac{1}{2}(x^2+x+1)^{-1/2}(2x+1) = \dfrac{2x+1}{2\sqrt{x^2+x+1}}$. $f'(x) = 0 \;\Rightarrow\; x = -\dfrac{1}{2}$.

$f'(x) > 0$ for $-\frac{1}{2} < x < 1$ and $f'(x) < 0$ for $-2 < x < -\frac{1}{2}$, so $f\left(-\frac{1}{2}\right) = \sqrt{3}/2$ is a local minimum value. Checking the endpoints, we find $f(-2) = f(1) = \sqrt{3}$. Thus, $f\left(-\frac{1}{2}\right) = \sqrt{3}/2$ is the absolute minimum value and $f(-2) = f(1) = \sqrt{3}$ is the absolute maximum value.

**5.** $f(x) = x + 2\cos x$, $[-\pi, \pi]$. $f'(x) = 1 - 2\sin x$. $f'(x) = 0 \;\Rightarrow\; \sin x = \frac{1}{2} \;\Rightarrow\; x = \frac{\pi}{6}, \frac{5\pi}{6}$. $f'(x) > 0$ for $\left(-\pi, \frac{\pi}{6}\right)$ and $\left(\frac{5\pi}{6}, \pi\right)$, and $f'(x) < 0$ for $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$, so $f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \sqrt{3} \approx 2.26$ is a local maximum value and $f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} - \sqrt{3} \approx 0.89$ is a local minimum value. Checking the endpoints, we find $f(-\pi) = -\pi - 2 \approx -5.14$ and $f(\pi) = \pi - 2 \approx 1.14$. Thus, $f(-\pi) = -\pi - 2$ is the absolute minimum value and $f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \sqrt{3}$ is the absolute maximum value.

**6.** $f(x) = x^2e^{-x}$, $[-1, 3]$. $f'(x) = x^2(-e^{-x}) + e^{-x}(2x) = xe^{-x}(-x+2)$. $f'(x) = 0 \;\Rightarrow\; x = 0$ or $x = 2$.

$f'(x) > 0$ for $0 < x < 2$ and $f'(x) < 0$ for $-1 < x < 0$ and $2 < x < 3$, so $f(0) = 0$ is a local minimum value and $f(2) = 4e^{-2} \approx 0.54$ is a local maximum value. Checking the endpoints, we find $f(-1) = e \approx 2.72$ and $f(3) = 9e^{-3} \approx 0.45$. Thus, $f(0) = 0$ is the absolute minimum value and $f(-1) = e$ is the absolute maximum value.

**7.** This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{\tan x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{e^x}{\sec^2 x} = \frac{1}{1} = 1$

**8.** This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0} \frac{\tan 4x}{x + \sin 2x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{4\sec^2 4x}{1 + 2\cos 2x} = \frac{4(1)}{1+2(1)} = \frac{4}{3}$

**9.** This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0} \frac{e^{2x} - e^{-2x}}{\ln(x+1)} \overset{\text{H}}{=} \lim_{x\to 0} \frac{2e^{2x} + 2e^{-2x}}{1/(x+1)} = \frac{2+2}{1} = 4$

**10.** This limit has the form $\frac{\infty}{\infty}$. $\displaystyle\lim_{x\to \infty} \frac{e^{2x} - e^{-2x}}{\ln(x+1)} \overset{\text{H}}{=} \lim_{x\to \infty} \frac{2e^{2x} + 2e^{-2x}}{1/(x+1)} = \lim_{x\to\infty} 2(x+1)(e^{2x} + e^{-2x}) = \infty$

since $2(x+1) \to \infty$ and $(e^{2x} + e^{-2x}) \to \infty$ as $x \to \infty$.

**11.** This limit has the form $\infty \cdot 0$.

$$\lim_{x\to-\infty}(x^2-x^3)e^{2x} = \lim_{x\to-\infty}\frac{x^2-x^3}{e^{-2x}} \quad \left[\tfrac{\infty}{\infty}\text{ form}\right] \quad \overset{\text{H}}{=} \lim_{x\to-\infty}\frac{2x-3x^2}{-2e^{-2x}} \quad \left[\tfrac{\infty}{\infty}\text{ form}\right]$$

$$\overset{\text{H}}{=} \lim_{x\to-\infty}\frac{2-6x}{4e^{-2x}} \quad \left[\tfrac{\infty}{\infty}\text{ form}\right] \quad \overset{\text{H}}{=} \lim_{x\to-\infty}\frac{-6}{-8e^{-2x}} = 0$$

**12.** This limit has the form $0 \cdot \infty$. $\displaystyle\lim_{x\to\pi^-}(x-\pi)\csc x = \lim_{x\to\pi^-}\frac{x-\pi}{\sin x} \quad \left[\tfrac{0}{0}\text{ form}\right] \overset{\text{H}}{=} \lim_{x\to\pi^-}\frac{1}{\cos x} = \frac{1}{-1} = -1$

**13.** This limit has the form $\infty - \infty$.

$$\lim_{x\to1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) = \lim_{x\to1^+}\left(\frac{x\ln x - x + 1}{(x-1)\ln x}\right) \overset{\text{H}}{=} \lim_{x\to1^+}\frac{x\cdot(1/x)+\ln x - 1}{(x-1)\cdot(1/x)+\ln x} = \lim_{x\to1^+}\frac{\ln x}{1-1/x+\ln x}$$

$$\overset{\text{H}}{=} \lim_{x\to1^+}\frac{1/x}{1/x^2+1/x} = \frac{1}{1+1} = \frac{1}{2}$$

**14.** $y = (\tan x)^{\cos x} \quad\Rightarrow\quad \ln y = \cos x\,\ln\tan x$, so

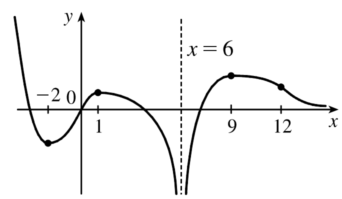
$$\lim_{x\to(\pi/2)^-}\ln y = \lim_{x\to(\pi/2)^-}\frac{\ln\tan x}{\sec x} \overset{\text{H}}{=} \lim_{x\to(\pi/2)^-}\frac{(1/\tan x)\sec^2 x}{\sec x\,\tan x} = \lim_{x\to(\pi/2)^-}\frac{\sec x}{\tan^2 x} = \lim_{x\to(\pi/2)^-}\frac{\cos x}{\sin^2 x} = \frac{0}{1^2} = 0,$$

so $\displaystyle\lim_{x\to(\pi/2)^-}(\tan x)^{\cos x} = \lim_{x\to(\pi/2)^-}e^{\ln y} = e^0 = 1.$

**15.** $f(0)=0$, $f'(-2)=f'(1)=f'(9)=0$, $\displaystyle\lim_{x\to\infty}f(x)=0$, $\displaystyle\lim_{x\to6}f(x)=-\infty$,

$f'(x)<0$ on $(-\infty,-2)$, $(1,6)$, and $(9,\infty)$, $f'(x)>0$ on $(-2,1)$ and $(6,9)$,

$f''(x)>0$ on $(-\infty,0)$ and $(12,\infty)$, $f''(x)<0$ on $(0,6)$ and $(6,12)$



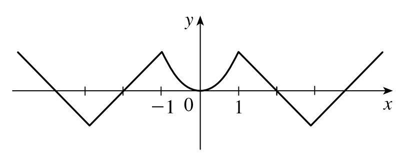

**16.** For $0<x<1$, $f'(x)=2x$, so $f(x)=x^2+C$. Since $f(0)=0$,

$f(x)=x^2$ on $[0,1]$. For $1<x<3$, $f'(x)=-1$, so $f(x)=-x+D$.

$1=f(1)=-1+D \quad\Rightarrow\quad D=2$, so $f(x)=2-x$. For $x>3$, $f'(x)=1$,

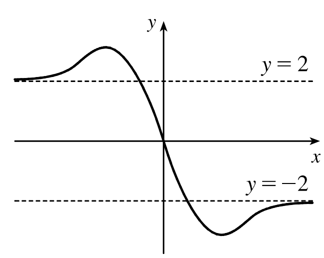
so $f(x)=x+E$. $-1=f(3)=3+E \quad\Rightarrow\quad E=-4$, so $f(x)=x-4$.

Since $f$ is even, its graph is symmetric about the $y$-axis.




**17.** $f$ is odd, $f'(x)<0$ for $0<x<2$, $f'(x)>0$ for $x>2$,

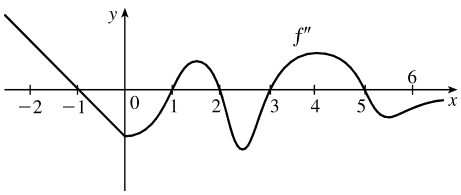
$f''(x)>0$ for $0<x<3$, $f''(x)<0$ for $x>3$, $\lim_{x\to\infty}f(x)=-2$



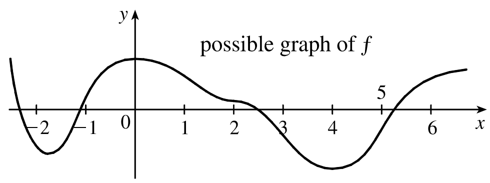

**18.** (a) Using the Test for Monotonic Functions we know that $f$ is increasing on $(-2,0)$ and $(4,\infty)$ because $f'>0$ on $(-2,0)$ and $(4,\infty)$, and that $f$ is decreasing on $(-\infty,-2)$ and $(0,4)$ because $f'<0$ on $(-\infty,-2)$ and $(0,4)$.

(b) Using the First Derivative Test, we know that $f$ has a local maximum at $x=0$ because $f'$ changes from positive to negative at $x=0$, and that $f$ has a local minimum at $x=4$ because $f'$ changes from negative to positive at $x=4$.

(c)




(d)

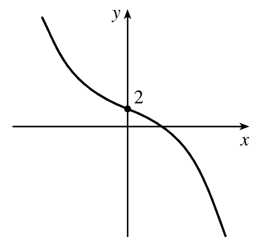



**19.** $y = f(x) = 2 - 2x - x^3$ **A.** $D = \mathbb{R}$ **B.** $y$-intercept: $f(0) = 2$. The $x$-intercept (approximately 0.770917) can be found using Newton's Method. **C.** No symmetry **D.** No asymptote

**E.** $f'(x) = -2 - 3x^2 = -(3x^2 + 2) < 0$, so $f$ is decreasing on $\mathbb{R}$.

**F.** No extreme value **G.** $f''(x) = -6x < 0$ on $(0, \infty)$ and $f''(x) > 0$ on $(-\infty, 0)$, so $f$ is CD on $(0, \infty)$ and CU on $(-\infty, 0)$. There is an IP at $(0, 2)$.

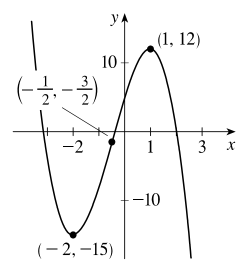
**H.**




**20.** $y = f(x) = -2x^3 - 3x^2 + 12x + 5$ **A.** $D = \mathbb{R}$ **B.** $y$-intercept: $f(0) = 5$; $x$-intercept: $f(x) = 0$ $\Leftrightarrow$ $x \approx -3.15, -0.39, 2.04$ **C.** No symmetry **D.** No asymptote

**E.** $f'(x) = -6x^2 - 6x + 12 = -6(x^2 + x - 2) = -6(x + 2)(x - 1)$. $f'(x) > 0$ for $-2 < x < 1$, so $f$ is increasing on $(-2, 1)$ and decreasing on $(-\infty, -2)$ and $(1, \infty)$. **F.** Local minimum value $f(-2) = -15$, local maximum value $f(1) = 12$ **G.** $f''(x) = -12x - 6 = -12\left(x + \frac{1}{2}\right)$. $f''(x) > 0$ for $x < -\frac{1}{2}$, so $f$ is CU on $\left(-\infty, -\frac{1}{2}\right)$ and CD on $\left(-\frac{1}{2}, \infty\right)$. There is an IP at $\left(-\frac{1}{2}, -\frac{3}{2}\right)$.

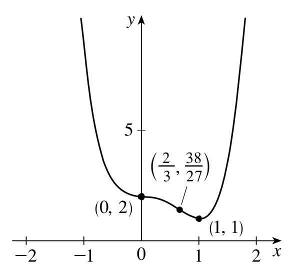
**H.**




**21.** $y = f(x) = 3x^4 - 4x^3 + 2$ **A.** $D = \mathbb{R}$ **B.** $y$-intercept: $f(0) = 2$; no $x$-intercept **C.** No symmetry **D.** No asymptote

**E.** $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$. $f'(x) > 0$ for $x > 1$, so $f$ is increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$. **F.** $f'(x)$ does not change sign at $x = 0$, so there is no local extremum there. $f(1) = 1$ is a local minimum value. **G.** $f''(x) = 36x^2 - 24x = 12x(3x - 2)$. $f''(x) < 0$ for $0 < x < \frac{2}{3}$, so $f$ is CD on $\left(0, \frac{2}{3}\right)$ and $f$ is CU on $(-\infty, 0)$ and $\left(\frac{2}{3}, \infty\right)$. There are inflection points at $(0, 2)$ and $\left(\frac{2}{3}, \frac{38}{27}\right)$.

**H.**




**22.** $y = f(x) = \dfrac{x}{1 - x^2}$ **A.** $D = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ **B.** $y$-intercept: $f(0) = 0$; $x$-intercept: 0

**C.** $f(-x) = -f(x)$, so $f$ is odd and the graph is symmetric about the origin. **D.** $\displaystyle\lim_{x \to \pm\infty} \frac{x}{1 - x^2} = 0$, so $y = 0$ is a HA.

$\displaystyle\lim_{x \to -1^-} \frac{x}{1 - x^2} = \infty$ and $\displaystyle\lim_{x \to -1^+} \frac{x}{1 - x^2} = -\infty$, so $x = -1$ is a VA. Similarly, $\displaystyle\lim_{x \to 1^-} \frac{x}{1 - x^2} = \infty$ and

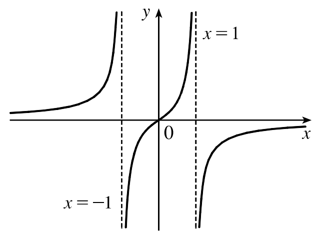
$\displaystyle\lim_{x \to 1^+} \frac{x}{1 - x^2} = -\infty$, so $x = 1$ is a VA. **E.** $f'(x) = \dfrac{(1 - x^2)(1) - x(-2x)}{(1 - x^2)^2} = \dfrac{1 + x^2}{(1 - x^2)^2} > 0$ for $x \neq \pm 1$, so $f$ is increasing on $(-\infty,-1)$, $(-1,1)$, and $(1,\infty)$. **F.** No local extrema

**G.** $f''(x) = \dfrac{(1-x^2)^2(2x)-(1+x^2)2(1-x^2)(-2x)}{[(1-x^2)^2]^2}$

$= \dfrac{2x(1-x^2)[(1-x^2)+2(1+x^2)]}{(1-x^2)^4} = \dfrac{2x(3+x^2)}{(1-x^2)^3}$

$f''(x)>0$ for $x<-1$ and $0<x<1$, and $f''(x)<0$ for $-1<x<0$ and $x>1$, so $f$ is CU on $(-\infty,-1)$ and $(0,1)$, and $f$ is CD on $(-1,0)$ and $(1,\infty)$. $(0,0)$ is an IP.

**H.**




**23.** $y=f(x)=\dfrac{1}{x(x-3)^2}$ **A.** $D=\{x \mid x\neq 0,3\}=(-\infty,0)\cup(0,3)\cup(3,\infty)$ **B.** No intercepts. **C.** No symmetry.

**D.** $\lim\limits_{x\to\pm\infty}\dfrac{1}{x(x-3)^2}=0$, so $y=0$ is a HA. $\lim\limits_{x\to 0^+}\dfrac{1}{x(x-3)^2}=\infty$, $\lim\limits_{x\to 0^-}\dfrac{1}{x(x-3)^2}=-\infty$, $\lim\limits_{x\to 3}\dfrac{1}{x(x-3)^2}=\infty$,

so $x=0$ and $x=3$ are VA. **E.** $f'(x)=-\dfrac{(x-3)^2+2x(x-3)}{x^2(x-3)^4}=\dfrac{3(1-x)}{x^2(x-3)^3} \;\Rightarrow\; f'(x)>0 \;\Leftrightarrow\; 1<x<3$,

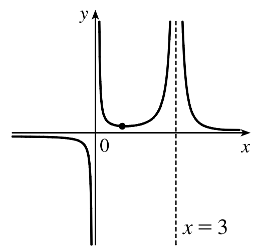
so $f$ is increasing on $(1,3)$ and decreasing on $(-\infty,0)$, $(0,1)$, and $(3,\infty)$.

**F.** Local minimum value $f(1)=\frac{1}{4}$ **G.** $f''(x)=\dfrac{6(2x^2-4x+3)}{x^3(x-3)^4}$.

Note that $2x^2-4x+3>0$ for all $x$ since it has negative discriminant.

So $f''(x)>0 \;\Leftrightarrow\; x>0 \;\Rightarrow\; f$ is CU on $(0,3)$ and $(3,\infty)$ and CD on $(-\infty,0)$. No IP

**H.**




**24.** $y=f(x)=\dfrac{1}{x^2}-\dfrac{1}{(x-2)^2}$ **A.** $D=\{x \mid x\neq 0,2\}$ **B.** $y$-intercept: none; $x$-intercept: $f(x)=0 \;\Rightarrow$

$\dfrac{1}{x^2}=\dfrac{1}{(x-2)^2} \;\Leftrightarrow\; (x-2)^2=x^2 \;\Leftrightarrow\; x^2-4x+4=x^2 \;\Leftrightarrow\; 4x=4 \;\Leftrightarrow\; x=1$ **C.** No symmetry

**D.** $\lim\limits_{x\to 0}f(x)=\infty$ and $\lim\limits_{x\to 2}f(x)=-\infty$, so $x=0$ and $x=2$ are VA; $\lim\limits_{x\to\pm\infty}f(x)=0$, so $y=0$ is a HA

**E.** $f'(x)=-\dfrac{2}{x^3}+\dfrac{2}{(x-2)^3}>0 \;\Rightarrow\; \dfrac{-(x-2)^3+x^3}{x^3(x-2)^3}>0 \;\Leftrightarrow\; \dfrac{-x^3+6x^2-12x+8+x^3}{x^3(x-2)^3}>0 \;\Leftrightarrow$

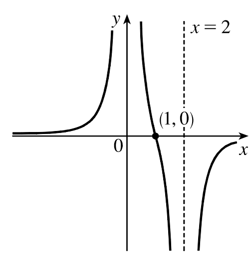
$\dfrac{2(3x^2-6x+4)}{x^3(x-2)^3}>0$. The numerator is positive (the discriminant of the quadratic is negative), so $f'(x)>0$ if $x<0$ or $x>2$, and hence, $f$ is increasing on $(-\infty,0)$ and $(2,\infty)$ and decreasing on $(0,2)$.

**F.** No local extreme values **G.** $f''(x)=\dfrac{6}{x^4}-\dfrac{6}{(x-2)^4}>0 \;\Rightarrow$

$\dfrac{(x-2)^4-x^4}{x^4(x-2)^4}>0 \;\Leftrightarrow\; \dfrac{x^4-8x^3+24x^2-32x+16-x^4}{x^4(x-2)^4}>0 \;\Leftrightarrow$

$\dfrac{-8(x^3-3x^2+4x-2)}{x^4(x-2)^4}>0 \;\Leftrightarrow\; \dfrac{-8(x-1)(x^2-2x+2)}{x^4(x-2)^4}>0$. So $f''$ is positive for $x<1$ $[x\neq 0]$ and negative for $x>1$ $[x\neq 2]$. Thus, $f$ is CU on $(-\infty,0)$ and $(0,1)$ and $f$ is CD on $(1,2)$ and $(2,\infty)$. IP at $(1,0)$

**H.**



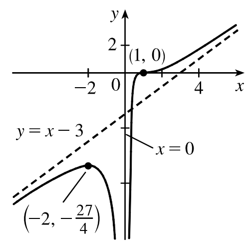
**25.** $y = f(x) = \dfrac{(x-1)^3}{x^2} = \dfrac{x^3 - 3x^2 + 3x - 1}{x^2} = x - 3 + \dfrac{3x-1}{x^2}$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$

**B.** $y$-intercept: none; $x$-intercept: $f(x) = 0 \iff x = 1$ **C.** No symmetry **D.** $\displaystyle\lim_{x\to 0^-} \frac{(x-1)^3}{x^2} = -\infty$ and

$\displaystyle\lim_{x\to 0^+} f(x) = -\infty$, so $x = 0$ is a VA. $f(x) - (x-3) = \dfrac{3x-1}{x^2} \to 0$ as $x \to \pm\infty$, so $y = x - 3$ is a SA.

**E.** $f'(x) = \dfrac{x^2 \cdot 3(x-1)^2 - (x-1)^3(2x)}{(x^2)^2} = \dfrac{x(x-1)^2[3x - 2(x-1)]}{x^4} = \dfrac{(x-1)^2(x+2)}{x^3}$. $f'(x) < 0$ for $-2 < x < 0$,

so $f$ is increasing on $(-\infty, -2)$, decreasing on $(-2, 0)$, and increasing on $(0, \infty)$. **H.**




**F.** Local maximum value $f(-2) = -\frac{27}{4}$ **G.** $f(x) = x - 3 + \dfrac{3}{x} - \dfrac{1}{x^2} \Rightarrow$

$f'(x) = 1 - \dfrac{3}{x^2} + \dfrac{2}{x^3} \Rightarrow f''(x) = \dfrac{6}{x^3} - \dfrac{6}{x^4} = \dfrac{6x-6}{x^4} = \dfrac{6(x-1)}{x^4}$.

$f''(x) > 0$ for $x > 1$, so $f$ is CD on $(-\infty, 0)$ and $(0, 1)$, and $f$ is CU on $(1, \infty)$.

There is an inflection point at $(1, 0)$.

**26.** $y = f(x) = \sqrt{1-x} + \sqrt{1+x}$ **A.** $1 - x \geq 0$ and $1 + x \geq 0 \Rightarrow x \leq 1$ and $x \geq -1$, so $D = [-1, 1]$.

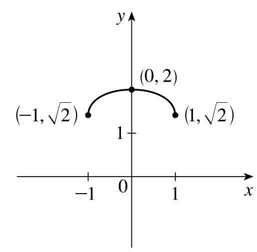
**B.** $y$-intercept: $f(0) = 1 + 1 = 2$; no $x$-intercept because $f(x) > 0$ for all $x$.

**C.** $f(-x) = f(x)$, so the curve is symmetric about the $y$-axis. **D.** No asymptote

**E.** $f'(x) = \frac{1}{2}(1-x)^{-1/2}(-1) + \frac{1}{2}(1+x)^{-1/2} = \dfrac{-1}{2\sqrt{1-x}} + \dfrac{1}{2\sqrt{1+x}} = \dfrac{-\sqrt{1+x} + \sqrt{1-x}}{2\sqrt{1-x}\sqrt{1+x}} > 0 \Rightarrow$

$-\sqrt{1+x} + \sqrt{1-x} > 0 \Rightarrow \sqrt{1-x} > \sqrt{1+x} \Rightarrow 1 - x > 1 + x \Rightarrow -2x > 0 \Rightarrow x < 0$, so $f'(x) > 0$ for

$-1 < x < 0$ and $f'(x) < 0$ for $0 < x < 1$. Thus, $f$ is increasing on $(-1, 0)$ **H.**




and decreasing on $(0, 1)$. **F.** Local maximum value $f(0) = 2$

**G.** $f''(x) = -\frac{1}{2}\left(-\frac{1}{2}\right)(1-x)^{-3/2}(-1) + \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-3/2}$

$$= \frac{-1}{4(1-x)^{3/2}} + \frac{-1}{4(1+x)^{3/2}} < 0$$

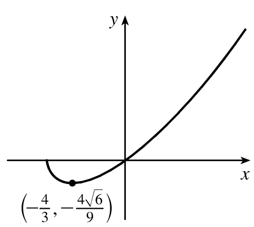
for all $x$ in the domain, so $f$ is CD on $(-1, 1)$. No IP

**27.** $y = f(x) = x\sqrt{2+x}$ **A.** $D = [-2, \infty)$ **B.** $y$-intercept: $f(0) = 0$; $x$-intercepts: $-2$ and $0$ **C.** No symmetry

**D.** No asymptote **E.** $f'(x) = \dfrac{x}{2\sqrt{2+x}} + \sqrt{2+x} = \dfrac{1}{2\sqrt{2+x}}[x + 2(2+x)] = \dfrac{3x+4}{2\sqrt{2+x}} = 0$ when $x = -\frac{4}{3}$, so $f$ is

decreasing on $\left(-2, -\frac{4}{3}\right)$ and increasing on $\left(-\frac{4}{3}, \infty\right)$. **F.** Local minimum value $f\left(-\frac{4}{3}\right) = -\frac{4}{3}\sqrt{\frac{2}{3}} = -\frac{4\sqrt{6}}{9} \approx -1.09$,

no local maximum **H.**




**G.** $f''(x) = \dfrac{2\sqrt{2+x} \cdot 3 - (3x+4)\dfrac{1}{\sqrt{2+x}}}{4(2+x)} = \dfrac{6(2+x) - (3x+4)}{4(2+x)^{3/2}}$

$$= \frac{3x+8}{4(2+x)^{3/2}}$$

$f''(x) > 0$ for $x > -2$, so $f$ is CU on $(-2, \infty)$. No IP

**28.** $y = f(x) = x^{2/3}(x-3)^2$ **A.** $D = \mathbb{R}$ **B.** $y$-intercept: $f(0) = 0$; $x$-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad x = 0, 3$

**C.** No symmetry **D.** No asymptote

**E.** $f'(x) = x^{2/3} \cdot 2(x-3) + (x-3)^2 \cdot \frac{2}{3}x^{-1/3} = \frac{2}{3}x^{-1/3}(x-3)[3x + (x-3)] = \frac{2}{3}x^{-1/3}(x-3)(4x-3)$.

$f'(x) > 0 \quad \Leftrightarrow \quad 0 < x < \frac{3}{4}$ or $x > 3$, so $f$ is decreasing on $(-\infty, 0)$, increasing on $\left(0, \frac{3}{4}\right)$, decreasing on $\left(\frac{3}{4}, 3\right)$, and increasing on $(3, \infty)$. **F.** Local minimum value $f(0) = f(3) = 0$; local maximum value

$f\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^{2/3}\left(-\frac{9}{4}\right)^2 = \frac{81}{16}\sqrt[3]{\frac{9}{16}} = \frac{81}{32}\sqrt[3]{\frac{9}{2}}$ $[\approx 4.18]$

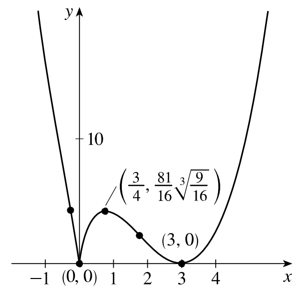
**G.** $f'(x) = \left(\frac{2}{3}x^{-1/3}\right)(4x^2 - 15x + 9) \quad \Rightarrow$

$$\begin{aligned} f''(x) &= \left(\tfrac{2}{3}x^{-1/3}\right)(8x-15) + (4x^2 - 15x + 9)\left(-\tfrac{2}{9}x^{-4/3}\right) \\ &= \tfrac{2}{9}x^{-4/3}[3x(8x-15) - (4x^2 - 15x + 9)] \\ &= \tfrac{2}{9}x^{-4/3}(20x^2 - 30x - 9) \end{aligned}$$

$f''(x) = 0 \quad \Leftrightarrow \quad x \approx -0.26$ or $1.76$. $f''(x)$ does not exist at $x = 0$.

$f$ is CU on $(-\infty, -0.26)$, CD on $(-0.26, 0)$, CD on $(0, 1.76)$, and CU on $(1.76, \infty)$. There are inflection points at $(-0.26, 4.28)$ and $(1.76, 2.25)$.

**H.**



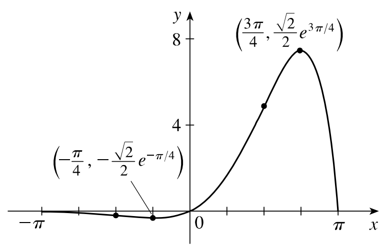

**29.** $y = f(x) = e^x \sin x$, $-\pi \le x \le \pi$ **A.** $D = [-\pi, \pi]$ **B.** $y$-intercept: $f(0) = 0$; $f(x) = 0 \quad \Leftrightarrow \quad \sin x = 0 \quad \Rightarrow$ $x = -\pi, 0, \pi$. **C.** No symmetry **D.** No asymptote **E.** $f'(x) = e^x \cos x + \sin x \cdot e^x = e^x(\cos x + \sin x)$.

$f'(x) = 0 \quad \Leftrightarrow \quad -\cos x = \sin x \quad \Leftrightarrow \quad -1 = \tan x \quad \Rightarrow \quad x = -\frac{\pi}{4}, \frac{3\pi}{4}$. $f'(x) > 0$ for $-\frac{\pi}{4} < x < \frac{3\pi}{4}$ and $f'(x) < 0$ for $-\pi < x < -\frac{\pi}{4}$ and $\frac{3\pi}{4} < x < \pi$, so $f$ is increasing on $\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$ and $f$ is decreasing on $\left(-\pi, -\frac{\pi}{4}\right)$ and $\left(\frac{3\pi}{4}, \pi\right)$.

**F.** Local minimum value $f\left(-\frac{\pi}{4}\right) = (-\sqrt{2}/2)e^{-\pi/4} \approx -0.32$ and local maximum value $f\left(\frac{3\pi}{4}\right) = \left(\sqrt{2}/2\right)e^{3\pi/4} \approx 7.46$

**G.** $f''(x) = e^x(-\sin x + \cos x) + (\cos x + \sin x)e^x = e^x(2\cos x) > 0 \quad \Rightarrow$ $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $f''(x) < 0 \quad \Rightarrow \quad -\pi < x < -\frac{\pi}{2}$ and $\frac{\pi}{2} < x < \pi$, so $f$ is CU on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $f$ is CD on $\left(-\pi, -\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$. There are inflection points at $\left(-\frac{\pi}{2}, -e^{-\pi/2}\right)$ and $\left(\frac{\pi}{2}, e^{\pi/2}\right)$.

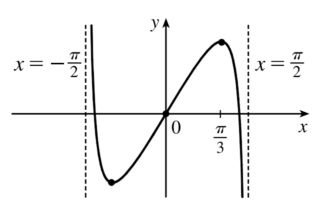
**H.**




**30.** $y = f(x) = 4x - \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. **B.** $y$-intercept $= f(0) = 0$ **C.** $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. **D.** $\displaystyle\lim_{x \to \pi/2^-}(4x - \tan x) = -\infty$, $\displaystyle\lim_{x \to -\pi/2^+}(4x - \tan x) = \infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA. **E.** $f'(x) = 4 - \sec^2 x > 0 \quad \Leftrightarrow \quad \sec x < 2 \quad \Leftrightarrow \quad \cos x > \frac{1}{2} \quad \Leftrightarrow \quad -\frac{\pi}{3} < x < \frac{\pi}{3}$, so $f$ is increasing on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ and decreasing on $\left(-\frac{\pi}{2}, -\frac{\pi}{3}\right)$ and $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$. **F.** $f\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} - \sqrt{3}$ is a local maximum value, $f\left(-\frac{\pi}{3}\right) = \sqrt{3} - \frac{4\pi}{3}$ is a local minimum value.

**G.** $f''(x) = -2\sec^2 x \tan x > 0 \quad \Leftrightarrow \quad \tan x < 0 \quad \Leftrightarrow \quad -\frac{\pi}{2} < x < 0$, so $f$ is CU on $\left(-\frac{\pi}{2}, 0\right)$ and CD on $\left(0, \frac{\pi}{2}\right)$. IP at $(0, 0)$

**H.**



**31.** $y = f(x) = \sin^{-1}(1/x)$ **A.** $D = \{x \mid -1 \le 1/x \le 1\} = (-\infty, -1] \cup [1, \infty)$. **B.** No intercept

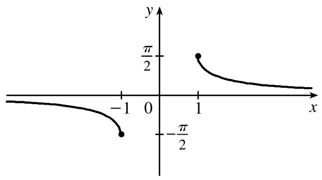
**C.** $f(-x) = -f(x)$, symmetric about the origin **D.** $\lim\limits_{x \to \pm\infty} \sin^{-1}(1/x) = \sin^{-1}(0) = 0$, so $y = 0$ is a HA.

**E.** $f'(x) = \dfrac{1}{\sqrt{1-(1/x)^2}}\left(-\dfrac{1}{x^2}\right) = \dfrac{-1}{\sqrt{x^4 - x^2}} < 0$, so $f$ is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

**F.** No local extreme value, but $f(1) = \frac{\pi}{2}$ is the absolute maximum value and $f(-1) = -\frac{\pi}{2}$ is the absolute minimum value.

**G.** $f''(x) = \dfrac{4x^3 - 2x}{2(x^4 - x^2)^{3/2}} = \dfrac{x(2x^2-1)}{(x^4-x^2)^{3/2}} > 0$ for $x > 1$ and $f''(x) < 0$ for $x < -1$, so $f$ is CU on $(1, \infty)$ and CD on $(-\infty, -1)$. No IP

**H.**

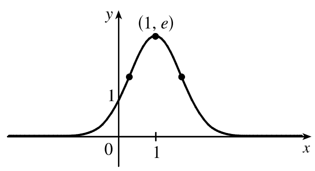



**32.** $y = f(x) = e^{2x - x^2}$ **A.** $D = \mathbb{R}$ **B.** $y$-intercept 1; no $x$-intercept **C.** No symmetry **D.** $\lim\limits_{x \to \pm\infty} e^{2x-x^2} = 0$, so $y = 0$ is a HA. **E.** $y = f(x) = e^{2x-x^2} \Rightarrow f'(x) = 2(1-x)e^{2x-x^2} > 0 \Leftrightarrow x < 1$, so $f$ is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. **F.** $f(1) = e$ is a local and absolute maximum value.

**G.** $f''(x) = 2(2x^2 - 4x + 1)e^{2x-x^2} = 0 \Leftrightarrow x = 1 \pm \frac{\sqrt{2}}{2}$.

$f''(x) > 0 \Leftrightarrow x < 1 - \frac{\sqrt{2}}{2}$ or $x > 1 + \frac{\sqrt{2}}{2}$, so $f$ is CU on $\left(-\infty, 1 - \frac{\sqrt{2}}{2}\right)$ and $\left(1 + \frac{\sqrt{2}}{2}, \infty\right)$, and CD on $\left(1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}\right)$. IP at $\left(1 \pm \frac{\sqrt{2}}{2}, \sqrt{e}\right)$

**H.**




**33.** $y = f(x) = (x-2)e^{-x}$ **A.** $D = \mathbb{R}$ **B.** $y$-intercept: $f(0) = -2$; $x$-intercept: $f(x) = 0 \Leftrightarrow x = 2$

**C.** No symmetry **D.** $\lim\limits_{x \to \infty} \dfrac{x-2}{e^x} \overset{\text{H}}{=} \lim\limits_{x \to \infty} \dfrac{1}{e^x} = 0$, so $y = 0$ is a HA. No VA

**E.** $f'(x) = (x-2)(-e^{-x}) + e^{-x}(1) = e^{-x}[-(x-2)+1] = (3-x)e^{-x}$.

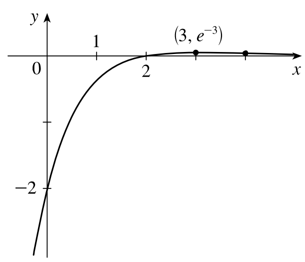
$f'(x) > 0$ for $x < 3$, so $f$ is increasing on $(-\infty, 3)$ and decreasing on $(3, \infty)$.

**F.** Local maximum value $f(3) = e^{-3}$, no local minimum value

**G.** $f''(x) = (3-x)(-e^{-x}) + e^{-x}(-1) = e^{-x}[-(3-x)+(-1)]$
$= (x-4)e^{-x} > 0$

for $x > 4$, so $f$ is CU on $(4, \infty)$ and CD on $(-\infty, 4)$. IP at $(4, 2e^{-4})$

**H.**



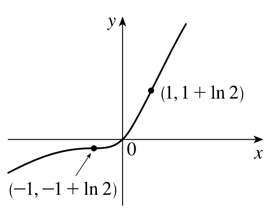

**34.** $y = f(x) = x + \ln(x^2+1)$ **A.** $D = \mathbb{R}$ **B.** $y$-intercept: $f(0) = 0 + \ln 1 = 0$; $x$-intercept: $f(x) = 0 \Leftrightarrow$ $\ln(x^2+1) = -x \Leftrightarrow x^2 + 1 = e^{-x} \Rightarrow x = 0$ since the graphs of $y = x^2 + 1$ and $y = e^{-x}$ intersect only at $x = 0$.

**C.** No symmetry **D.** No asymptote **E.** $f'(x) = 1 + \dfrac{2x}{x^2+1} = \dfrac{x^2+2x+1}{x^2+1} = \dfrac{(x+1)^2}{x^2+1}$. $f'(x) > 0$ if $x \ne -1$ and $f$ is increasing on $\mathbb{R}$. **F.** No local extreme values

**G.** $f''(x) = \dfrac{(x^2+1)2 - 2x(2x)}{(x^2+1)^2} = \dfrac{2[(x^2+1) - 2x^2]}{(x^2+1)^2} = \dfrac{2(1-x^2)}{(x^2+1)^2}$.

$f''(x) > 0 \Leftrightarrow -1 < x < 1$ and $f''(x) < 0 \Leftrightarrow x < -1$ or $x > 1$, so $f$ is CU on $(-1, 1)$ and $f$ is CD on $(-\infty, -1)$ and $(1, \infty)$. IP at $(-1, -1 + \ln 2)$ and $(1, 1 + \ln 2)$

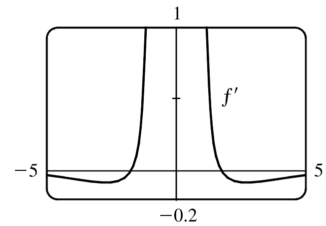
**H.**

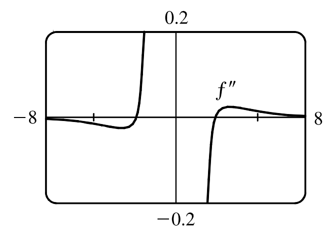


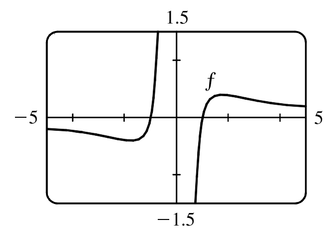
**35.** $f(x) = \dfrac{x^2 - 1}{x^3} \quad \Rightarrow \quad f'(x) = \dfrac{x^3(2x) - (x^2 - 1)3x^2}{x^6} = \dfrac{3 - x^2}{x^4} \quad \Rightarrow$

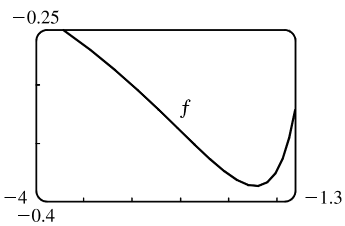
$f''(x) = \dfrac{x^4(-2x) - (3 - x^2)4x^3}{x^8} = \dfrac{2x^2 - 12}{x^5}$

*Estimates:* From the graphs of $f'$ and $f''$, it appears that $f$ is increasing on $(-1.73, 0)$ and $(0, 1.73)$ and decreasing on $(-\infty, -1.73)$ and $(1.73, \infty)$; $f$ has a local maximum of about $f(1.73) = 0.38$ and a local minimum of about $f(-1.7) = -0.38$; $f$ is CU on $(-2.45, 0)$ and $(2.45, \infty)$, and CD on $(-\infty, -2.45)$ and $(0, 2.45)$; and $f$ has inflection points at about $(-2.45, -0.34)$ and $(2.45, 0.34)$.



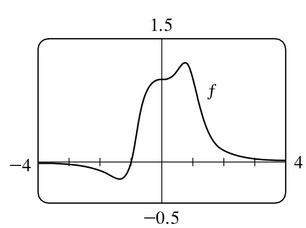



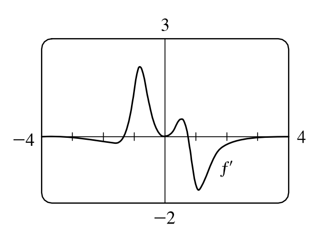



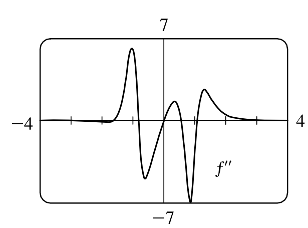

*Exact:* Now $f'(x) = \dfrac{3 - x^2}{x^4}$ is positive for $0 < x^2 < 3$, that is, $f$ is increasing on $(-\sqrt{3}, 0)$ and $(0, \sqrt{3})$; and $f'(x)$ is negative (and so $f$ is decreasing) on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$. $f'(x) = 0$ when $x = \pm\sqrt{3}$. $f'$ goes from positive to negative at $x = \sqrt{3}$, so $f$ has a local maximum of $f(\sqrt{3}) = \frac{(\sqrt{3})^2 - 1}{(\sqrt{3})^3} = \frac{2\sqrt{3}}{9}$; and since $f$ is odd, we know that maxima on the interval $(0, \infty)$ correspond to minima on $(-\infty, 0)$, so $f$ has a local minimum of $f(-\sqrt{3}) = -\frac{2\sqrt{3}}{9}$. Also, $f''(x) = \dfrac{2x^2 - 12}{x^5}$ is positive (so $f$ is CU) on $(-\sqrt{6}, 0)$ and $(\sqrt{6}, \infty)$, and negative (so $f$ is CD) on $(-\infty, -\sqrt{6})$ and $(0, \sqrt{6})$. There are IP at $\left(\sqrt{6}, \frac{5\sqrt{6}}{36}\right)$ and $\left(-\sqrt{6}, -\frac{5\sqrt{6}}{36}\right)$.

**36.** $f(x) = \dfrac{x^3 + 1}{x^6 + 1} \quad \Rightarrow \quad f'(x) = -\dfrac{3x^2(x^6 + 2x^3 - 1)}{(x^6 + 1)^2} \quad \Rightarrow \quad f''(x) = \dfrac{6x(2x^{12} + 7x^9 - 9x^6 - 5x^3 + 1)}{(x^6 + 1)^3}$.

$f(x) = 0 \quad \Leftrightarrow \quad x = -1$. $f'(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $x \approx -1.34, 0.75$. $f''(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $x \approx -1.64, -0.82, 0.54, 1.09$. From the graphs of $f$ and $f'$, it appears that $f$ is decreasing on $(-\infty, -1.34)$, increasing on $(-1.34, 0.75)$, and decreasing on $(0.75, \infty)$. $f$ has a local minimum value of $f(-1.34) \approx -0.21$ and a local maximum value of $f(0.75) \approx 1.21$. From the graphs of $f$ and $f''$, it appears that $f$ is CD on $(-\infty, -1.64)$, CU on $(-1.64, -0.82)$, CD on $(-0.82, 0)$, CU on $(0, 0.54)$, CD on $(0.54, 1.09)$ and CU on $(1.09, \infty)$. There are inflection points at about $(-1.64, -0.17)$, $(-0.82, 0.34)$, $(0.54, 1.13)$, $(1.09, 0.86)$, and at $(0, 1)$.

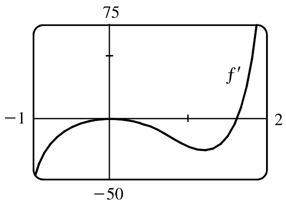


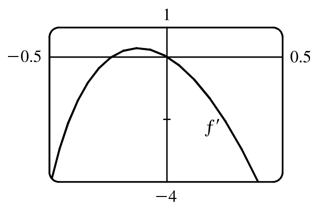



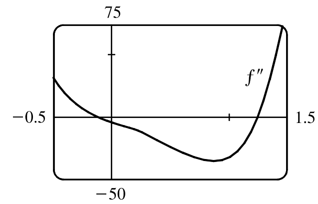



**37.** $f(x) = 3x^6 - 5x^5 + x^4 - 5x^3 - 2x^2 + 2 \quad \Rightarrow \quad f'(x) = 18x^5 - 25x^4 + 4x^3 - 15x^2 - 4x \quad \Rightarrow$

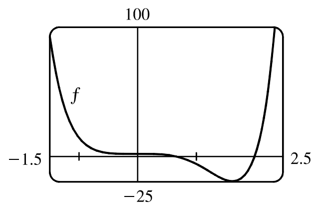
$f''(x) = 90x^4 - 100x^3 + 12x^2 - 30x - 4$

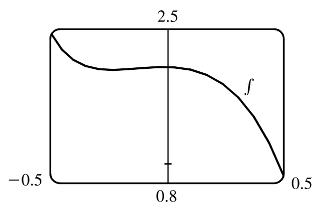


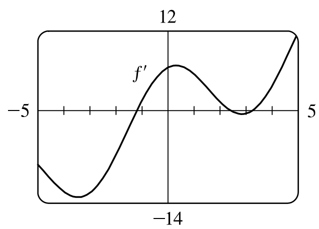



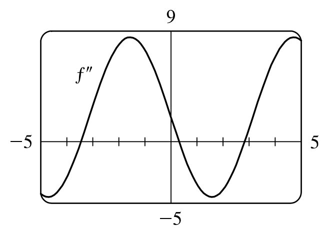

From the graphs of $f'$ and $f''$, it appears that $f$ is increasing on $(-0.23, 0)$ and $(1.62, \infty)$ and decreasing on $(-\infty, -0.23)$ and $(0, 1.62)$; $f$ has a local maximum of $f(0) = 2$ and local minima of about $f(-0.23) = 1.96$ and $f(1.62) = -19.2$; $f$ is CU on $(-\infty, -0.12)$ and $(1.24, \infty)$ and CD on $(-0.12, 1.24)$; and $f$ has inflection points at about $(-0.12, 1.98)$ and $(1.24, -12.1)$.



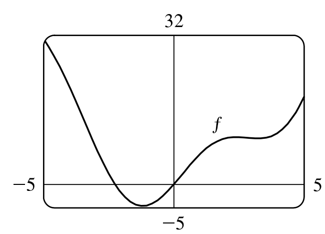




**38.** $f(x) = x^2 + 6.5 \sin x,\ -5 \leq x \leq 5 \quad \Rightarrow \quad f'(x) = 2x + 6.5 \cos x \quad \Rightarrow \quad f''(x) = 2 - 6.5 \sin x. \quad f(x) = 0 \quad \Leftrightarrow$ $x \approx -2.25$ and $x = 0$; $\quad f'(x) = 0 \quad \Leftrightarrow \quad x \approx -1.19, 2.40, 3.24$; $\quad f''(x) = 0 \quad \Leftrightarrow \quad x \approx -3.45, 0.31, 2.83$.

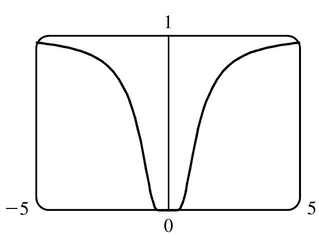



From the graphs of $f'$ and $f''$, it appears that $f$ is decreasing on $(-5, -1.19)$ and $(2.40, 3.24)$ and increasing on $(-1.19, 2.40)$ and $(3.24, 5)$; $f$ has a local maximum of about $f(2.40) = 10.15$ and local minima of about $f(-1.19) = -4.62$ and $f(3.24) = 9.86$; $f$ is CU on $(-3.45, 0.31)$ and $(2.83, 5)$ and CD on $(-5, -3.45)$ and $(0.31, 2.83)$; and $f$ has inflection points at about $(-3.45, 13.93)$, $(0.31, 2.10)$, and $(2.83, 10.00)$.




**39.**



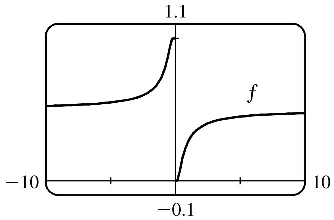

From the graph, we estimate the points of inflection to be about $(\pm 0.82, 0.22)$.

$f(x) = e^{-1/x^2} \quad \Rightarrow \quad f'(x) = 2x^{-3}e^{-1/x^2} \quad \Rightarrow$

$f''(x) = 2[x^{-3}(2x^{-3})e^{-1/x^2} + e^{-1/x^2}(-3x^{-4})] = 2x^{-6}e^{-1/x^2}\left(2 - 3x^2\right)$.

This is 0 when $2 - 3x^2 = 0 \quad \Leftrightarrow \quad x = \pm\sqrt{\frac{2}{3}}$, so the inflection points are $\left(\pm\sqrt{\frac{2}{3}}, e^{-3/2}\right)$.

**40.** (a)




(b) $f(x) = \dfrac{1}{1+e^{1/x}}$.

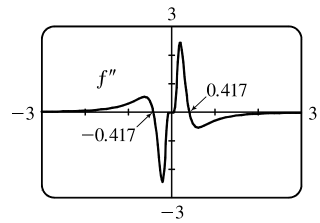
$\displaystyle\lim_{x\to\infty} f(x) = \frac{1}{1+1} = \frac{1}{2}$, $\displaystyle\lim_{x\to-\infty} f(x) = \frac{1}{1+1} = \frac{1}{2}$,

as $x \to 0^+$, $1/x \to \infty$, so $e^{1/x} \to \infty \;\Rightarrow\; \displaystyle\lim_{x\to 0^+} f(x) = 0$,

as $x \to 0^-$, $1/x \to -\infty$, so $e^{1/x} \to 0 \;\Rightarrow\; \displaystyle\lim_{x\to 0^-} f(x) = \frac{1}{1+0} = 1$

(c) From the graph of $f$, estimates for the IP are $(-0.4, 0.9)$ and $(0.4, 0.08)$.

(d) $f''(x) = -\dfrac{e^{1/x}[e^{1/x}(2x-1)+2x+1]}{x^4(e^{1/x}+1)^3}$



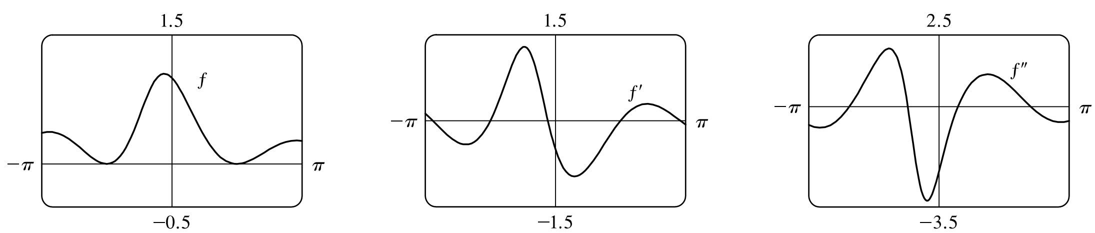

(e) From the graph, we see that $f''$ changes sign at $x = \pm 0.417$ ($x = 0$ is not in the domain of $f$). IP are approximately $(0.417, 0.083)$ and $(-0.417, 0.917)$.

**41.** $f(x) = \dfrac{\cos^2 x}{\sqrt{x^2+x+1}}$, $-\pi \le x \le \pi \;\Rightarrow\; f'(x) = -\dfrac{\cos x\,[(2x+1)\cos x + 4(x^2+x+1)\sin x]}{2(x^2+x+1)^{3/2}} \;\Rightarrow$

$$f''(x) = -\frac{(8x^4+16x^3+16x^2+8x+9)\cos^2 x - 8(x^2+x+1)(2x+1)\sin x\,\cos x - 8(x^2+x+1)^2\sin^2 x}{4(x^2+x+1)^{5/2}}$$

$f(x) = 0 \;\Leftrightarrow\; x = \pm\frac{\pi}{2}$; $f'(x) = 0 \;\Leftrightarrow\; x \approx -2.96, -1.57, -0.18, 1.57, 3.01$;

$f''(x) = 0 \;\Leftrightarrow\; x \approx -2.16, -0.75, 0.46$, and $2.21$.



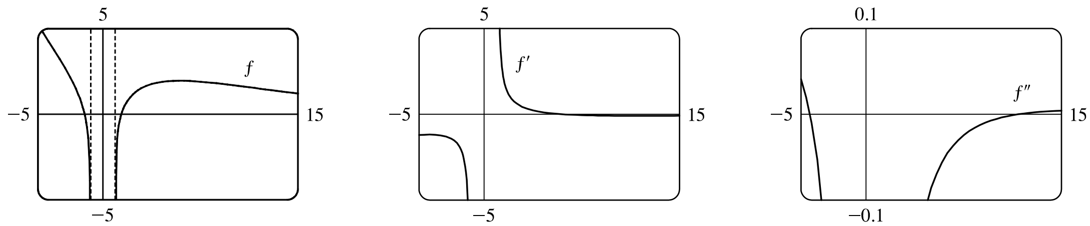

The $x$-coordinates of the maximum points are the values at which $f'$ changes from positive to negative, that is, $-2.96$, $-0.18$, and $3.01$. The $x$-coordinates of the minimum points are the values at which $f'$ changes from negative to positive, that is, $-1.57$ and $1.57$. The $x$-coordinates of the inflection points are the values at which $f''$ changes sign, that is, $-2.16$, $-0.75$, $0.46$, and $2.21$.

**42.** $f(x) = e^{-0.1x}\ln(x^2-1) \;\Rightarrow\; f'(x) = \dfrac{e^{-0.1x}\left[(x^2-1)\ln(x^2-1) - 20x\right]}{10(1-x^2)} \;\Rightarrow$

$$f''(x) = \frac{e^{-0.1x}\left[(x^2-1)^2\ln(x^2-1) - 40(x^3+5x^2-x+5)\right]}{100(x^2-1)^2}.$$

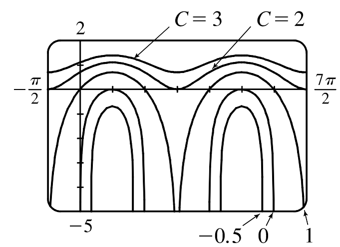
The domain of $f$ is $(-\infty,-1)\cup(1,\infty)$. $f(x) = 0 \;\Leftrightarrow\; x = \pm\sqrt{2}$; $f'(x) = 0 \;\Leftrightarrow\; x \approx 5.87$;

$f''(x) = 0 \;\Leftrightarrow\; x \approx -4.31$ and $11.74$.




$f'$ changes from positive to negative at $x \approx 5.87$, so $5.87$ is the $x$-coordinate of the maximum point. There is no minimum point. The $x$-coordinates of the inflection points are the values at which $f''$ changes sign, that is, $-4.31$ and $11.74$.

**43.** The family of functions $f(x) = \ln(\sin x + C)$ all have the same period and all have maximum values at $x = \frac{\pi}{2} + 2\pi n$. Since the domain of ln is $(0, \infty)$, $f$ has a graph only if $\sin x + C > 0$ somewhere. Since $-1 \leq \sin x \leq 1$, this happens if $C > -1$, that is, $f$ has no graph if $C \leq -1$. Similarly, if $C > 1$, then $\sin x + C > 0$ and $f$ is continuous on $(-\infty, \infty)$. As $C$ increases, the graph of $f$ is shifted vertically upward and flattens out. If $-1 < C \leq 1$, $f$ is defined where $\sin x + C > 0$ $\Leftrightarrow$ $\sin x > -C$ $\Leftrightarrow$ $\sin^{-1}(-C) < x < \pi - \sin^{-1}(-C)$. Since the period is $2\pi$, the domain of $f$ is $\big(2n\pi + \sin^{-1}(-C), (2n+1)\pi - \sin^{-1}(-C)\big)$, $n$ an integer.

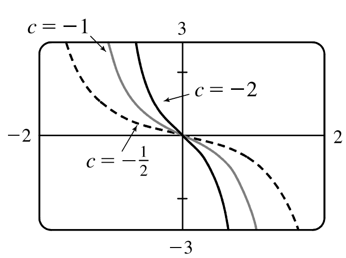


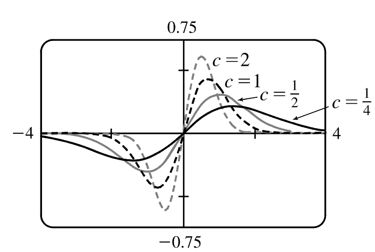

**44.** We exclude the case $c = 0$, since in that case $f(x) = 0$ for all $x$. To find the maxima and minima, we differentiate:

$$f(x) = cxe^{-cx^2} \quad \Rightarrow \quad f'(x) = c\left[xe^{-cx^2}(-2cx) + e^{-cx^2}(1)\right] = ce^{-cx^2}(-2cx^2 + 1)$$

This is 0 where $-2cx^2 + 1 = 0$ $\Leftrightarrow$ $x = \pm 1/\sqrt{2c}$. So if $c > 0$, there are two maxima or minima, whose $x$-coordinates approach 0 as $c$ increases. The negative root gives a minimum and the positive root gives a maximum, by the First Derivative Test. By substituting back into the equation, we see that $f\left(\pm 1/\sqrt{2c}\right) = c\left(\pm 1/\sqrt{2c}\right) e^{-c\left(\pm 1/\sqrt{2c}\right)^2} = \pm\sqrt{c/2e}$. So as $c$ increases, the extreme points become more pronounced. Note that if $c > 0$, then $\lim\limits_{x \to \pm\infty} f(x) = 0$. If $c < 0$, then there are no extreme values, and $\lim\limits_{x \to \pm\infty} f(x) = \mp\infty$.

To find the points of inflection, we differentiate again: $f'(x) = ce^{-cx^2}\left(-2cx^2 + 1\right)$ $\Rightarrow$ $f''(x) = c\left[e^{-cx^2}(-4cx) + (-2cx^2 + 1)(-2cxe^{-cx^2})\right] = -2c^2xe^{-cx^2}(3 - 2cx^2)$. This is 0 at $x = 0$ and where $3 - 2cx^2 = 0$ $\Leftrightarrow$ $x = \pm\sqrt{3/(2c)}$ $\Rightarrow$ IP at $\left(\pm\sqrt{3/(2c)}, \pm\sqrt{3c/2}\, e^{-3/2}\right)$. If $c > 0$ there are three inflection points, and as $c$ increases, the $x$-coordinates of the nonzero inflection points approach 0. If $c < 0$, there is only one inflection point, the origin.



**45.** Let $f(x) = 3x + 2\cos x + 5$. Then $f(0) = 7 > 0$ and $f(-\pi) = -3\pi - 2 + 5 = -3\pi + 3 = -3(\pi - 1) < 0$, and since $f$ is continuous on $\mathbb{R}$ (hence on $[-\pi, 0]$), the Intermediate Value Theorem assures us that there is at least one zero of $f$ in $[-\pi, 0]$. Now $f'(x) = 3 - 2\sin x > 0$ implies that $f$ is increasing on $\mathbb{R}$, so there is exactly one zero of $f$, and hence, exactly one real root of the equation $3x + 2\cos x + 5 = 0$.

**46.** By the Mean Value Theorem, $f'(c) = \dfrac{f(4) - f(0)}{4 - 0} \quad\Leftrightarrow\quad 4f'(c) = f(4) - 1$ for some $c$ with $0 < c < 4$. Since $2 \le f'(c) \le 5$, we have $4(2) \le 4f'(c) \le 4(5) \quad\Leftrightarrow\quad 4(2) \le f(4) - 1 \le 4(5) \quad\Leftrightarrow\quad 8 \le f(4) - 1 \le 20 \quad\Leftrightarrow$ $9 \le f(4) \le 21$.

**47.** Since $f$ is continuous on $[32, 33]$ and differentiable on $(32, 33)$, then by the Mean Value Theorem there exists a number $c$ in $(32, 33)$ such that $f'(c) = \frac{1}{5}c^{-4/5} = \dfrac{\sqrt[5]{33} - \sqrt[5]{32}}{33 - 32} = \sqrt[5]{33} - 2$, but $\frac{1}{5}c^{-4/5} > 0 \quad\Rightarrow\quad \sqrt[5]{33} - 2 > 0 \quad\Rightarrow\quad \sqrt[5]{33} > 2$. Also $f'$ is decreasing, so that $f'(c) < f'(32) = \frac{1}{5}(32)^{-4/5} = 0.0125 \quad\Rightarrow\quad 0.0125 > f'(c) = \sqrt[5]{33} - 2 \quad\Rightarrow\quad \sqrt[5]{33} < 2.0125$. Therefore, $2 < \sqrt[5]{33} < 2.0125$.

**48.** Since the point $(1, 3)$ is on the curve $y = ax^3 + bx^2$, we have $3 = a(1)^3 + b(1)^2 \quad\Rightarrow\quad 3 = a + b$ **(1)**.
$y' = 3ax^2 + 2bx \quad\Rightarrow\quad y'' = 6ax + 2b$. $\;y'' = 0$ [for inflection points] $\quad\Leftrightarrow\quad x = \dfrac{-2b}{6a} = -\dfrac{b}{3a}$. Since we want $x = 1$, $1 = -\dfrac{b}{3a} \quad\Rightarrow\quad b = -3a$. Combining with **(1)** gives us $3 = a - 3a \quad\Leftrightarrow\quad 3 = -2a \quad\Leftrightarrow\quad a = -\frac{3}{2}$. Hence, $b = -3\left(-\frac{3}{2}\right) = \frac{9}{2}$ and the curve is $y = -\frac{3}{2}x^3 + \frac{9}{2}x^2$.

**49.** (a) $g(x) = f(x^2) \quad\Rightarrow\quad g'(x) = 2xf'(x^2)$ by the Chain Rule. Since $f'(x) > 0$ for all $x \ne 0$, we must have $f'(x^2) > 0$ for $x \ne 0$, so $g'(x) = 0 \quad\Leftrightarrow\quad x = 0$. Now $g'(x)$ changes sign (from negative to positive) at $x = 0$, since one of its factors, $f'(x^2)$, is positive for all $x$, and its other factor, $2x$, changes from negative to positive at this point, so by the First Derivative Test, $f$ has a local and absolute minimum at $x = 0$.

(b) $g'(x) = 2xf'(x^2) \quad\Rightarrow\quad g''(x) = 2[xf''(x^2)(2x) + f'(x^2)] = 4x^2f''(x^2) + 2f'(x^2)$ by the Product Rule and the Chain Rule. But $x^2 > 0$ for all $x \ne 0$, $f''(x^2) > 0$ [since $f$ is CU for $x > 0$], and $f'(x^2) > 0$ for all $x \ne 0$, so since all of its factors are positive, $g''(x) > 0$ for $x \ne 0$. Whether $g''(0)$ is positive or 0 doesn't matter [since the sign of $g''$ does not change there]; $g$ is concave upward on $\mathbb{R}$.

**50.** Call the two integers $x$ and $y$. Then $x + 4y = 1000$, so $x = 1000 - 4y$. Their product is $P = xy = (1000 - 4y)y$, so our problem is to maximize the function $P(y) = 1000y - 4y^2$, where $0 < y < 250$ and $y$ is an integer. $P'(y) = 1000 - 8y$, so $P'(y) = 0 \quad\Leftrightarrow\quad y = 125$. $\;P''(y) = -8 < 0$, so $P(125) = 62{,}500$ is an absolute maximum. Since the optimal $y$ turned out to be an integer, we have found the desired pair of numbers, namely $x = 1000 - 4(125) = 500$ and $y = 125$.

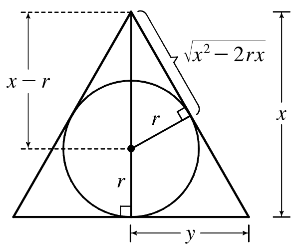
**51.** If $B = 0$, the line is vertical and the distance from $x = -\dfrac{C}{A}$ to $(x_1, y_1)$ is $\left|x_1 + \dfrac{C}{A}\right| = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$, so assume $B \ne 0$. The square of the distance from $(x_1, y_1)$ to the line is $f(x) = (x - x_1)^2 + (y - y_1)^2$ where $Ax + By + C = 0$, so we minimize $f(x) = (x - x_1)^2 + \left(-\dfrac{A}{B}x - \dfrac{C}{B} - y_1\right)^2 \quad \Rightarrow \quad f'(x) = 2\,(x - x_1) + 2\left(-\dfrac{A}{B}x - \dfrac{C}{B} - y_1\right)\left(-\dfrac{A}{B}\right)$.

$f'(x) = 0 \quad \Rightarrow \quad x = \dfrac{B^2x_1 - ABy_1 - AC}{A^2 + B^2}$ and this gives a minimum since $f''(x) = 2\left(1 + \dfrac{A^2}{B^2}\right) > 0$. Substituting this value of $x$ into $f(x)$ and simplifying gives $f(x) = \dfrac{(Ax_1 + By_1 + C)^2}{A^2 + B^2}$, so the minimum distance is

$\sqrt{f(x)} = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

**52.** On the hyperbola $xy = 8$, if $d(x)$ is the distance from the point $(x, y) = (x, 8/x)$ to the point $(3, 0)$, then $[d(x)]^2 = (x - 3)^2 + 64/x^2 = f(x)$. $f'(x) = 2(x - 3) - 128/x^3 = 0 \quad \Rightarrow \quad x^4 - 3x^3 - 64 = 0 \quad \Rightarrow$ $(x - 4)\left(x^3 + x^2 + 4x + 16\right) = 0 \quad \Rightarrow \quad x = 4$ since the solution must have $x > 0$. Then $y = \frac{8}{4} = 2$, so the point is $(4, 2)$.

**53.**




By similar triangles, $\dfrac{y}{x} = \dfrac{r}{\sqrt{x^2 - 2rx}}$, so the area of the triangle is

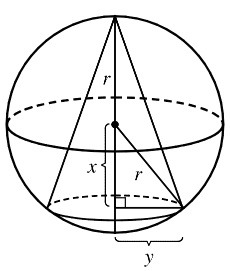
$A(x) = \frac{1}{2}(2y)x = xy = \dfrac{rx^2}{\sqrt{x^2 - 2rx}} \quad \Rightarrow$

$A'(x) = \dfrac{2rx\sqrt{x^2 - 2rx} - rx^2(x - r)/\sqrt{x^2 - 2rx}}{x^2 - 2rx} = \dfrac{rx^2\,(x - 3r)}{(x^2 - 2rx)^{3/2}} = 0$

when $x = 3r$.

$A'(x) < 0$ when $2r < x < 3r$, $A'(x) > 0$ when $x > 3r$. So $x = 3r$ gives a minimum and $A(3r) = \dfrac{r(9r^2)}{\sqrt{3}\,r} = 3\sqrt{3}\,r^2$.

**54.**



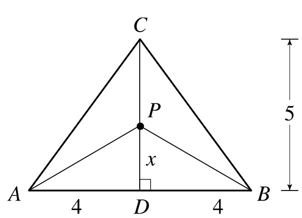

The volume of the cone is $V = \frac{1}{3}\pi y^2(r + x) = \frac{1}{3}\pi(r^2 - x^2)(r + x)$, $-r \le x \le r$.

$$\begin{aligned} V'(x) &= \tfrac{\pi}{3}[(r^2 - x^2)\,(1) + (r + x)(-2x)] = \tfrac{\pi}{3}[(r + x)(r - x - 2x)] \\ &= \tfrac{\pi}{3}(r + x)(r - 3x) = 0 \text{ when } x = -r \text{ or } x = r/3. \end{aligned}$$

Now $V(r) = 0 = V(-r)$, so the maximum occurs at $x = r/3$ and the volume is

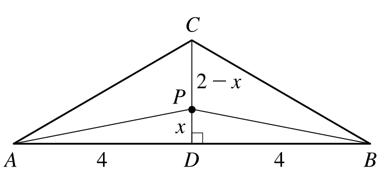
$V\left(\dfrac{r}{3}\right) = \dfrac{\pi}{3}\left(r^2 - \dfrac{r^2}{9}\right)\left(\dfrac{4r}{3}\right) = \dfrac{32\pi r^3}{81}$.

**55.**




We minimize $L(x) = |PA| + |PB| + |PC| = 2\sqrt{x^2 + 16} + (5 - x)$, $0 \le x \le 5$. $L'(x) = 2x/\sqrt{x^2 + 16} - 1 = 0 \quad \Leftrightarrow \quad 2x = \sqrt{x^2 + 16} \quad \Leftrightarrow$ $4x^2 = x^2 + 16 \quad \Leftrightarrow \quad x = \frac{4}{\sqrt{3}}$. $L(0) = 13$, $L\left(\frac{4}{\sqrt{3}}\right) \approx 11.9$, $L(5) \approx 12.8$, so the minimum occurs when $x = \frac{4}{\sqrt{3}} \approx 2.3$.

**56.**

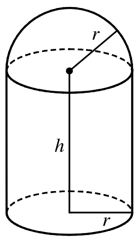



If $|CD| = 2$, the last part of $L(x)$ changes from $(5 - x)$ to $(2 - x)$ with $0 \le x \le 2$. But we still get $L'(x) = 0 \quad \Leftrightarrow \quad x = \frac{4}{\sqrt{3}}$, which isn't in the interval $[0, 2]$. Now $L(0) = 10$ and $L(2) = 2\sqrt{20} = 4\sqrt{5} \approx 8.9$. The minimum occurs when $P = C$.

**57.** $v = K\sqrt{\dfrac{L}{C} + \dfrac{C}{L}} \quad \Rightarrow \quad \dfrac{dv}{dL} = \dfrac{K}{2\sqrt{(L/C) + (C/L)}}\left(\dfrac{1}{C} - \dfrac{C}{L^2}\right) = 0 \quad \Leftrightarrow \quad \dfrac{1}{C} = \dfrac{C}{L^2} \quad \Leftrightarrow \quad L^2 = C^2 \quad \Leftrightarrow \quad L = C.$

This gives the minimum velocity since $v' < 0$ for $0 < L < C$ and $v' > 0$ for $L > C$.

**58.**




We minimize the surface area $S = \pi r^2 + 2\pi rh + \frac{1}{2}(4\pi r^2) = 3\pi r^2 + 2\pi rh$.

Solving $V = \pi r^2 h + \frac{2}{3}\pi r^3$ for $h$, we get $h = \dfrac{V - \frac{2}{3}\pi r^3}{\pi r^2} = \dfrac{V}{\pi r^2} - \frac{2}{3}r$, so

$S(r) = 3\pi r^2 + 2\pi r\left[\dfrac{V}{\pi r^2} - \frac{2}{3}r\right] = \frac{5}{3}\pi r^2 + \dfrac{2V}{r}$.

$S'(r) = -\dfrac{2V}{r^2} + \frac{10}{3}\pi r = \dfrac{\frac{10}{3}\pi r^3 - 2V}{r^2} = 0 \quad \Leftrightarrow \quad \frac{10}{3}\pi r^3 = 2V \quad \Leftrightarrow \quad r^3 = \dfrac{3V}{5\pi} \quad \Leftrightarrow \quad r = \sqrt[3]{\dfrac{3V}{5\pi}}.$

This gives an absolute minimum since $S'(r) < 0$ for $0 < r < \sqrt[3]{\dfrac{3V}{5\pi}}$ and $S'(r) > 0$ for $r > \sqrt[3]{\dfrac{3V}{5\pi}}$. Thus,

$$h = \frac{V - \frac{2}{3}\pi \cdot \dfrac{3V}{5\pi}}{\pi\sqrt[3]{\dfrac{(3V)^2}{(5\pi)^2}}} = \frac{\left(V - \frac{2}{5}V\right)\sqrt[3]{(5\pi)^2}}{\pi\sqrt[3]{(3V)^2}} = \frac{3V\sqrt[3]{(5\pi)^2}}{5\pi\sqrt[3]{(3V)^2}} = \sqrt[3]{\frac{3V}{5\pi}} = r$$

**59.** Let $x$ denote the number of \$1 decreases in ticket price. Then the ticket price is $\$12 - \$1(x)$, and the average attendance is $11{,}000 + 1000(x)$. Now the revenue per game is

$$\begin{aligned} R(x) &= (\text{price per person}) \times (\text{number of people per game}) \\ &= (12 - x)(11{,}000 + 1000x) = -1000x^2 + 1000x + 132{,}000 \end{aligned}$$

for $0 \le x \le 4$ [since the seating capacity is 15,000] $\Rightarrow$ $R'(x) = -2000x + 1000 = 0$ $\Leftrightarrow$ $x = 0.5$. This is a maximum since $R''(x) = -2000 < 0$ for all $x$. Now we must check the value of $R(x) = (12 - x)(11{,}000 + 1000x)$ at $x = 0.5$ and at the endpoints of the domain to see which value of $x$ gives the maximum value of $R$.

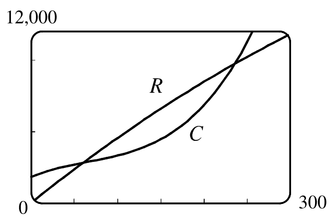
$R(0) = (12)(11{,}000) = 132{,}000$, $R(0.5) = (11.5)(11{,}500) = 132{,}250$, and $R(4) = (8)(15{,}000) = 120{,}000$. Thus, the maximum revenue of \$132,250 per game occurs when the average attendance is 11,500 and the ticket price is \$11.50.

**60.** (a) $C(x) = 1800 + 25x - 0.2x^2 + 0.001x^3$ and

$R(x) = xp(x) = 48.2x - 0.03x^2$.

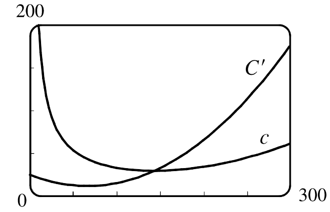
The profit is maximized when $C'(x) = R'(x)$.

From the figure, we estimate that the tangents are parallel when $x \approx 160$.




(b) $C'(x) = 25 - 0.4x + 0.003x^2$ and $R'(x) = 48.2 - 0.06x$. $C'(x) = R'(x) \quad \Rightarrow \quad 0.003x^2 - 0.34x - 23.2 = 0 \quad \Rightarrow$ $x_1 \approx 161.3$ $(x > 0)$. $R''(x) = -0.06$ and $C''(x) = -0.4 + 0.006x$, so $R''(x_1) = -0.06 < C''(x_1) \approx 0.57 \quad \Rightarrow$ profit is maximized by producing 161 units.

(c) $c(x) = \dfrac{C(x)}{x} = \dfrac{1800}{x} + 25 - 0.2x + 0.001x^2$ is the average cost. Since the average cost is minimized when the marginal cost equals the average cost, we graph $c(x)$ and $C'(x)$ and estimate the point of intersection. From the figure, $C'(x) = c(x) \quad \Leftrightarrow \quad x \approx 144$.




**61.** $f(x) = x^5 - x^4 + 3x^2 - 3x - 2 \quad \Rightarrow \quad f'(x) = 5x^4 - 4x^3 + 6x - 3$, so $x_{n+1} = x_n - \dfrac{x_n^5 - x_n^4 + 3x_n^2 - 3x_n - 2}{5x_n^4 - 4x_n^3 + 6x_n - 3}$.

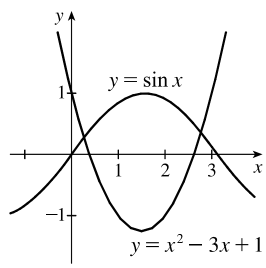
Now $x_1 = 1 \quad \Rightarrow \quad x_2 = 1.5 \quad \Rightarrow \quad x_3 \approx 1.343860 \quad \Rightarrow \quad x_4 \approx 1.300320 \quad \Rightarrow \quad x_5 \approx 1.297396 \quad \Rightarrow$

$x_6 \approx 1.297383 \approx x_7$, so the root in $[1, 2]$ is $1.297383$, to six decimal places.

**62.** Graphing $y = \sin x$ and $y = x^2 - 3x + 1$ shows that there are two roots, one about 0.3 and the other about 2.8. $f(x) = \sin x - x^2 + 3x - 1 \quad \Rightarrow$

$f'(x) = \cos x - 2x + 3 \quad \Rightarrow \quad x_{n+1} = x_n - \dfrac{\sin x_n - x_n^2 + 3x_n - 1}{\cos x_n - 2x_n + 3}$.

Now $x_1 = 0.3 \quad \Rightarrow \quad x_2 \approx 0.268552 \quad \Rightarrow \quad x_3 \approx 0.268881 \approx x_4$ and $x_1 = 2.8 \quad \Rightarrow \quad x_2 \approx 2.770354 \quad \Rightarrow \quad x_3 \approx 2.770058 \approx x_4$, so to six decimal places, the roots are 0.268881 and 2.770058.

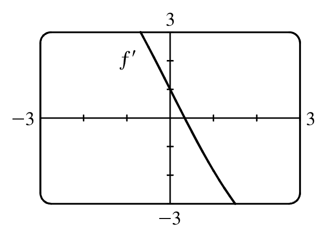



**63.** $f(t) = \cos t + t - t^2 \quad \Rightarrow \quad f'(t) = -\sin t + 1 - 2t$. $f'(t)$ exists for all $t$, so to find the maximum of $f$, we can examine the zeros of $f'$.

From the graph of $f'$, we see that a good choice for $t_1$ is $t_1 = 0.3$.

Use $g(t) = -\sin t + 1 - 2t$ and $g'(t) = -\cos t - 2$ to obtain

$t_2 \approx 0.33535293$, $t_3 \approx 0.33541803 \approx t_4$. Since $f''(t) = -\cos t - 2 < 0$ for all $t$, $f(0.33541803) \approx 1.16718557$ is the absolute maximum.




**64.** $y = f(x) = x \sin x, \ \ 0 \leq x \leq 2\pi$. **A.** $D = [0, 2\pi]$ **B.** $y$-intercept: $f(0) = 0$; $x$-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $\sin x = 0 \quad \Leftrightarrow \quad x = 0, \pi$, or $2\pi$. **C.** There is no symmetry on $D$, but if $f$ is defined for all real numbers $x$, then $f$ is an even function. **D.** No asymptote **E.** $f'(x) = x \cos x + \sin x$. To find critical numbers in $(0, 2\pi)$, we graph $f'$ and see that there are two critical numbers, about 2 and 4.9. To find them more precisely, we use Newton's method, setting

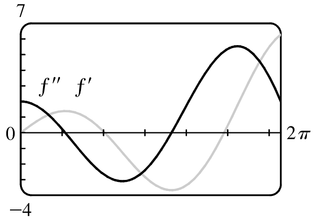
$g(x) = f'(x) = x \cos x + \sin x$, so that $g'(x) = f''(x) = 2 \cos x - x \sin x$ and $x_{n+1} = x_n - \dfrac{x_n \cos x_n + \sin x_n}{2 \cos x_n - x_n \sin x_n}$.

$x_1 = 2 \quad \Rightarrow \quad x_2 \approx 2.029048$, $x_3 \approx 2.028758 \approx x_4$ and $x_1 = 4.9 \quad \Rightarrow \quad x_2 \approx 4.913214$, $x_3 \approx 4.913180 \approx x_4$, so the critical numbers, to six decimal places, are $r_1 = 2.028758$ and $r_2 = 4.913180$. By checking sample values of $f'$ in $(0, r_1)$, $(r_1, r_2)$, and $(r_2, 2\pi)$, we see that $f$ is increasing on $(0, r_1)$, decreasing on $(r_1, r_2)$, and increasing on $(r_2, 2\pi)$. **F.** Local maximum value $f(r_1) \approx 1.819706$, local minimum value $f(r_2) \approx -4.814470$. **G.** $f''(x) = 2 \cos x - x \sin x$. To find points where $f''(x) = 0$, we graph $f''$ and find that $f''(x) = 0$ at about 1 and 3.6. To find the values more precisely, we use Newton's method. Set $h(x) = f''(x) = 2\cos x - x\sin x$. Then $h'(x) = -3\sin x - x\cos x$, so

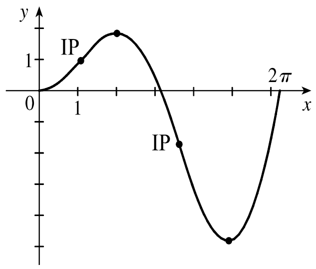
$x_{n+1} = x_n - \dfrac{2\cos x_n - x_n \sin x_n}{-3\sin x_n - x_n\cos x_n}$. $x_1 = 1 \Rightarrow x_2 \approx 1.078028$, $x_3 \approx 1.076874 \approx x_4$ and $x_1 = 3.6 \Rightarrow$

$x_2 \approx 3.643996$, $x_3 \approx 3.643597 \approx x_4$, so the zeros of $f''$, to six decimal places, are $r_3 = 1.076874$ and $r_4 = 3.643597$.

By checking sample values of $f''$ in $(0, r_3)$, $(r_3, r_4)$, and $(r_4, 2\pi)$, we see that $f$ is CU on $(0, r_3)$, CD on $(r_3, r_4)$, and CU on $(r_4, 2\pi)$. $f$ has inflection points at $(r_3, f(r_3) \approx 0.948166)$ and $(r_4, f(r_4) \approx -1.753240)$.




**H.**




**65.** $f(x) = 4\sqrt{x} - 6x^2 + 3 = 4x^{1/2} - 6x^2 + 3 \Rightarrow F(x) = 4\left(\frac{2}{3}x^{3/2}\right) - 6\left(\frac{1}{3}x^3\right) + 3x + C = \frac{8}{3}x^{3/2} - 2x^3 + 3x + C$

**66.** $g(x) = \dfrac{1}{x} + \dfrac{1}{x^2+1} \Rightarrow G(x) = \begin{cases} \ln x + \tan^{-1} x + C_1 & \text{if } x > 0 \\ \ln(-x) + \tan^{-1} x + C_2 & \text{if } x < 0 \end{cases}$

**67.** $f(t) = 2\sin t - 3e^t \Rightarrow F(t) = -2\cos t - 3e^t + C$

**68.** $f(x) = x^{-3} + \cosh x \Rightarrow F(x) = \begin{cases} -1/(2x^2) + \sinh x + C_1 & \text{if } x > 0 \\ -1/(2x^2) + \sinh x + C_2 & \text{if } x < 0 \end{cases}$

**69.** $f'(t) = 2t - 3\sin t \Rightarrow f(t) = t^2 + 3\cos t + C$.

$f(0) = 3 + C$ and $f(0) = 5 \Rightarrow C = 2$, so $f(t) = t^2 + 3\cos t + 2$.

**70.** $f'(u) = \dfrac{u^2 + \sqrt{u}}{u} = u + u^{-1/2} \Rightarrow f(u) = \frac{1}{2}u^2 + 2u^{1/2} + C$.

$f(1) = \frac{1}{2} + 2 + C$ and $f(1) = 3 \Rightarrow C = \frac{1}{2}$, so $f(u) = \frac{1}{2}u^2 + 2\sqrt{u} + \frac{1}{2}$.

**71.** $f''(x) = 1 - 6x + 48x^2 \Rightarrow f'(x) = x - 3x^2 + 16x^3 + C$. $f'(0) = C$ and $f'(0) = 2 \Rightarrow C = 2$, so $f'(x) = x - 3x^2 + 16x^3 + 2$ and hence, $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + D$.

$f(0) = D$ and $f(0) = 1 \Rightarrow D = 1$, so $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + 1$.

**72.** $f''(x) = 5x^3 + 6x^2 + 2 \Rightarrow f'(x) = \frac{5}{4}x^4 + 2x^3 + 2x + C \Rightarrow f(x) = \frac{1}{4}x^5 + \frac{1}{2}x^4 + x^2 + Cx + D$. Now $f(0) = D$ and $f(0) = 3$, so $D = 3$. Also, $f(1) = \frac{1}{4} + \frac{1}{2} + 1 + C + 3 = C + \frac{19}{4}$ and $f(1) = -2$, so $C + \frac{19}{4} = -2 \Rightarrow C = -\frac{27}{4}$. Thus, $f(x) = \frac{1}{4}x^5 + \frac{1}{2}x^4 + x^2 - \frac{27}{4}x + 3$.

**73.** $v(t) = s'(t) = 2t - \dfrac{1}{1+t^2} \Rightarrow s(t) = t^2 - \tan^{-1} t + C$.

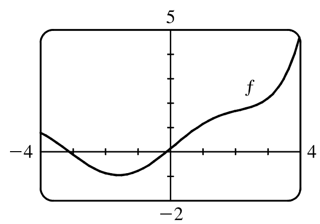
$s(0) = 0 - 0 + C = C$ and $s(0) = 1 \Rightarrow C = 1$, so $s(t) = t^2 - \tan^{-1} t + 1$.

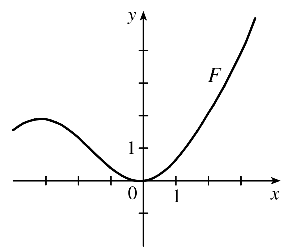
**74.** $a(t) = v'(t) = \sin t + 3\cos t \quad \Rightarrow \quad v(t) = -\cos t + 3\sin t + C$.

$v(0) = -1 + 0 + C$ and $v(0) = 2 \quad \Rightarrow \quad C = 3$, so $v(t) = -\cos t + 3\sin t + 3$ and $s(t) = -\sin t - 3\cos t + 3t + D$.

$s(0) = -3 + D$ and $s(0) = 0 \quad \Rightarrow \quad D = 3$, and $s(t) = -\sin t - 3\cos t + 3t + 3$.

**75.** (a) Since $f$ is 0 just to the left of the $y$-axis, we must have a minimum of $F$ at the same place since we are increasing through $(0, 0)$ on $F$. There must be a local maximum to the left of $x = -3$, since $f$ changes from positive to negative there.



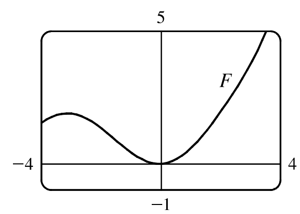

(b) $f(x) = 0.1e^x + \sin x \quad \Rightarrow$

$F(x) = 0.1e^x - \cos x + C$. $F(0) = 0 \quad \Rightarrow$

$0.1 - 1 + C = 0 \quad \Rightarrow \quad C = 0.9$, so

$F(x) = 0.1e^x - \cos x + 0.9$.

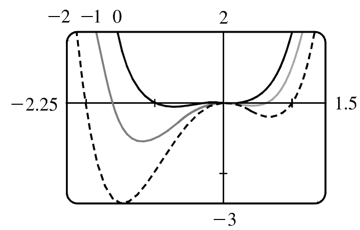
(c)

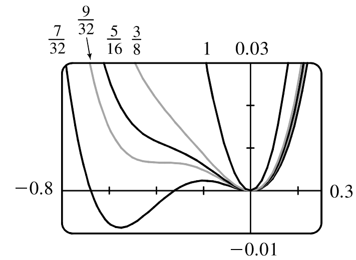



**76.** $f(x) = x^4 + x^3 + cx^2 \quad \Rightarrow \quad f'(x) = 4x^3 + 3x^2 + 2cx$. This is 0 when $x\left(4x^2 + 3x + 2c\right) = 0 \quad \Leftrightarrow \quad x = 0$ or $4x^2 + 3x + 2c = 0$. Using the quadratic formula, we find that the roots of this last equation are $x = \dfrac{-3 \pm \sqrt{9 - 32c}}{8}$. Now if $9 - 32c < 0 \quad \Leftrightarrow \quad c > \frac{9}{32}$, then $(0, 0)$ is the only critical point, a minimum. If $c = \frac{9}{32}$, then there are two critical points (a minimum at $x = 0$, and a horizontal tangent with no maximum or minimum at $x = -\frac{3}{8}$) and if $c < \frac{9}{32}$, then there are three critical points except when $c = 0$, in which case the root with the $+$ sign coincides with the critical point at $x = 0$. For $0 < c < \frac{9}{32}$, there is a minimum at $x = -\dfrac{3}{8} - \dfrac{\sqrt{9 - 32c}}{8}$, a maximum at $x = -\dfrac{3}{8} + \dfrac{\sqrt{9 - 32c}}{8}$, and a minimum at $x = 0$. For $c = 0$, there is a minimum at $x = -\frac{3}{4}$ and a horizontal tangent with no extremum at $x = 0$, and for $c < 0$, there is a maximum at $x = 0$, and there are minima at $x = -\dfrac{3}{8} \pm \dfrac{\sqrt{9 - 32c}}{8}$. Now we calculate $f''(x) = 12x^2 + 6x + 2c$. The roots of this equation are $x = \dfrac{-6 \pm \sqrt{36 - 4 \cdot 12 \cdot 2c}}{24}$. So if $36 - 96c \leq 0 \quad \Leftrightarrow \quad c \geq \frac{3}{8}$, then there is no inflection point. If $c < \frac{3}{8}$, then there are two inflection points at $x = -\dfrac{1}{4} \pm \dfrac{\sqrt{9 - 24c}}{12}$.

[continued]

| Value of $c$ | No. of CP | No. of IP |
|---|---|---|
| $c < 0$ | 3 | 2 |
| $c = 0$ | 2 | 2 |
| $0 < c < \frac{9}{32}$ | 3 | 2 |
| $c = \frac{9}{32}$ | 2 | 2 |
| $\frac{9}{32} < c < \frac{3}{8}$ | 1 | 2 |
| $c \geq \frac{3}{8}$ | 1 | 0 |

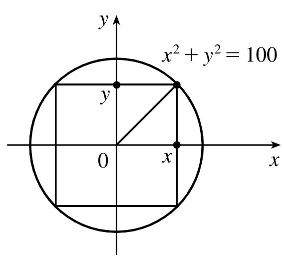



**77.** Choosing the positive direction to be upward, we have $a(t) = -9.8 \quad\Rightarrow\quad v(t) = -9.8t + v_0$, but $v(0) = 0 = v_0 \quad\Rightarrow$ $v(t) = -9.8t = s'(t) \quad\Rightarrow\quad s(t) = -4.9t^2 + s_0$, but $s(0) = s_0 = 500 \quad\Rightarrow\quad s(t) = -4.9t^2 + 500$. When $s = 0$, $-4.9t^2 + 500 = 0 \quad\Rightarrow\quad t_1 = \sqrt{\frac{500}{4.9}} \approx 10.1 \quad\Rightarrow\quad v(t_1) = -9.8\sqrt{\frac{500}{4.9}} \approx -98.995\text{ m/s}$. Since the canister has been designed to withstand an impact velocity of 100 m/s, the canister will *not burst*.

**78.** Let $s_A(t)$ and $s_B(t)$ be the position functions for cars $A$ and $B$ and let $f(t) = s_A(t) - s(t)$. Since $A$ passed $B$ twice, there must be three values of $t$ such that $f(t) = 0$. Then by three applications of Rolle's Theorem (see Exercise 4.2.22), there is a number $c$ such that $f''(c) = 0$. So $s_A''(c) = s_B''(c)$; that is, $A$ and $B$ had equal accelerations at $t = c$. We assume that $f$ is continuous on $[0, T]$ and twice differentiable on $(0, T)$, where $T$ is the total time of the race.

**79.** (a)




The cross-sectional area of the rectangular beam is

$A = 2x \cdot 2y = 4xy = 4x\sqrt{100 - x^2}$, $0 \leq x \leq 10$, so

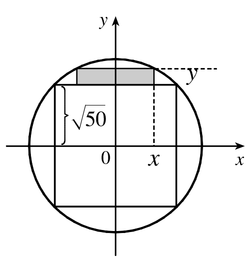
$$\frac{dA}{dx} = 4x\left(\tfrac{1}{2}\right)(100 - x^2)^{-1/2}(-2x) + (100 - x^2)^{1/2} \cdot 4$$

$$= \frac{-4x^2}{(100 - x^2)^{1/2}} + 4(100 - x^2)^{1/2} = \frac{4[-x^2 + (100 - x^2)]}{(100 - x^2)^{1/2}}.$$

$\dfrac{dA}{dx} = 0$ when $-x^2 + (100 - x^2) = 0 \quad\Rightarrow\quad x^2 = 50 \quad\Rightarrow\quad x = \sqrt{50} \approx 7.07 \quad\Rightarrow\quad y = \sqrt{100 - \left(\sqrt{50}\right)^2} = \sqrt{50}$.

Since $A(0) = A(10) = 0$, the rectangle of maximum area is a square.

(b)




The cross-sectional area of each rectangular plank (shaded in the figure) is

$A = 2x\left(y - \sqrt{50}\right) = 2x\left[\sqrt{100 - x^2} - \sqrt{50}\right]$, $0 \leq x \leq \sqrt{50}$, so

$$\frac{dA}{dx} = 2\left(\sqrt{100 - x^2} - \sqrt{50}\right) + 2x\left(\tfrac{1}{2}\right)(100 - x^2)^{-1/2}(-2x)$$

$$= 2(100 - x^2)^{1/2} - 2\sqrt{50} - \frac{2x^2}{(100 - x^2)^{1/2}}$$

Set $\dfrac{dA}{dx} = 0$: $\quad (100 - x^2) - \sqrt{50}\,(100 - x^2)^{1/2} - x^2 = 0 \quad\Rightarrow\quad 100 - 2x^2 = \sqrt{50}\,(100 - x^2)^{1/2} \quad\Rightarrow$

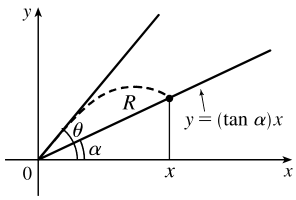
$10{,}000 - 400x^2 + 4x^4 = 50(100 - x^2) \quad\Rightarrow\quad 4x^4 - 350x^2 + 5000 = 0 \quad\Rightarrow\quad 2x^4 - 175x^2 + 2500 = 0 \quad\Rightarrow$

$x^2 = \dfrac{175 \pm \sqrt{10{,}625}}{4} \approx 69.52$ or $17.98 \quad\Rightarrow\quad x \approx 8.34$ or $4.24$. But $8.34 > \sqrt{50}$, so $x_1 \approx 4.24 \quad\Rightarrow$

$y - \sqrt{50} = \sqrt{100 - x_1^2} - \sqrt{50} \approx 1.99$. Each plank should have dimensions about $8\frac{1}{2}$ inches by 2 inches.

(c) From the figure in part (a), the width is $2x$ and the depth is $2y$, so the strength is $S = k(2x)(2y)^2 = 8kxy^2 = 8kx(100 - x^2) = 800kx - 8kx^3$, $0 \le x \le 10$. $dS/dx = 800k - 24kx^2 = 0$ when $24kx^2 = 800k \;\Rightarrow\; x^2 = \frac{100}{3} \;\Rightarrow\; x = \frac{10}{\sqrt{3}} \;\Rightarrow\; y = \sqrt{\frac{200}{3}} = \frac{10\sqrt{2}}{\sqrt{3}} = \sqrt{2}\,x$. Since $S(0) = S(10) = 0$, the maximum strength occurs when $x = \frac{10}{\sqrt{3}}$. The dimensions should be $\frac{20}{\sqrt{3}} \approx 11.55$ inches by $\frac{20\sqrt{2}}{\sqrt{3}} \approx 16.33$ inches.

**80.** (a)




$y = (\tan\theta)x - \dfrac{g}{2v^2\cos^2\theta}x^2$. The parabola intersects the line when

$$(\tan\alpha)x = (\tan\theta)x - \frac{g}{2v^2\cos^2\theta}x^2 \quad\Rightarrow$$

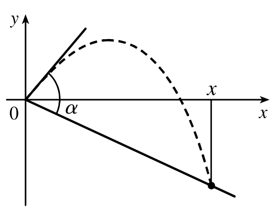
$$x = \frac{(\tan\theta - \tan\alpha)2v^2\cos^2\theta}{g} \quad\Rightarrow$$

$$R(\theta) = \frac{x}{\cos\alpha} = \left(\frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha}\right)\frac{2v^2\cos^2\theta}{g\cos\alpha} = \left(\frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha}\right)(\cos\theta\,\cos\alpha)\frac{2v^2\cos\theta}{g\cos^2\alpha}$$
$$= (\sin\theta\,\cos\alpha - \sin\alpha\,\cos\theta)\,\frac{2v^2\cos\theta}{g\cos^2\alpha} = \sin(\theta - \alpha)\,\frac{2v^2\cos\theta}{g\cos^2\alpha}$$

(b) $$R'(\theta) = \frac{2v^2}{g\cos^2\alpha}\left[\cos\theta\cdot\cos(\theta-\alpha) + \sin(\theta-\alpha)(-\sin\theta)\right] = \frac{2v^2}{g\cos^2\alpha}\cos[\theta + (\theta - \alpha)]$$
$$= \frac{2v^2}{g\cos^2\alpha}\cos(2\theta - \alpha) = 0$$

when $\cos(2\theta - \alpha) = 0 \;\Rightarrow\; 2\theta - \alpha = \frac{\pi}{2} \;\Rightarrow\; \theta = \dfrac{\pi/2 + \alpha}{2} = \dfrac{\pi}{4} + \dfrac{\alpha}{2}$. The First Derivative Test shows that this gives a maximum value for $R(\theta)$. [This could be done without calculus by applying the formula for $\sin x\,\cos y$ to $R(\theta)$.]

(c)




Replacing $\alpha$ by $-\alpha$ in part (a), we get $R(\theta) = \dfrac{2v^2\cos\theta\,\sin(\theta + \alpha)}{g\cos^2\alpha}$. Proceeding as in part (b), or simply by replacing $\alpha$ by $-\alpha$ in the result of part (b), we see that $R(\theta)$ is maximized when $\theta = \dfrac{\pi}{4} - \dfrac{\alpha}{2}$.

**81.** $$\lim_{E\to 0^+} P(E) = \lim_{E\to 0^+}\left(\frac{e^E + e^{-E}}{e^E - e^{-E}} - \frac{1}{E}\right)$$
$$= \lim_{E\to 0^+}\frac{E\left(e^E + e^{-E}\right) - 1\left(e^E - e^{-E}\right)}{\left(e^E - e^{-E}\right)E} = \lim_{E\to 0^+}\frac{Ee^E + Ee^{-E} - e^E + e^{-E}}{Ee^E - Ee^{-E}} \qquad \left[\text{form is } \tfrac{0}{0}\right]$$
$$\overset{\text{H}}{=} \lim_{E\to 0^+}\frac{Ee^E + e^E\cdot 1 + E\left(-e^{-E}\right) + e^{-E}\cdot 1 - e^E + \left(-e^{-E}\right)}{Ee^E + e^E\cdot 1 - \left[E(-e^{-E}) + e^{-E}\cdot 1\right]}$$
$$= \lim_{E\to 0^+}\frac{Ee^E - Ee^{-E}}{Ee^E + e^E + Ee^{-E} - e^{-E}} = \lim_{E\to 0^+}\frac{e^E - e^{-E}}{e^E + \dfrac{e^E}{E} + e^{-E} - \dfrac{e^{-E}}{E}} \qquad [\text{divide by } E]$$
$$= \frac{0}{2 + L}, \quad \text{where } L = \lim_{E\to 0^+}\frac{e^E - e^{-E}}{E} \quad \left[\text{form is } \tfrac{0}{0}\right] \quad \overset{\text{H}}{=} \lim_{E\to 0^+}\frac{e^E + e^{-E}}{1} = \frac{1+1}{1} = 2$$

Thus, $\displaystyle\lim_{E\to 0^+} P(E) = \frac{0}{2+2} = 0$.

**82.** $\lim_{c\to 0^+} s(t) = \lim_{c\to 0^+}\left(\dfrac{m}{c}\ln\cosh\sqrt{\dfrac{gc}{mt}}\right) = m\lim_{c\to 0^+}\dfrac{\ln\cosh\sqrt{ac}}{c}$ [let $a = g/(mt)$]

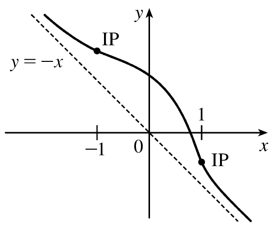
$$\overset{\text{H}}{=} m\lim_{c\to 0^+}\frac{\dfrac{1}{\cosh\sqrt{ac}}\left(\sinh\sqrt{ac}\right)\left(\dfrac{\sqrt{a}}{2\sqrt{c}}\right)}{1} = \frac{m\sqrt{a}}{2}\lim_{c\to 0^+}\frac{\tanh\sqrt{ac}}{\sqrt{c}}$$

$$\overset{\text{H}}{=} \frac{m\sqrt{a}}{2}\lim_{c\to 0^+}\frac{\operatorname{sech}^2\sqrt{ac}\left[\sqrt{a}/\left(2\sqrt{c}\right)\right]}{1/\left(2\sqrt{c}\right)} = \frac{ma}{2}\lim_{c\to 0^+}\operatorname{sech}^2\sqrt{ac} = \frac{ma}{2}(1)^2 = \frac{mg}{2mt} = \frac{g}{2t}$$

**83.** We first show that $\dfrac{x}{1+x^2} < \tan^{-1}x$ for $x > 0$. Let $f(x) = \tan^{-1}x - \dfrac{x}{1+x^2}$. Then

$f'(x) = \dfrac{1}{1+x^2} - \dfrac{1(1+x^2) - x(2x)}{(1+x^2)^2} = \dfrac{(1+x^2)-(1-x^2)}{(1+x^2)^2} = \dfrac{2x^2}{(1+x^2)^2} > 0$ for $x > 0$. So $f(x)$ is increasing on $(0,\infty)$. Hence, $0 < x \;\Rightarrow\; 0 = f(0) < f(x) = \tan^{-1}x - \dfrac{x}{1+x^2}$. So $\dfrac{x}{1+x^2} < \tan^{-1}x$ for $0 < x$. We next show that $\tan^{-1}x < x$ for $x > 0$. Let $h(x) = x - \tan^{-1}x$. Then $h'(x) = 1 - \dfrac{1}{1+x^2} = \dfrac{x^2}{1+x^2} > 0$. Hence, $h(x)$ is increasing on $(0,\infty)$. So for $0 < x$, $0 = h(0) < h(x) = x - \tan^{-1}x$. Hence, $\tan^{-1}x < x$ for $x > 0$, and we conclude that $\dfrac{x}{1+x^2} < \tan^{-1}x < x$ for $x > 0$.

**84.** If $f'(x) < 0$ for all $x$, $f''(x) > 0$ for $|x| > 1$, $f''(x) < 0$ for $|x| < 1$, and $\lim_{x\to\pm\infty}[f(x)+x] = 0$, then $f$ is decreasing everywhere, concave up on $(-\infty,-1)$ and $(1,\infty)$, concave down on $(-1,1)$, and approaches the line $y = -x$ as $x \to \pm\infty$. An example of such a graph is sketched.

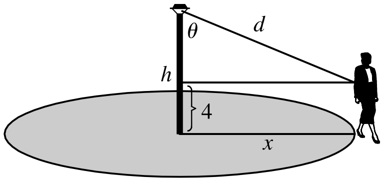



**85.** (a) $I = \dfrac{k\cos\theta}{d^2} = \dfrac{k(h/d)}{d^2} = k\dfrac{h}{d^3} = k\dfrac{h}{\left(\sqrt{40^2+h^2}\right)^3} = k\dfrac{h}{(1600+h^2)^{3/2}} \quad\Rightarrow$

$$\frac{dI}{dh} = k\frac{(1600+h^2)^{3/2} - h\frac{3}{2}(1600+h^2)^{1/2}\cdot 2h}{[(1600+h^2)^{3/2}]^2} = \frac{k(1600+h^2)^{1/2}(1600+h^2-3h^2)}{(1600+h^2)^3}$$
$$= \frac{k(1600-2h^2)}{(1600+h^2)^{5/2}} \qquad [k \text{ is the constant of proportionality}]$$

Set $dI/dh = 0$: $1600 - 2h^2 = 0 \;\Rightarrow\; h^2 = 800 \;\Rightarrow\; h = \sqrt{800} = 20\sqrt{2}$. By the First Derivative Test, $I$ has a local maximum at $h = 20\sqrt{2} \approx 28$ ft.

(b)




$\dfrac{dx}{dt} = 4$ ft/s

$$I = \frac{k\cos\theta}{d^2} = \frac{k[(h-4)/d]}{d^2} = \frac{k(h-4)}{d^3}$$
$$= \frac{k(h-4)}{[(h-4)^2+x^2]^{3/2}} = k(h-4)\left[(h-4)^2+x^2\right]^{-3/2}$$

[continued]

$$\frac{dI}{dt} = \frac{dI}{dx}\cdot\frac{dx}{dt} = k(h-4)\left(-\tfrac{3}{2}\right)\left[(h-4)^2+x^2\right]^{-5/2}\cdot 2x\cdot\frac{dx}{dt}$$

$$= k(h-4)(-3x)\left[(h-4)^2+x^2\right]^{-5/2}\cdot 4 = \frac{-12xk(h-4)}{[(h-4)^2+x^2]^{5/2}}$$

$$\left.\frac{dI}{dt}\right|_{x=40} = -\frac{480k(h-4)}{[(h-4)^2+1600]^{5/2}}$$

**86.** (a) $V'(t)$ is the rate of change of the volume of the water with respect to time. $H'(t)$ is the rate of change of the height of the water with respect to time. Since the volume and the height are increasing, $V'(t)$ and $H'(t)$ are positive.

(b) $V'(t)$ is constant, so $V''(t)$ is zero (the slope of a constant function is 0).

(c) At first, the height $H$ of the water increases quickly because the tank is narrow. But as the sphere widens, the rate of increase of the height slows down, reaching a minimum at $t = t_2$. Thus, the height is increasing at a decreasing rate on $(0, t_2)$, so its graph is concave downward and $H''(t_1) < 0$. As the sphere narrows for $t > t_2$, the rate of increase of the height begins to increase, and the graph of $H$ is concave upward. Therefore, $H''(t_2) = 0$ and $H''(t_3) > 0$.

# PROBLEMS PLUS

**1.** Let $y = f(x) = e^{-x^2}$. The area of the rectangle under the curve from $-x$ to $x$ is $A(x) = 2xe^{-x^2}$ where $x \geq 0$. We maximize $A(x)$: $A'(x) = 2e^{-x^2} - 4x^2e^{-x^2} = 2e^{-x^2}\left(1 - 2x^2\right) = 0 \quad \Rightarrow \quad x = \frac{1}{\sqrt{2}}$. This gives a maximum since $A'(x) > 0$ for $0 \leq x < \frac{1}{\sqrt{2}}$ and $A'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$. We next determine the points of inflection of $f(x)$. Notice that $f'(x) = -2xe^{-x^2} = -A(x)$. So $f''(x) = -A'(x)$ and hence, $f''(x) < 0$ for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and $f''(x) > 0$ for $x < -\frac{1}{\sqrt{2}}$ and $x > \frac{1}{\sqrt{2}}$. So $f(x)$ changes concavity at $x = \pm\frac{1}{\sqrt{2}}$, and the two vertices of the rectangle of largest area are at the inflection points.

**2.** Let $f(x) = \sin x - \cos x$ on $[0, 2\pi]$ since $f$ has period $2\pi$. $f'(x) = \cos x + \sin x = 0 \quad \Leftrightarrow \quad \cos x = -\sin x \quad \Leftrightarrow$ $\tan x = -1 \quad \Leftrightarrow \quad x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. Evaluating $f$ at its critical numbers and endpoints, we get $f(0) = -1$, $f\left(\frac{3\pi}{4}\right) = \sqrt{2}$, $f\left(\frac{7\pi}{4}\right) = -\sqrt{2}$, and $f(2\pi) = -1$. So $f$ has absolute maximum value $\sqrt{2}$ and absolute minimum value $-\sqrt{2}$. Thus, $-\sqrt{2} \leq \sin x - \cos x \leq \sqrt{2} \quad \Rightarrow \quad |\sin x - \cos x| \leq \sqrt{2}$.

**3.** $f(x)$ has the form $e^{g(x)}$, so it will have an absolute maximum (minimum) where $g$ has an absolute maximum (minimum).

$$g(x) = 10|x-2| - x^2 = \begin{cases} 10(x-2) - x^2 & \text{if } x - 2 > 0 \\ 10[-(x-2)] - x^2 & \text{if } x - 2 < 0 \end{cases} = \begin{cases} -x^2 + 10x - 20 & \text{if } x > 2 \\ -x^2 - 10x + 20 & \text{if } x < 2 \end{cases} \quad \Rightarrow$$

$$g'(x) = \begin{cases} -2x + 10 & \text{if } x > 2 \\ -2x - 10 & \text{if } x < 2 \end{cases}$$

$g'(x) = 0$ if $x = -5$ or $x = 5$, and $g'(2)$ does not exist, so the critical numbers of $g$ are $-5$, 2, and 5. Since $g''(x) = -2$ for all $x \neq 2$, $g$ is concave downward on $(-\infty, 2)$ and $(2, \infty)$, and $g$ will attain its absolute maximum at one of the critical numbers. Since $g(-5) = 45$, $g(2) = -4$, and $g(5) = 5$, we see that $f(-5) = e^{45}$ is the absolute maximum value of $f$. Also, $\lim\limits_{x\to\infty} g(x) = -\infty$, so $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} e^{g(x)} = 0$. But $f(x) > 0$ for all $x$, so there is no absolute minimum value of $f$.

**4.** $x^2y^2\left(4 - x^2\right)\left(4 - y^2\right) = x^2\left(4 - x^2\right)y^2\left(4 - y^2\right) = f(x)f(y)$, where $f(t) = t^2\left(4 - t^2\right)$. We will show that $0 \leq f(t) \leq 4$ for $|t| \leq 2$, which gives $0 \leq f(x)f(y) \leq 16$ for $|x| \leq 2$ and $|y| \leq 2$.

$f(t) = 4t^2 - t^4 \quad \Rightarrow \quad f'(t) = 8t - 4t^3 = 4t\left(2 - t^2\right) = 0 \quad \Rightarrow \quad t = 0$ or $\pm\sqrt{2}$.

$f(0) = 0$, $f\left(\pm\sqrt{2}\right) = 2(4 - 2) = 4$, and $f(2) = 0$. So 0 is the absolute minimum value of $f(t)$ on $[-2, 2]$ and 4 is the absolute maximum value of $f(t)$ on $[-2, 2]$. We conclude that $0 \leq f(t) \leq 4$ for $|t| \leq 2$ and hence, $0 \leq f(x)f(y) \leq 4^2$ or $0 \leq x^2\left(4 - x^2\right)y^2\left(4 - y^2\right) \leq 16$.

**5.** $y = \dfrac{\sin x}{x} \quad \Rightarrow \quad y' = \dfrac{x\cos x - \sin x}{x^2} \quad \Rightarrow \quad y'' = \dfrac{-x^2\sin x - 2x\cos x + 2\sin x}{x^3}$. If $(x, y)$ is an inflection point,

then $y'' = 0 \quad \Rightarrow \quad (2 - x^2)\sin x = 2x\cos x \quad \Rightarrow \quad (2 - x^2)^2\sin^2 x = 4x^2\cos^2 x \quad \Rightarrow$

$(2 - x^2)^2\sin^2 x = 4x^2(1 - \sin^2 x) \quad \Rightarrow \quad (4 - 4x^2 + x^4)\sin^2 x = 4x^2 - 4x^2\sin^2 x \quad \Rightarrow$

$(4 + x^4)\sin^2 x = 4x^2 \quad \Rightarrow \quad (x^4 + 4)\dfrac{\sin^2 x}{x^2} = 4 \quad \Rightarrow \quad y^2(x^4 + 4) = 4$ since $y = \dfrac{\sin x}{x}$.

**6.** Let $P\left(a, 1 - a^2\right)$ be the point of contact. The equation of the tangent line at $P$ is $y - \left(1 - a^2\right) = (-2a)(x - a) \quad \Rightarrow$

$y - 1 + a^2 = -2ax + 2a^2 \quad \Rightarrow \quad y = -2ax + a^2 + 1$. To find the $x$-intercept, put $y = 0$: $2ax = a^2 + 1 \quad \Rightarrow$

$x = \dfrac{a^2 + 1}{2a}$. To find the $y$-intercept, put $x = 0$: $y = a^2 + 1$. Therefore, the area of the triangle is

$\dfrac{1}{2}\left(\dfrac{a^2 + 1}{2a}\right)\left(a^2 + 1\right) = \dfrac{\left(a^2 + 1\right)^2}{4a}$. Therefore, we minimize the function $A(a) = \dfrac{\left(a^2 + 1\right)^2}{4a}$, $a > 0$.

$A'(a) = \dfrac{(4a)2\left(a^2 + 1\right)(2a) - \left(a^2 + 1\right)^2(4)}{16a^2} = \dfrac{\left(a^2 + 1\right)\left[4a^2 - \left(a^2 + 1\right)\right]}{4a^2} = \dfrac{\left(a^2 + 1\right)\left(3a^2 - 1\right)}{4a^2}$.

$A'(a) = 0$ when $3a^2 - 1 = 0 \quad \Rightarrow \quad a = \frac{1}{\sqrt{3}}$. $A'(a) < 0$ for $a < \frac{1}{\sqrt{3}}$, $A'(a) > 0$ for $a > \frac{1}{\sqrt{3}}$. So by the First Derivative

Test, there is an absolute minimum when $a = \frac{1}{\sqrt{3}}$. The required point is $\left(\frac{1}{\sqrt{3}}, \frac{2}{3}\right)$ and the corresponding minimum area

is $A\left(\frac{1}{\sqrt{3}}\right) = \frac{4\sqrt{3}}{9}$.

**7.** Let $L = \displaystyle\lim_{x \to 0} \frac{ax^2 + \sin bx + \sin cx + \sin dx}{3x^2 + 5x^4 + 7x^6}$. Now $L$ has the indeterminate form of type $\frac{0}{0}$, so we can apply l'Hospital's

Rule. $L = \displaystyle\lim_{x \to 0} \frac{2ax + b\cos bx + c\cos cx + d\cos dx}{6x + 20x^3 + 42x^5}$. The denominator approaches 0 as $x \to 0$, so the numerator must also

approach 0 (because the limit exists). But the numerator approaches $0 + b + c + d$, so $b + c + d = 0$. Apply l'Hospital's Rule

again. $L = \displaystyle\lim_{x \to 0} \frac{2a - b^2\sin bx - c^2\sin cx - d^2\sin dx}{6 + 60x^2 + 210x^4} = \frac{2a - 0}{6 + 0} = \frac{2a}{6}$, which must equal 8.

$\dfrac{2a}{6} = 8 \quad \Rightarrow \quad a = 24$. Thus, $a + b + c + d = a + (b + c + d) = 24 + 0 = 24$.

**8.** We first present some preliminary results that we will invoke when calculating the limit.

**(1)** If $y = (1 + ax)^x$, then $\ln y = x\ln(1 + ax)$, and $\displaystyle\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} x\ln(1 + ax) = 0$. Thus, $\displaystyle\lim_{x \to 0^+}(1 + ax)^x = e^0 = 1$.

**(2)** If $y = (1 + ax)^x$, then $\ln y = x\ln(1 + ax)$, and implicitly differentiating gives us $\dfrac{y'}{y} = x \cdot \dfrac{a}{1 + ax} + \ln(1 + ax) \quad \Rightarrow$

$y' = y\left[\dfrac{ax}{1 + ax} + \ln(1 + ax)\right]$. Thus, $y = (1 + ax)^x \quad \Rightarrow \quad y' = (1 + ax)^x\left[\dfrac{ax}{1 + ax} + \ln(1 + ax)\right]$.

**(3)** If $y = \dfrac{ax}{1 + ax}$, then $y' = \dfrac{(1 + ax)a - ax(a)}{(1 + ax)^2} = \dfrac{a + a^2x - a^2x}{(1 + ax)^2} = \dfrac{a}{(1 + ax)^2}$.

$$\begin{aligned}
\lim_{x\to\infty}\frac{(x+2)^{1/x}-x^{1/x}}{(x+3)^{1/x}-x^{1/x}} &= \lim_{x\to\infty}\frac{x^{1/x}[(1+2/x)^{1/x}-1]}{x^{1/x}[(1+3/x)^{1/x}-1]} && \text{[factor out } x^{1/x}\text{]}\\
&= \lim_{x\to\infty}\frac{(1+2/x)^{1/x}-1}{(1+3/x)^{1/x}-1}\\
&= \lim_{t\to 0^+}\frac{(1+2t)^t-1}{(1+3t)^t-1} && \text{[let } t=1/x\text{, form } 0/0 \text{ by } \mathbf{(1)}\text{]}\\
&\overset{\text{H}}{=} \lim_{t\to 0^+}\frac{(1+2t)^t\left[\dfrac{2t}{1+2t}+\ln(1+2t)\right]}{(1+3t)^t\left[\dfrac{3t}{1+3t}+\ln(1+3t)\right]} && \text{[by } \mathbf{(2)}\text{]}\\
&= \lim_{t\to 0^+}\frac{(1+2t)^t}{(1+3t)^t}\cdot\lim_{t\to 0^+}\frac{\dfrac{2t}{1+2t}+\ln(1+2t)}{\dfrac{3t}{1+3t}+\ln(1+3t)}\\
&= \frac{1}{1}\cdot\lim_{t\to 0^+}\frac{\dfrac{2t}{1+2t}+\ln(1+2t)}{\dfrac{3t}{1+3t}+\ln(1+3t)} && \text{[by } \mathbf{(1)}\text{, now form } 0/0\text{]}\\
&\overset{\text{H}}{=} \lim_{t\to 0^+}\frac{\dfrac{2}{(1+2t)^2}+\dfrac{2}{1+2t}}{\dfrac{3}{(1+3t)^2}+\dfrac{3}{1+3t}} && \text{[by } \mathbf{(3)}\text{]}\\
&= \frac{2+2}{3+3}=\frac{4}{6}=\frac{2}{3}
\end{aligned}$$

**9.** Differentiating $x^2+xy+y^2=12$ implicitly with respect to $x$ gives $2x+y+x\,\dfrac{dy}{dx}+2y\,\dfrac{dy}{dx}=0$, so $\dfrac{dy}{dx}=-\dfrac{2x+y}{x+2y}$.

At a highest or lowest point, $\dfrac{dy}{dx}=0 \;\Leftrightarrow\; y=-2x$. Substituting $-2x$ for $y$ in the original equation gives $x^2+x(-2x)+(-2x)^2=12$, so $3x^2=12$ and $x=\pm 2$. If $x=2$, then $y=-2x=-4$, and if $x=-2$ then $y=4$. Thus, the highest and lowest points are $(-2,4)$ and $(2,-4)$.

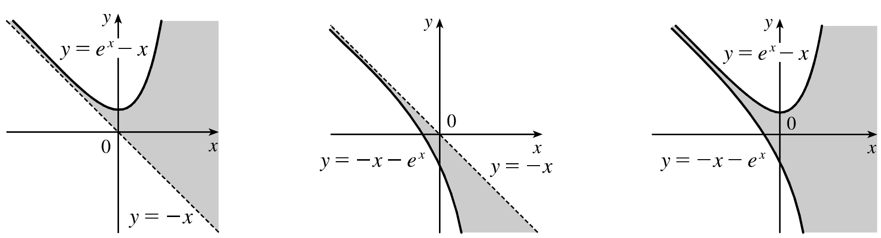
**10.** *Case (i) (first graph):* For $x+y\geq 0$, that is, $y\geq -x$, $|x+y|=x+y\leq e^x \;\Rightarrow\; y\leq e^x-x$.

Note that $y=e^x-x$ is always above the line $y=-x$ and that $y=-x$ is a slant asymptote.

*Case (ii) (second graph):* For $x+y<0$, that is, $y<-x$, $|x+y|=-x-y\leq e^x \;\Rightarrow\; y\geq -x-e^x$.

Note that $-x-e^x$ is always below the line $y=-x$ and $y=-x$ is a slant asymptote.

Putting the two pieces together gives the third graph.



**11.** (a) $y = x^2 \quad \Rightarrow \quad y' = 2x$, so the slope of the tangent line at $P(a, a^2)$ is $2a$ and the slope of the normal line is $-\dfrac{1}{2a}$ for $a \neq 0$. An equation of the normal line is $y - a^2 = -\dfrac{1}{2a}(x - a)$. Substitute $x^2$ for $y$ to find the $x$-coordinates of the two points of intersection of the parabola and the normal line. $x^2 - a^2 = -\dfrac{x}{2a} + \dfrac{1}{2} \quad \Leftrightarrow \quad x^2 + \left(\dfrac{1}{2a}\right)x - \dfrac{1}{2} - a^2 = 0$. We know that $a$ is a root of this quadratic equation, so $x - a$ is a factor, and we have $(x - a)\left(x + \dfrac{1}{2a} + a\right) = 0$, and hence, $x = -a - \dfrac{1}{2a}$ is the $x$-coordinate of the point $Q$. We want to minimize the $y$-coordinate of $Q$, which is

$\left(-a - \dfrac{1}{2a}\right)^2 = a^2 + 1 + \dfrac{1}{4a^2} = y(a)$. Now $y'(a) = 2a - \dfrac{1}{2a^3} = \dfrac{4a^4 - 1}{2a^3} = \dfrac{(2a^2 + 1)(2a^2 - 1)}{2a^3} = 0 \quad \Rightarrow$

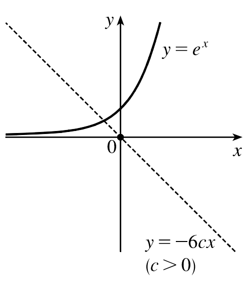
$a = \dfrac{1}{\sqrt{2}}$ for $a > 0$. Since $y''(a) = 2 + \dfrac{3}{2a^4} > 0$, we see that $a = \dfrac{1}{\sqrt{2}}$ gives us the minimum value of the $y$-coordinate of $Q$.

(b) The square $S$ of the distance from $P(a, a^2)$ to $Q\left(-a - \dfrac{1}{2a}, \left(-a - \dfrac{1}{2a}\right)^2\right)$ is given by

$$\begin{aligned}
S &= \left(-a - \frac{1}{2a} - a\right)^2 + \left[\left(-a - \frac{1}{2a}\right)^2 - a^2\right]^2 = \left(-2a - \frac{1}{2a}\right)^2 + \left[\left(a^2 + 1 + \frac{1}{4a^2}\right) - a^2\right]^2 \\
&= \left(4a^2 + 2 + \frac{1}{4a^2}\right) + \left(1 + \frac{1}{4a^2}\right)^2 = \left(4a^2 + 2 + \frac{1}{4a^2}\right) + 1 + \frac{2}{4a^2} + \frac{1}{16a^4} \\
&= 4a^2 + 3 + \frac{3}{4a^2} + \frac{1}{16a^4}
\end{aligned}$$

$S' = 8a - \dfrac{6}{4a^3} - \dfrac{4}{16a^5} = 8a - \dfrac{3}{2a^3} - \dfrac{1}{4a^5} = \dfrac{32a^6 - 6a^2 - 1}{4a^5} = \dfrac{(2a^2 - 1)(4a^2 + 1)^2}{4a^5}$. The only real positive zero of the equation $S' = 0$ is $a = \dfrac{1}{\sqrt{2}}$. Since $S'' = 8 + \dfrac{9}{2a^4} + \dfrac{5}{4a^6} > 0$, $a = \dfrac{1}{\sqrt{2}}$ corresponds to the shortest possible length of the line segment $PQ$.

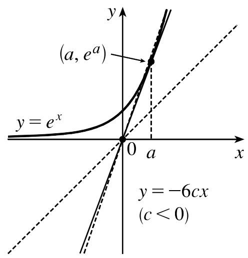
**12.**




$y = cx^3 + e^x \quad \Rightarrow \quad y' = 3cx^2 + e^x \quad \Rightarrow \quad y'' = 6cx + e^x$. The curve will have inflection points when $y''$ changes sign. $y'' = 0 \quad \Rightarrow \quad -6cx = e^x$, so $y''$ will change sign when the line $y = -6cx$ intersects the curve $y = e^x$ (but is not tangent to it). Note that if $c = 0$, the curve is just $y = e^x$, which has no inflection point.

The first figure shows that for $c > 0$, $y = -6cx$ will intersect $y = e^x$ once, so $y = cx^3 + e^x$ will have one inflection point.

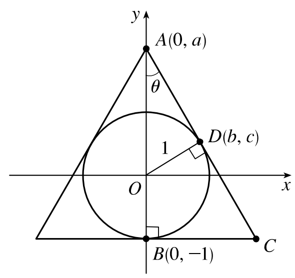
[continued]




The second figure shows that for $c < 0$, the line $y = -6cx$ can intersect the curve $y = e^x$ in two points (two inflection points), be tangent to it (no inflection point), or not intersect it (no inflection point). The tangent line at $(a, e^a)$ has slope $e^a$, but from the diagram we see that the slope is $\dfrac{e^a}{a}$. So $\dfrac{e^a}{a} = e^a \quad \Rightarrow \quad a = 1$. Thus, the slope is $e$. The line $y = -6cx$ must have slope greater than $e$, so $-6c > e \quad \Rightarrow \quad c < -e/6$.

Therefore, the curve $y = cx^3 + e^x$ will have one inflection point if $c > 0$ and two inflection points if $c < -e/6$.

**13.**




$\overline{AC}$ is tangent to the unit circle at $D$. To find the slope of $\overline{AC}$ at $D$, use implicit differentiation. $x^2 + y^2 = 1 \quad \Rightarrow \quad 2x + 2y\,y' = 0 \quad \Rightarrow \quad y\,y' = -x \quad \Rightarrow \quad y' = -\dfrac{x}{y}$.

Thus, the tangent line at $D(b, c)$ has equation $y = -\dfrac{b}{c}x + a$. At $D$, $x = b$ and $y = c$, so $c = -\dfrac{b}{c}(b) + a \quad \Rightarrow \quad a = c + \dfrac{b^2}{c} = \dfrac{c^2 + b^2}{c} = \dfrac{1}{c}$, and hence $c = \dfrac{1}{a}$.

Since $b^2 + c^2 = 1$, $b = \sqrt{1 - c^2} = \sqrt{1 - 1/a^2} = \sqrt{\dfrac{a^2 - 1}{a^2}} = \dfrac{\sqrt{a^2 - 1}}{a}$, and now we have both $b$ and $c$ in terms of $a$. At $C$, $y = -1$, so $-1 = -\dfrac{b}{c}x + a \quad \Rightarrow \quad \dfrac{b}{c}x = a + 1 \quad \Rightarrow$

$x = \dfrac{c}{b}(a + 1) = \dfrac{1/a}{\sqrt{a^2 - 1}/a}(a + 1) = \dfrac{a + 1}{\sqrt{(a + 1)(a - 1)}} = \sqrt{\dfrac{a + 1}{a - 1}}$, and $C$ has coordinates $\left(\sqrt{\dfrac{a + 1}{a - 1}}, -1\right)$. Let $S$ be the square of the distance from $A$ to $C$. Then $S(a) = \left(0 - \sqrt{\dfrac{a + 1}{a - 1}}\right)^2 + (a + 1)^2 = \dfrac{a + 1}{a - 1} + (a + 1)^2 \quad \Rightarrow$

$$\begin{aligned} S'(a) &= \frac{(a - 1)(1) - (a + 1)(1)}{(a - 1)^2} + 2(a + 1) = \frac{-2 + 2(a + 1)(a - 1)^2}{(a - 1)^2} \\ &= \frac{-2 + 2(a^3 - a^2 - a + 1)}{(a - 1)^2} = \frac{2a^3 - 2a^2 - 2a}{(a - 1)^2} = \frac{2a(a^2 - a - 1)}{(a - 1)^2} \end{aligned}$$

Using the quadratic formula, we find that the solutions of $a^2 - a - 1 = 0$ are $a = \dfrac{1 \pm \sqrt{5}}{2}$, so $a_1 = \dfrac{1 + \sqrt{5}}{2}$ (the "golden mean") since $a > 0$. For $1 < a < a_1$, $S'(a) < 0$, and for $a > a_1$, $S'(a) > 0$, so $a_1$ minimizes $S$.

*Note:* The minimum length of the equal sides is $\sqrt{S(a_1)} = \cdots = \sqrt{\dfrac{11 + 5\sqrt{5}}{2}} \approx 3.33$ and the corresponding length of the third side is $2\sqrt{\dfrac{a_1 + 1}{a_1 - 1}} = \cdots = 2\sqrt{2 + \sqrt{5}} \approx 4.12$, so the triangle is *not* equilateral.

4.PP.13: changed "minimmum" to "corresponding" as circled here.

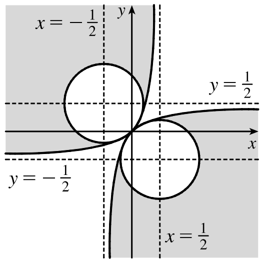
*Another method*: In $\Delta ABC$, $\cos\theta = \dfrac{a + 1}{\overline{AC}}$, so $\overline{AC} = \dfrac{a + 1}{\cos\theta}$. In $\Delta ADO$, $\sin\theta = \dfrac{1}{a}$, so $\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - 1/a^2} = \dfrac{1}{a}\sqrt{a^2 - 1}$. Thus $\overline{AC} = \dfrac{a + 1}{(1/a)\sqrt{a^2 - 1}} = \dfrac{a(a + 1)}{\sqrt{a^2 - 1}} = f(a)$. Now find the minimum of $f$.

**14.** To sketch the region $\{(x,y) \mid 2xy \le |x-y| \le x^2+y^2\}$, we consider two cases.

*Case 1:* $x \ge y$ This is the case in which $(x,y)$ lies on or below the line $y=x$. The double inequality becomes $2xy \le x-y \le x^2+y^2$. The right-hand inequality holds if and only if $x^2-x+y^2+y \ge 0 \Leftrightarrow$ $\left(x-\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2 \ge \frac{1}{2} \Leftrightarrow (x,y)$ lies on or outside the circle with radius $\frac{1}{\sqrt{2}}$ centered at $\left(\frac{1}{2},-\frac{1}{2}\right)$.

The left-hand inequality holds if and only if $2xy-x+y \le 0 \Leftrightarrow xy-\frac{1}{2}x+\frac{1}{2}y \le 0 \Leftrightarrow$ $\left(x+\frac{1}{2}\right)\left(y-\frac{1}{2}\right) \le -\frac{1}{4} \Leftrightarrow (x,y)$ lies on or below the hyperbola $\left(x+\frac{1}{2}\right)\left(y-\frac{1}{2}\right)=-\frac{1}{4}$, which passes through the origin and approaches the lines $y=\frac{1}{2}$ and $x=-\frac{1}{2}$ asymptotically.

*Case 2:* $y \ge x$ This is the case in which $(x,y)$ lies on or above the line $y=x$. The double inequality becomes $2xy \le y-x \le x^2+y^2$. The right-hand inequality holds if and only if $x^2+x+y^2-y \ge 0 \Leftrightarrow$ $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2 \ge \frac{1}{2} \Leftrightarrow (x,y)$ lies on or outside the circle of radius $\frac{1}{\sqrt{2}}$ centered at $\left(-\frac{1}{2},\frac{1}{2}\right)$. The left-hand inequality holds if and only if $2xy+x-y \le 0 \Leftrightarrow xy+\frac{1}{2}x-\frac{1}{2}y \le 0 \Leftrightarrow \left(x-\frac{1}{2}\right)\left(y+\frac{1}{2}\right) \le -\frac{1}{4} \Leftrightarrow (x,y)$ lies on or above the left-hand branch of the hyperbola $\left(x-\frac{1}{2}\right)\left(y+\frac{1}{2}\right)=-\frac{1}{4}$, which passes through the origin and approaches the lines $y=-\frac{1}{2}$ and $x=\frac{1}{2}$ asymptotically. Therefore, the region of interest consists of the points on or above the left branch of the hyperbola $\left(x-\frac{1}{2}\right)\left(y+\frac{1}{2}\right)=-\frac{1}{4}$ that are on or outside the circle $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2}$, together with the points on or below the right branch of the hyperbola $\left(x+\frac{1}{2}\right)\left(y-\frac{1}{2}\right)=-\frac{1}{4}$ that are on or outside the circle $\left(x-\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2=\frac{1}{2}$. Note that the inequalities are unchanged when $x$ and $y$ are interchanged, so the region is symmetric about the line $y=x$. So we need only have analyzed case 1 and then reflected that region about the line $y=x$, instead of considering case 2.




**15.** $A=\left(x_1,x_1^2\right)$ and $B=\left(x_2,x_2^2\right)$, where $x_1$ and $x_2$ are the solutions of the quadratic equation $x^2=mx+b$. Let $P=\left(x,x^2\right)$ and set $A_1=(x_1,0)$, $B_1=(x_2,0)$, and $P_1=(x,0)$. Let $f(x)$ denote the area of triangle $PAB$. Then $f(x)$ can be expressed in terms of the areas of three trapezoids as follows:

$$\begin{aligned} f(x) &= \text{area}\,(A_1ABB_1)-\text{area}\,(A_1APP_1)-\text{area}\,(B_1BPP_1) \\ &= \tfrac{1}{2}\left(x_1^2+x_2^2\right)(x_2-x_1)-\tfrac{1}{2}\left(x_1^2+x^2\right)(x-x_1)-\tfrac{1}{2}\left(x^2+x_2^2\right)(x_2-x) \end{aligned}$$

After expanding and canceling terms, we get

$f(x)=\frac{1}{2}\left(x_2x_1^2-x_1x_2^2-xx_1^2+x_1x^2-x_2x^2+xx_2^2\right)=\frac{1}{2}\left[x_1^2(x_2-x)+x_2^2(x-x_1)+x^2(x_1-x_2)\right]$

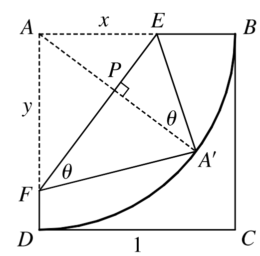
$f'(x)=\frac{1}{2}\left[-x_1^2+x_2^2+2x(x_1-x_2)\right]$. $f''(x)=\frac{1}{2}[2(x_1-x_2)]=x_1-x_2<0$ since $x_2>x_1$.

$f'(x)=0 \Rightarrow 2x(x_1-x_2)=x_1^2-x_2^2 \Rightarrow x_P=\frac{1}{2}(x_1+x_2)$.

$$\begin{aligned} f(x_P) &= \tfrac{1}{2}\left(x_1^2\left[\tfrac{1}{2}(x_2-x_1)\right] + x_2^2\left[\tfrac{1}{2}(x_2-x_1)\right] + \tfrac{1}{4}(x_1+x_2)^2(x_1-x_2)\right) \\ &= \tfrac{1}{2}\left[\tfrac{1}{2}(x_2-x_1)\left(x_1^2+x_2^2\right) - \tfrac{1}{4}(x_2-x_1)(x_1+x_2)^2\right] = \tfrac{1}{8}(x_2-x_1)\left[2\left(x_1^2+x_2^2\right) - \left(x_1^2+2x_1x_2+x_2^2\right)\right] \\ &= \tfrac{1}{8}(x_2-x_1)\left(x_1^2-2x_1x_2+x_2^2\right) = \tfrac{1}{8}(x_2-x_1)(x_1-x_2)^2 = \tfrac{1}{8}(x_2-x_1)(x_2-x_1)^2 = \tfrac{1}{8}(x_2-x_1)^3 \end{aligned}$$

To put this in terms of $m$ and $b$, we solve the system $y = x_1^2$ and $y = mx_1 + b$, giving us $x_1^2 - mx_1 - b = 0 \quad\Rightarrow$ $x_1 = \frac{1}{2}\left(m - \sqrt{m^2+4b}\right)$. Similarly, $x_2 = \frac{1}{2}\left(m + \sqrt{m^2+4b}\right)$. The area is then $\frac{1}{8}(x_2-x_1)^3 = \frac{1}{8}\left(\sqrt{m^2+4b}\right)^3$, and is attained at the point $P\left(x_P, x_P^2\right) = P\left(\frac{1}{2}m, \frac{1}{4}m^2\right)$.

*Note:* Another way to get an expression for $f(x)$ is to use the formula for an area of a triangle in terms of the coordinates of the vertices: $f(x) = \frac{1}{2}\left[\left(x_2x_1^2 - x_1x_2^2\right) + \left(x_1x^2 - xx_1^2\right) + \left(xx_2^2 - x_2x^2\right)\right]$.




**16.** Let $x = |AE|$, $y = |AF|$ as shown. The area $\mathcal{A}$ of the $\Delta AEF$ is $\mathcal{A} = \frac{1}{2}xy$. We need to find a relationship between $x$ and $y$, so that we can take the derivative $d\mathcal{A}/dx$ and then find the maximum and minimum areas. Now let $A'$ be the point on which $A$ ends up after the fold has been performed, and let $P$ be the intersection of $AA'$ and $EF$. Note that $AA'$ is perpendicular to $EF$ since we are reflecting $A$ through the line $EF$ to get to $A'$, and that $|AP| = |PA'|$ for the same reason.

But $|AA'| = 1$, since $AA'$ is a radius of the circle. Since $|AP| + |PA'| = |AA'|$, we have $|AP| = \frac{1}{2}$. Another way to express the area of the triangle is $\mathcal{A} = \frac{1}{2}\,|EF|\,|AP| = \frac{1}{2}\sqrt{x^2+y^2}\left(\frac{1}{2}\right) = \frac{1}{4}\sqrt{x^2+y^2}$. Equating the two expressions for $\mathcal{A}$, we get $\frac{1}{2}xy = \frac{1}{4}\sqrt{x^2+y^2} \quad\Rightarrow\quad 4x^2y^2 = x^2+y^2 \quad\Rightarrow\quad y^2\left(4x^2-1\right) = x^2 \quad\Rightarrow\quad y = x/\sqrt{4x^2-1}$.

(Note that we could also have derived this result from the similarity of $\triangle A'PE$ and $\triangle A'FE$; that is,
$\dfrac{|A'P|}{|PE|} = \dfrac{|A'F|}{|A'E|} \quad\Rightarrow\quad \dfrac{\frac{1}{2}}{\sqrt{x^2-\left(\frac{1}{2}\right)^2}} = \dfrac{y}{x} \quad\Rightarrow\quad y = \dfrac{\frac{1}{2}x}{\sqrt{4x^2-1}\,/2} = \dfrac{x}{\sqrt{4x^2-1}}$.) Now we can substitute for $y$ and calculate $\dfrac{d\mathcal{A}}{dx}$: $\mathcal{A} = \dfrac{1}{2}\dfrac{x^2}{\sqrt{4x^2-1}} \quad\Rightarrow\quad \dfrac{d\mathcal{A}}{dx} = \dfrac{1}{2}\left[\dfrac{\sqrt{4x^2-1}\,(2x) - x^2\left(\frac{1}{2}\right)\left(4x^2-1\right)^{-1/2}(8x)}{4x^2-1}\right]$. This is 0 when
$2x\sqrt{4x^2-1} - 4x^3\left(4x^2-1\right)^{-1/2} = 0 \quad\Leftrightarrow\quad 2x\left(4x^2-1\right)^{-1/2}\left[\left(4x^2-1\right) - 2x^2\right] = 0 \quad\Rightarrow\quad \left(4x^2-1\right) - 2x^2 = 0$ $(x > 0) \quad\Leftrightarrow\quad 2x^2 = 1 \quad\Rightarrow\quad x = \frac{1}{\sqrt{2}}$. So this is one possible value for an extremum. We must also test the endpoints of the interval over which $x$ ranges. The largest value that $x$ can attain is 1, and the smallest value of $x$ occurs when $y = 1 \quad\Leftrightarrow$ $1 = x/\sqrt{4x^2-1} \quad\Leftrightarrow\quad x^2 = 4x^2-1 \quad\Leftrightarrow\quad 3x^2 = 1 \quad\Leftrightarrow\quad x = \frac{1}{\sqrt{3}}$. This will give the same value of $\mathcal{A}$ as will $x = 1$, since the geometric situation is the same (reflected through the line $y = x$). We calculate
$\mathcal{A}\left(\frac{1}{\sqrt{2}}\right) = \dfrac{1}{2}\dfrac{\left(1/\sqrt{2}\right)^2}{\sqrt{4\left(1/\sqrt{2}\right)^2-1}} = \dfrac{1}{4} = 0.25$, and $\mathcal{A}(1) = \dfrac{1}{2}\dfrac{1^2}{\sqrt{4(1)^2-1}} = \dfrac{1}{2\sqrt{3}} \approx 0.29$. So the maximum area is $\mathcal{A}(1) = \mathcal{A}\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{2\sqrt{3}}$ and the minimum area is $\mathcal{A}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4}$.

[continued]

*Another method:* Use the angle $\theta$ (see diagram above) as a variable:

$\mathcal{A} = \frac{1}{2}xy = \frac{1}{2}\left(\frac{1}{2}\sec\theta\right)\left(\frac{1}{2}\csc\theta\right) = \dfrac{1}{8\sin\theta\,\cos\theta} = \dfrac{1}{4\sin 2\theta}$. $\mathcal{A}$ is minimized when $\sin 2\theta$ is maximal, that is, when $\sin 2\theta = 1 \quad\Rightarrow\quad 2\theta = \frac{\pi}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{4}$. Also note that $A'E = x = \frac{1}{2}\sec\theta \le 1 \quad\Rightarrow\quad \sec\theta \le 2 \quad\Rightarrow\quad \cos\theta \ge \frac{1}{2} \quad\Rightarrow\quad \theta \le \frac{\pi}{3}$, and similarly, $A'F = y = \frac{1}{2}\csc\theta \le 1 \quad\Rightarrow\quad \csc\theta \le 2 \quad\Rightarrow\quad \sin\theta \le \frac{1}{2} \quad\Rightarrow\quad \theta \ge \frac{\pi}{6}$.

As above, we find that $\mathcal{A}$ is maximized at these endpoints: $\mathcal{A}\left(\frac{\pi}{6}\right) = \dfrac{1}{4\sin\frac{\pi}{3}} = \dfrac{1}{2\sqrt{3}} = \dfrac{1}{4\sin\frac{2\pi}{3}} = \mathcal{A}\left(\frac{\pi}{3}\right)$;

and minimized at $\theta = \frac{\pi}{4}$: $\mathcal{A}\left(\frac{\pi}{4}\right) = \dfrac{1}{4\sin\frac{\pi}{2}} = \dfrac{1}{4}$.

**17.** Suppose that the curve $y = a^x$ intersects the line $y = x$. Then $a^{x_0} = x_0$ for some $x_0 > 0$, and hence $a = x_0^{1/x_0}$. We find the maximum value of $g(x) = x^{1/x}$, $x > 0$, because if $a$ is larger than the maximum value of this function, then the curve $y = a^x$ does not intersect the line $y = x$. $g'(x) = e^{(1/x)\ln x}\left(-\dfrac{1}{x^2}\ln x + \dfrac{1}{x}\cdot\dfrac{1}{x}\right) = x^{1/x}\left(\dfrac{1}{x^2}\right)(1 - \ln x)$. This is 0 only where $x = e$, and for $0 < x < e$, $f'(x) > 0$, while for $x > e$, $f'(x) < 0$, so $g$ has an absolute maximum of $g(e) = e^{1/e}$. So if $y = a^x$ intersects $y = x$, we must have $0 < a \le e^{1/e}$. Conversely, suppose that $0 < a \le e^{1/e}$. Then $a^e \le e$, so the graph of $y = a^x$ lies below or touches the graph of $y = x$ at $x = e$. Also $a^0 = 1 > 0$, so the graph of $y = a^x$ lies above that of $y = x$ at $x = 0$. Therefore, by the Intermediate Value Theorem, the graphs of $y = a^x$ and $y = x$ must intersect somewhere between $x = 0$ and $x = e$.

**18.** If $L = \lim\limits_{x\to\infty}\left(\dfrac{x+a}{x-a}\right)^x$, then $L$ has the indeterminate form $1^\infty$, so

$$\ln L = \lim_{x\to\infty}\ln\left(\frac{x+a}{x-a}\right)^x = \lim_{x\to\infty} x\ln\left(\frac{x+a}{x-a}\right) = \lim_{x\to\infty}\frac{\ln(x+a) - \ln(x-a)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\dfrac{1}{x+a} - \dfrac{1}{x-a}}{-1/x^2}$$

$$= \lim_{x\to\infty}\left[\frac{(x-a)-(x+a)}{(x+a)(x-a)}\cdot\frac{-x^2}{1}\right] = \lim_{x\to\infty}\frac{2ax^2}{x^2-a^2} = \lim_{x\to\infty}\frac{2a}{1-a^2/x^2} = 2a$$

Hence, $\ln L = 2a$, so $L = e^{2a}$. From the original equation, we want $L = e^1 \quad\Rightarrow\quad 2a = 1 \quad\Rightarrow\quad a = \frac{1}{2}$.

**19.** Note that $f(0) = 0$, so for $x \ne 0$, $\left|\dfrac{f(x) - f(0)}{x - 0}\right| = \left|\dfrac{f(x)}{x}\right| = \dfrac{|f(x)|}{|x|} \le \dfrac{|\sin x|}{|x|} = \dfrac{\sin x}{x}$.

Therefore, $|f'(0)| = \left|\lim\limits_{x\to 0}\dfrac{f(x) - f(0)}{x - 0}\right| = \lim\limits_{x\to 0}\left|\dfrac{f(x) - f(0)}{x - 0}\right| \le \lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$.

But $f(x) = a_1\sin x + a_2\sin 2x + \cdots + a_n\sin nx \quad\Rightarrow\quad f'(x) = a_1\cos x + 2a_2\cos 2x + \cdots + na_n\cos nx$, so $|f'(0)| = |a_1 + 2a_2 + \cdots + na_n| \le 1$.

*Another solution:* We are given that $\left|\sum_{k=1}^n a_k\sin kx\right| \le |\sin x|$. So for $x$ close to 0, and $x \ne 0$, we have $\left|\sum\limits_{k=1}^n a_k\dfrac{\sin kx}{\sin x}\right| \le 1 \quad\Rightarrow\quad \lim\limits_{x\to 0}\left|\sum\limits_{k=1}^n a_k\dfrac{\sin kx}{\sin x}\right| \le 1 \quad\Rightarrow\quad \left|\sum\limits_{k=1}^n a_k\lim\limits_{x\to 0}\dfrac{\sin kx}{\sin x}\right| \le 1$. But by l'Hospital's Rule, $\lim\limits_{x\to 0}\dfrac{\sin kx}{\sin x} = \lim\limits_{x\to 0}\dfrac{k\cos kx}{\cos x} = k$, so $\left|\sum\limits_{k=1}^n ka_k\right| \le 1$.

**20.** Let the circle have radius $r$, so $|OP| = |OQ| = r$, where $O$ is the center of the circle. Now $\angle POR$ has measure $\frac{1}{2}\theta$, and $\angle OPR$ is a right angle, so $\tan\frac{1}{2}\theta = \dfrac{|PR|}{r}$ and the area of $\triangle OPR$ is $\frac{1}{2}|OP|\,|PR| = \frac{1}{2}r^2\tan\frac{1}{2}\theta$. The area of the sector cut by $OP$ and $OR$ is $\frac{1}{2}r^2\left(\frac{1}{2}\theta\right) = \frac{1}{4}r^2\theta$. Let $S$ be the intersection of $PQ$ and $OR$. Then $\sin\frac{1}{2}\theta = \dfrac{|PS|}{r}$ and $\cos\frac{1}{2}\theta = \dfrac{|OS|}{r}$, and the area of $\triangle OSP$ is $\frac{1}{2}|OS|\,|PS| = \frac{1}{2}\left(r\cos\frac{1}{2}\theta\right)\left(r\sin\frac{1}{2}\theta\right) = \frac{1}{2}r^2\sin\frac{1}{2}\theta\,\cos\frac{1}{2}\theta = \frac{1}{4}r^2\sin\theta$.

So $B(\theta) = 2\left(\frac{1}{2}r^2\tan\frac{1}{2}\theta - \frac{1}{4}r^2\theta\right) = r^2\left(\tan\frac{1}{2}\theta - \frac{1}{2}\theta\right)$ and $A(\theta) = 2\left(\frac{1}{4}r^2\theta - \frac{1}{4}r^2\sin\theta\right) = \frac{1}{2}r^2(\theta - \sin\theta)$. Thus,

$$\begin{aligned}\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} &= \lim_{\theta\to 0^+}\frac{\frac{1}{2}r^2(\theta-\sin\theta)}{r^2\left(\tan\frac{1}{2}\theta - \frac{1}{2}\theta\right)} = \lim_{\theta\to 0^+}\frac{\theta-\sin\theta}{2\left(\tan\frac{1}{2}\theta - \frac{1}{2}\theta\right)} \overset{\text{H}}{=} \lim_{\theta\to 0^+}\frac{1-\cos\theta}{2\left(\frac{1}{2}\sec^2\frac{1}{2}\theta - \frac{1}{2}\right)} \\ &= \lim_{\theta\to 0^+}\frac{1-\cos\theta}{\sec^2\frac{1}{2}\theta - 1} = \lim_{\theta\to 0^+}\frac{1-\cos\theta}{\tan^2\frac{1}{2}\theta} \overset{\text{H}}{=} \lim_{\theta\to 0^+}\frac{\sin\theta}{2\left(\tan\frac{1}{2}\theta\right)\left(\sec^2\frac{1}{2}\theta\right)\frac{1}{2}} \\ &= \lim_{\theta\to 0^+}\frac{\sin\theta\cos^3\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} = \lim_{\theta\to 0^+}\frac{\left(2\sin\frac{1}{2}\theta\,\cos\frac{1}{2}\theta\right)\cos^3\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} = 2\lim_{\theta\to 0^+}\cos^4\left(\tfrac{1}{2}\theta\right) = 2(1)^4 = 2\end{aligned}$$

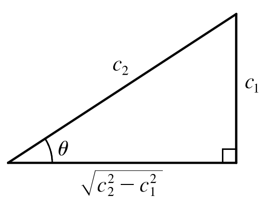
**21.** (a) Distance = rate × time, so time = distance/rate. $T_1 = \dfrac{D}{c_1}$, $T_2 = \dfrac{2\,|PR|}{c_1} + \dfrac{|RS|}{c_2} = \dfrac{2h\sec\theta}{c_1} + \dfrac{D - 2h\tan\theta}{c_2}$,

$$T_3 = \frac{2\sqrt{h^2 + D^2/4}}{c_1} = \frac{\sqrt{4h^2 + D^2}}{c_1}.$$

(b) $\dfrac{dT_2}{d\theta} = \dfrac{2h}{c_1}\cdot\sec\theta\,\tan\theta - \dfrac{2h}{c_2}\sec^2\theta = 0$ when $2h\sec\theta\left(\dfrac{1}{c_1}\tan\theta - \dfrac{1}{c_2}\sec\theta\right) = 0 \;\Rightarrow$

$\dfrac{1}{c_1}\dfrac{\sin\theta}{\cos\theta} - \dfrac{1}{c_2}\dfrac{1}{\cos\theta} = 0 \;\Rightarrow\; \dfrac{\sin\theta}{c_1\cos\theta} = \dfrac{1}{c_2\cos\theta} \;\Rightarrow\; \sin\theta = \dfrac{c_1}{c_2}$. The First Derivative Test shows that this gives a minimum.

(c) Using part (a) with $D = 1$ and $T_1 = 0.26$, we have $T_1 = \dfrac{D}{c_1} \;\Rightarrow\; c_1 = \frac{1}{0.26} \approx 3.85$ km/s. $T_3 = \dfrac{\sqrt{4h^2+D^2}}{c_1} \;\Rightarrow$ $4h^2 + D^2 = T_3^2c_1^2 \;\Rightarrow\; h = \frac{1}{2}\sqrt{T_3^2c_1^2 - D^2} = \frac{1}{2}\sqrt{(0.34)^2(1/0.26)^2 - 1^2} \approx 0.42$ km. To find $c_2$, we use $\sin\theta = \dfrac{c_1}{c_2}$ from part (b) and $T_2 = \dfrac{2h\sec\theta}{c_1} + \dfrac{D - 2h\tan\theta}{c_2}$ from part (a). From the figure,



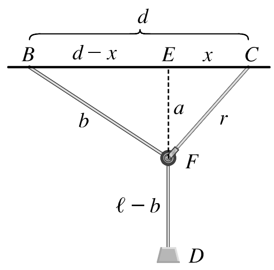

$\sin\theta = \dfrac{c_1}{c_2} \;\Rightarrow\; \sec\theta = \dfrac{c_2}{\sqrt{c_2^2 - c_1^2}}$ and $\tan\theta = \dfrac{c_1}{\sqrt{c_2^2 - c_1^2}}$, so

$T_2 = \dfrac{2hc_2}{c_1\sqrt{c_2^2 - c_1^2}} + \dfrac{D\sqrt{c_2^2 - c_1^2} - 2hc_1}{c_2\sqrt{c_2^2 - c_1^2}}$. Using the values for $T_2$ [given as 0.32], $h$, $c_1$, and $D$, we can graph $\mathrm{Y}_1 = T_2$ and $\mathrm{Y}_2 = \dfrac{2hc_2}{c_1\sqrt{c_2^2 - c_1^2}} + \dfrac{D\sqrt{c_2^2 - c_1^2} - 2hc_1}{c_2\sqrt{c_2^2 - c_1^2}}$ and find their intersection points. Doing so gives us $c_2 \approx 4.10$ and 7.66, but if $c_2 = 4.10$, then $\theta = \arcsin(c_1/c_2) \approx 69.6^\circ$, which implies that point $S$ is to the left of point $R$ in the diagram. So $c_2 = 7.66$ km/s.

**22.** A straight line intersects the curve $y = f(x) = x^4 + cx^3 + 12x^2 - 5x + 2$ in four distinct points if and only if the graph of $f$ has two inflection points. $f'(x) = 4x^3 + 3cx^2 + 24x - 5$ and $f''(x) = 12x^2 + 6cx + 24$.

$f''(x) = 0 \quad \Leftrightarrow \quad x = \dfrac{-6c \pm \sqrt{(6c)^2 - 4(12)(24)}}{2(12)}$. There are two distinct roots for $f''(x) = 0$ (and hence two inflection points) if and only if the discriminant is positive; that is, $36c^2 - 1152 > 0 \quad \Leftrightarrow \quad c^2 > 32 \quad \Leftrightarrow \quad |c| > \sqrt{32}$. Thus, the desired values of $c$ are $c < -4\sqrt{2}$ or $c > 4\sqrt{2}$.

**23.**




Let $a = |EF|$ and $b = |BF|$ as shown in the figure.

Since $\ell = |BF| + |FD|$, $|FD| = \ell - b$. Now

$$\begin{aligned} |ED| &= |EF| + |FD| = a + \ell - b \\ &\sqrt{r^2 - x^2} + \ell - \sqrt{(d-x)^2 + a^2} \\ &= \sqrt{r^2 - x^2} + \ell - \sqrt{(d-x)^2 + \left(\sqrt{r^2 - x^2}\right)^2} \\ &= \sqrt{r^2 - x^2} + \ell - \sqrt{d^2 - 2dx + x^2 + r^2 - x^2} \end{aligned}$$

Let $f(x) = \sqrt{r^2 - x^2} + \ell - \sqrt{d^2 + r^2 - 2dx}$.

$f'(x) = \frac{1}{2}\left(r^2 - x^2\right)^{-1/2}(-2x) - \frac{1}{2}\left(d^2 + r^2 - 2dx\right)^{-1/2}(-2d) = \dfrac{-x}{\sqrt{r^2 - x^2}} + \dfrac{d}{\sqrt{d^2 + r^2 - 2dx}}$.

$f'(x) = 0 \quad \Rightarrow \quad \dfrac{x}{\sqrt{r^2 - x^2}} = \dfrac{d}{\sqrt{d^2 + r^2 - 2dx}} \quad \Rightarrow \quad \dfrac{x^2}{r^2 - x^2} = \dfrac{d^2}{d^2 + r^2 - 2dx} \quad \Rightarrow$

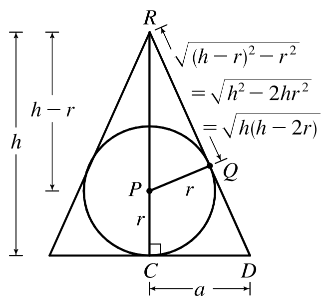
$d^2x^2 + r^2x^2 - 2dx^3 = d^2r^2 - d^2x^2 \quad \Rightarrow \quad 0 = 2dx^3 - 2d^2x^2 - r^2x^2 + d^2r^2 \quad \Rightarrow$

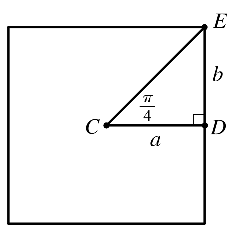
$0 = 2dx^2(x - d) - r^2\left(x^2 - d^2\right) \quad \Rightarrow \quad 0 = 2dx^2(x - d) - r^2(x + d)(x - d) \quad \Rightarrow \quad 0 = (x - d)\left[2dx^2 - r^2(x + d)\right]$

But $d > r > x$, so $x \neq d$. Thus, we solve $2dx^2 - r^2x - dr^2 = 0$ for $x$:

$x = \dfrac{-\left(-r^2\right) \pm \sqrt{\left(-r^2\right)^2 - 4(2d)\left(-dr^2\right)}}{2(2d)} = \dfrac{r^2 \pm \sqrt{r^4 + 8d^2r^2}}{4d}$. Because $\sqrt{r^4 + 8d^2r^2} > r^2$, the "negative" can be discarded. Thus, $x = \dfrac{r^2 + \sqrt{r^2}\sqrt{r^2 + 8d^2}}{4d} = \dfrac{r^2 + r\sqrt{r^2 + 8d^2}}{4d} \quad [r > 0] \quad = \dfrac{r}{4d}\left(r + \sqrt{r^2 + 8d^2}\right)$. The maximum value of $|ED|$ occurs at this value of $x$.

**24.**




Let $a = \overline{CD}$ denote the distance from the center $C$ of the base to the midpoint $D$ of a side of the base.

Since $\Delta PQR$ is similar to $\Delta DCR$, $\dfrac{a}{h} = \dfrac{r}{\sqrt{h(h-2r)}} \quad \Rightarrow \quad a = \dfrac{rh}{\sqrt{h(h-2r)}} = r\dfrac{\sqrt{h}}{\sqrt{h-2r}}$.

Let $b$ denote one-half the length of a side of the base. The area $A$ of the base is

$A = 8(\text{area of } \Delta CDE) = 8\left(\frac{1}{2}ab\right) = 4a\left(a\tan\frac{\pi}{4}\right) = 4a^2$.

The volume of the pyramid is $V = \frac{1}{3}Ah = \frac{1}{3}\left(4a^2\right)h = \dfrac{4}{3}\left(r\dfrac{\sqrt{h}}{\sqrt{h-2r}}\right)^2 h = \frac{4}{3}r^2\dfrac{h^2}{h-2r}$, with domain $h > 2r$.

Now $\dfrac{dV}{dh} = \dfrac{4}{3}r^2 \cdot \dfrac{(h-2r)(2h)-h^2(1)}{(h-2r)^2} = \dfrac{4}{3}r^2\dfrac{h^2-4hr}{(h-2r)^2} = \dfrac{4}{3}r^2\dfrac{h(h-4r)}{(h-2r)^2}$

and
$$\begin{aligned}\frac{d^2V}{dh^2} &= \frac{4}{3}r^2 \cdot \frac{(h-2r)^2(2h-4r) - \left(h^2-4hr\right)(2)(h-2r)(1)}{\left[(h-2r)^2\right]^2} \\ &= \frac{4}{3}r^2 \cdot \frac{2(h-2r)\left[\left(h^2-4hr+4r^2\right)-\left(h^2-4hr\right)\right]}{(h-2r)^2} \\ &= \frac{8}{3}r^2 \cdot \frac{4r^2}{(h-2r)^3} = \frac{32}{3}r^4 \cdot \frac{1}{(h-2r)^3}.\end{aligned}$$

The first derivative is equal to zero for $h = 4r$ and the second derivative is positive for $h > 2r$, so the volume of the pyramid is minimized when $h = 4r$.

To extend our solution to a regular $n$-gon, we make the following changes:

(1) the number of sides of the base is $n$

(2) the number of triangles in the base is $2n$

(3) $\angle DCE = \dfrac{\pi}{n}$

(4) $b = a\tan\dfrac{\pi}{n}$

We then obtain the following results: $A = na^2\tan\dfrac{\pi}{n}$, $V = \dfrac{nr^2}{3}\cdot\tan\left(\dfrac{\pi}{n}\right)\cdot\dfrac{h^2}{h-2r}$, $\dfrac{dV}{dh} = \dfrac{nr^2}{3}\cdot\tan\left(\dfrac{\pi}{n}\right)\cdot\dfrac{h(h-4r)}{(h-2r)^2}$,

and $\dfrac{d^2V}{dh^2} = \dfrac{8nr^4}{3}\cdot\tan\left(\dfrac{\pi}{n}\right)\cdot\dfrac{1}{(h-2r)^3}$. Notice that the answer, $h = 4r$, is independent of the number of sides of the base of the polygon!

**25.** $V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$. But $\dfrac{dV}{dt}$ is proportional to the surface area, so $\dfrac{dV}{dt} = k\cdot 4\pi r^2$ for some constant $k$.

Therefore, $4\pi r^2\dfrac{dr}{dt} = k\cdot 4\pi r^2 \quad \Leftrightarrow \quad \dfrac{dr}{dt} = k =$ constant. An antiderivative of $k$ with respect to $t$ is $kt$, so $r = kt + C$.

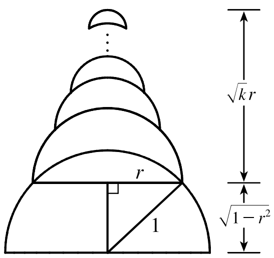
When $t = 0$, the radius $r$ must equal the original radius $r_0$, so $C = r_0$, and $r = kt + r_0$. To find $k$ we use the fact that when $t = 3$, $r = 3k + r_0$ and $V = \frac{1}{2}V_0 \quad \Rightarrow \quad \frac{4}{3}\pi(3k+r_0)^3 = \frac{1}{2}\cdot\frac{4}{3}\pi r_0^3 \quad \Rightarrow \quad (3k+r_0)^3 = \frac{1}{2}r_0^3 \quad \Rightarrow$

$3k + r_0 = \frac{1}{\sqrt[3]{2}}r_0 \quad \Rightarrow \quad k = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)$. Since $r = kt + r_0$, $r = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0$. When the snowball has melted completely we have $r = 0 \quad \Rightarrow \quad \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0 = 0$ which gives $t = \dfrac{3\sqrt[3]{2}}{\sqrt[3]{2}-1}$. Hence, it takes

$\dfrac{3\sqrt[3]{2}}{\sqrt[3]{2}-1} - 3 = \dfrac{3}{\sqrt[3]{2}-1} \approx 11$ h 33 min longer.

**26.** By ignoring the bottom hemisphere of the initial spherical bubble, we can rephrase the problem as follows: Prove that the maximum height of a stack of $n$ hemispherical bubbles is $\sqrt{n}$ if the radius of the bottom hemisphere is 1. We proceed by induction. The case $n = 1$ is obvious since $\sqrt{1}$ is the height of the first hemisphere. Suppose the assertion is true for $n = k$ and let's suppose we have $k + 1$ hemispherical bubbles forming a stack of maximum height. Suppose the second hemisphere (counting from the bottom) has radius $r$. Then by our induction hypothesis (scaled to the setting of a bottom hemisphere of radius $r$), the height of the stack formed by the top $k$ bubbles is $\sqrt{k}\,r$. (If it were shorter, then the total stack of $k + 1$ bubbles wouldn't have maximum height.)

The height of the whole stack is $H(r) = \sqrt{k}\,r + \sqrt{1 - r^2}$. (See the figure.) We want to choose $r$ so as to maximize $H(r)$. Note that $0 < r < 1$.




We calculate $H'(r) = \sqrt{k} - \dfrac{r}{\sqrt{1 - r^2}}$ and $H''(r) = \dfrac{-1}{(1 - r^2)^{3/2}}$.

$H'(r) = 0 \quad\Leftrightarrow\quad r^2 = k(1 - r^2) \quad\Leftrightarrow\quad (k + 1)r^2 = k \quad\Leftrightarrow\quad r = \sqrt{\dfrac{k}{k + 1}}$.

This is the only critical number in $(0, 1)$ and it represents a local maximum (hence an absolute maximum) since $H''(r) < 0$ on $(0, 1)$. When $r = \sqrt{\dfrac{k}{k+1}}$,

$H(r) = \sqrt{k}\dfrac{\sqrt{k}}{\sqrt{k+1}} + \sqrt{1 - \dfrac{k}{k+1}} = \dfrac{k}{\sqrt{k+1}} + \dfrac{1}{\sqrt{k+1}} = \sqrt{k+1}$. Thus, the assertion is true for $n = k + 1$ when it is true for $n = k$. By induction, it is true for all positive integers $n$.

*Note:* In general, a maximally tall stack of $n$ hemispherical bubbles consists of bubbles with radii

$$1, \sqrt{\frac{n-1}{n}}, \sqrt{\frac{n-2}{n}}, \ldots, \sqrt{\frac{2}{n}}, \sqrt{\frac{1}{n}}.$$