# 5 □ INTEGRALS

5.1 Areas and Distances
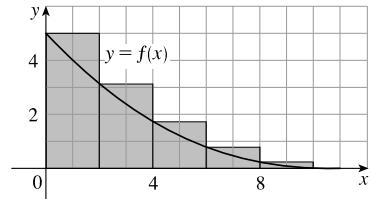
1. (a) Since f is *decreasing* , we can obtain a *lower* estimate by using *right* endpoints. We are instructed to use five rectangles, so $n = 5$.

$$R_5 = \sum_{i=1}^{5} f(x_i) \, \Delta x \qquad \left[\Delta x = \frac{b-a}{n} = \frac{10-0}{5} = 2 \right]$$

$$= f(x_1) \cdot 2 + f(x_2) \cdot 2 + f(x_3) \cdot 2 + f(x_4) \cdot 2 + f(x_5) \cdot 2$$

$$= 2[f(2) + f(4) + f(6) + f(8) + f(10)]$$

$$\approx 2(3.2 + 1.8 + 0.8 + 0.2 + 0)$$

$$= 2(6) = 12$$
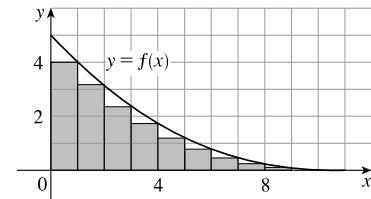

Since f is *decreasing* , we can obtain an *upper* estimate by using *left* endpoints.
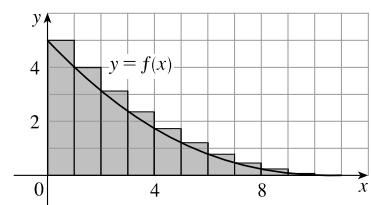
$$L_5 = \sum_{i=1}^{5} f(x_{i-1}) \, \Delta x$$

$$= f(x_0) \cdot 2 + f(x_1) \cdot 2 + f(x_2) \cdot 2 + f(x_3) \cdot 2 + f(x_4) \cdot 2$$

$$= 2[f(0) + f(2) + f(4) + f(6) + f(8)]$$

$$\approx 2(5 + 3.2 + 1.8 + 0.8 + 0.2)$$

$$= 2(11) = 22$$


(b) $$R_{10} = \sum_{i=1}^{10} f(x_i) \, \Delta x \qquad \left[\Delta x = \frac{10-0}{10} = 1 \right]$$

$$= 1[f(x_1) + f(x_2) + \cdots + f(x_{10})]$$

$$= f(1) + f(2) + \cdots + f(10)$$

$$\approx 4 + 3.2 + 2.5 + 1.8 + 1.3 + 0.8 + 0.5 + 0.2 + 0.1 + 0$$

$$= 14.4$$

$$L_{10} = \sum_{i=1}^{10} f(x_{i-1}) \, \Delta x$$

$$= f(0) + f(1) + \cdots + f(9)$$

$$= R_{10} + 1 \cdot f(0) - 1 \cdot f(10) \qquad \left[\begin{array}{l} \text{add leftmost upper rectangle,} \\ \text{subtract rightmost lower rectangle} \end{array} \right]$$

$$= 14.4 + 5 - 0$$

$$= 19.4$$

2. (a) (i) $L_6 = \sum\limits_{i=1}^{6} f(x_{i-1})\,\Delta x$ $[\Delta x = \frac{12-0}{6} = 2]$

$= 2[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$

$= 2[f(0) + f(2) + f(4) + f(6) + f(8) + f(10)]$

$\approx 2(9 + 8.8 + 8.2 + 7.3 + 5.9 + 4.1)$

$= 2(43.3) = 86.6$

(ii) $R_6 = L_6 + 2 \cdot f(12) - 2 \cdot f(0)$

$\approx 86.6 + 2(1) - 2(9) = 70.6$
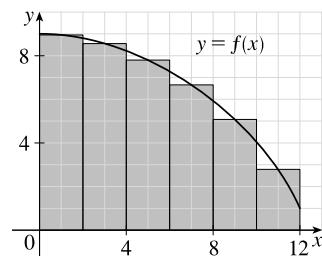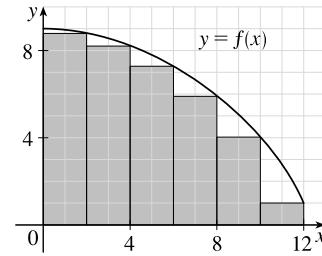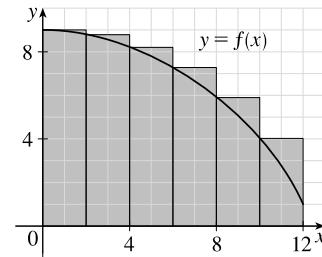
[Add area of rightmost lower rectangle

and subtract area of leftmost upper rectangle.]

(iii) $M_6 = \sum\limits_{i=1}^{6} f(x_i)\,\Delta x$

$= 2[f(1) + f(3) + f(5) + f(7) + f(9) + f(11)]$

$\approx 2(8.9 + 8.5 + 7.8 + 6.6 + 5.1 + 2.8)$

$= 2(39.7) = 79.4$

(b) Since f is *decreasing*, we obtain an *overestimate* by using *left* endpoints; that is, L_6.

(c) Since f is *decreasing*, we obtain an *underestimate* by using *right* endpoints; that is, R_6.
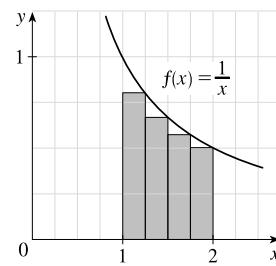
(d) M_6 gives the best estimate, since the area of each rectangle appears to be closer to the true area than the overestimates and underestimates in L_6 and R_6.

3. (a) $R_4 = \sum\limits_{i=1}^{4} f(x_i)\,\Delta x$ $\left[\Delta x = \frac{2-1}{4} = \frac{1}{4}\right] = \left[\sum\limits_{i=1}^{4} f(x_i)\right]\Delta x$

$= [f(x_1) + f(x_2) + f(x_3) + f(x_4)]\,\Delta x$

$= \left[\dfrac{1}{5/4} + \dfrac{1}{6/4} + \dfrac{1}{7/4} + \dfrac{1}{8/4}\right]\dfrac{1}{4} = \left[\frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2}\right]\frac{1}{4} \approx 0.6345$

Since f is *decreasing* on $[1, 2]$, an *underestimate* is obtained by using the

right endpoint approximation, R_4.
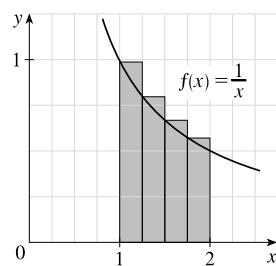
(b) $L_4 = \sum\limits_{i=1}^{4} f(x_{i-1})\,\Delta x = \left[\sum\limits_{i=1}^{4} f(x_{i-1})\right]\Delta x$

$= [f(x_0) + f(x_1) + f(x_2) + f(x_3)]\,\Delta x$

$= \left[\dfrac{1}{1} + \dfrac{1}{5/4} + \dfrac{1}{6/4} + \dfrac{1}{7/4}\right]\dfrac{1}{4} = \left[1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right]\frac{1}{4} \approx 0.7595$

L_4 is an overestimate. Alternatively, we could just add the area of the

leftmost upper rectangle and subtract the area of the rightmost lower

rectangle; that is, $L_4 = R_4 + f(1) \cdot \frac{1}{4} - f(2) \cdot \frac{1}{4}$.

4. (a) $R_4 = \sum\limits_{i=1}^{4} f(x_i)\, \Delta x \qquad \left[\Delta x = \dfrac{\pi/2 - 0}{4} = \dfrac{\pi}{8}\right] = \left[\sum\limits_{i=1}^{4} f(x_i)\right] \Delta x$

$\qquad = [f(x_1) + f(x_2) + f(x_3) + f(x_4)]\, \Delta x$

$\qquad = \left[\sin \tfrac{\pi}{8} + \sin \tfrac{2\pi}{8} + \sin \tfrac{3\pi}{8} + \sin \tfrac{4\pi}{8}\right] \tfrac{\pi}{8}$

$\qquad \approx 1.1835$

Since f is *increasing* on $\left[0, \tfrac{\pi}{2}\right]$, R_4 is an *overestimate*.

(b) $L_4 = \sum\limits_{i=1}^{4} f(x_{i-1})\, \Delta x = \left[\sum\limits_{i=1}^{4} f(x_{i-1})\right] \Delta x$

$\qquad = [f(x_0) + f(x_1) + f(x_2) + f(x_3)]\, \Delta x$

$\qquad = \left[\sin 0 + \sin \tfrac{\pi}{8} + \sin \tfrac{2\pi}{8} + \sin \tfrac{3\pi}{8}\right] \tfrac{\pi}{8}$

$\qquad \approx 0.7908$
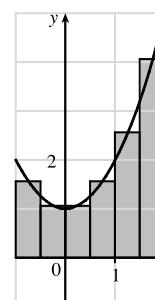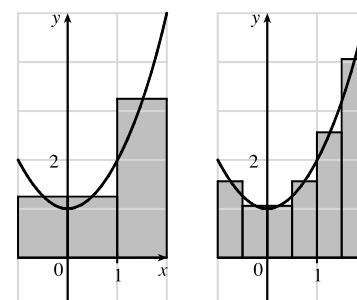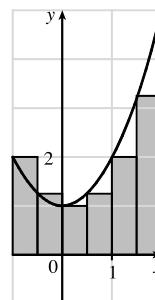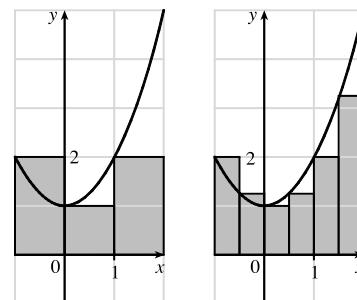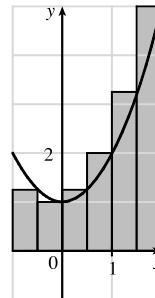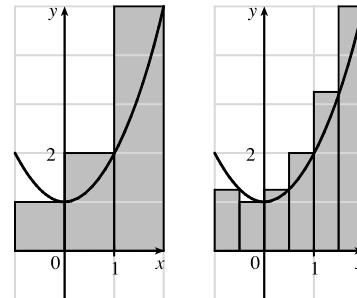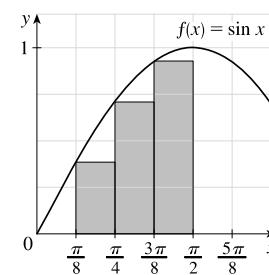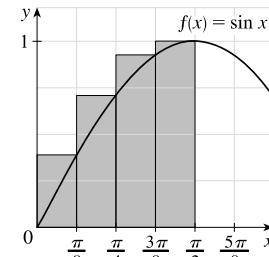
Since f is *increasing* on $\left[0, \tfrac{\pi}{2}\right]$, L_4 is an *underestimate*.

5. (a) $f(x) = 1 + x^2$ and $\Delta x = \dfrac{2 - (-1)}{3} = 1 \;\Rightarrow$

$\qquad R_3 = 1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2) = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 5 = 8.$

$\qquad \Delta x = \dfrac{2 - (-1)}{6} = 0.5 \;\Rightarrow$

$\qquad R_6 = 0.5[f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)]$

$\qquad\quad = 0.5(1.25 + 1 + 1.25 + 2 + 3.25 + 5)$

$\qquad\quad = 0.5(13.75) = 6.875$

(b) $L_3 = 1 \cdot f(-1) + 1 \cdot f(0) + 1 \cdot f(1) = 1 \cdot 2 + 1 \cdot 1 + 1 \cdot 2 = 5$

$\qquad L_6 = 0.5[f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5)]$

$\qquad\quad = 0.5(2 + 1.25 + 1 + 1.25 + 2 + 3.25)$

$\qquad\quad = 0.5(10.75) = 5.375$
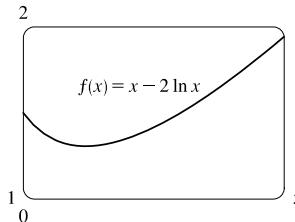
(c) $M_3 = 1 \cdot f(-0.5) + 1 \cdot f(0.5) + 1 \cdot f(1.5)$

$\qquad\quad = 1 \cdot 1.25 + 1 \cdot 1.25 + 1 \cdot 3.25 = 5.75$

$\qquad M_6 = 0.5[f(-0.75) + f(-0.25) + f(0.25)$

$\qquad\qquad\quad +\ f(0.75) + f(1.25) + f(1.75)]$

$\qquad\quad = 0.5(1.5625 + 1.0625 + 1.0625 + 1.5625 + 2.5625 + 4.0625)$

$\qquad\quad = 0.5(11.875) = 5.9375$

(d) M_6 appears to be the best estimate.

6. (a)

(b) $f(x) = x - 2\ln x$ and $\Delta x = \dfrac{5-1}{4} = 1 \Rightarrow$

(i) $R_4 = 1 \cdot f(2) + 1 \cdot f(3) + 1 \cdot f(4) + 1 \cdot f(5)$

$= (2 - 2\ln 2) + (3 - 2\ln 3) + (4 - 2\ln 4) + (5 - 2\ln 5)$

≈ 4.425

(ii) $M_4 = 1 \cdot f(1.5) + 1 \cdot f(2.5) + 1 \cdot f(3.5) + 1 \cdot f(4.5)$

$= (1.5 - 2\ln 1.5) + (2.5 - 2\ln 2.5)$

$\qquad + (3.5 - 2\ln 3.5) + (4.5 - 2\ln 4.5)$

≈ 3.843

(c) (i) $R_8 = \frac{1}{2}[f(1.5) + f(2) + \cdots + f(5)]$

$= \frac{1}{2}[(1.5 - 2\ln 1.5) + (2 - 2\ln 2) + \cdots + (5 - 2\ln 5)]$

≈ 4.134
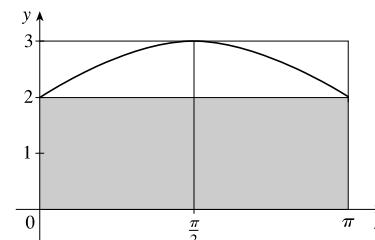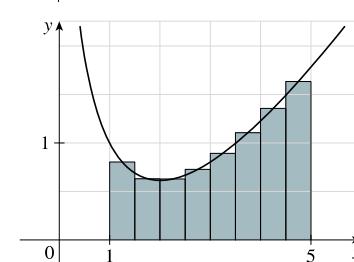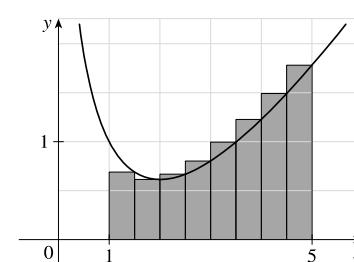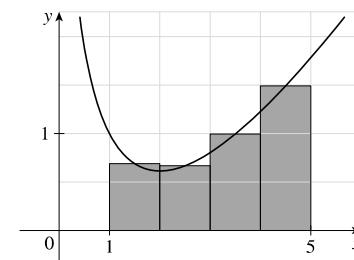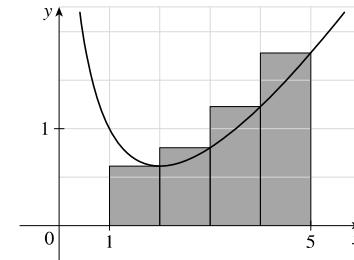
(ii) $M_8 = \frac{1}{2}[f(1.25) + f(1.75) + \cdots + f(4.75)]$

$= \frac{1}{2}[(1.25 - 2\ln 1.25) + (1.75 - 2\ln 1.75) + \cdots$

$\qquad\qquad\qquad + (4.75 - 2\ln 4.75)]$

≈ 3.889

7. $f(x) = 2 + \sin x,\ 0 \le x \le \pi,\ \Delta x = \pi/n.$

$\boldsymbol{n = 2}$: The maximum values of f on both subintervals occur at $x = \frac{\pi}{2}$, so

$upper\ sum = f\left(\frac{\pi}{2}\right) \cdot \frac{\pi}{2} + f\left(\frac{\pi}{2}\right) \cdot \frac{\pi}{2} = 3 \cdot \frac{\pi}{2} + 3 \cdot \frac{\pi}{2}$

$\qquad\qquad = 3\pi \approx 9.42.2$

The minimum values of f on the subintervals occur at $x = 0$ and

$x = \pi$, so

$lower\ sum = f(0) \cdot \frac{\pi}{2} + f(\pi) \cdot \frac{\pi}{2} = 2 \cdot \frac{\pi}{2} + 2 \cdot \frac{\pi}{2} = 2\pi \approx 6.28.$
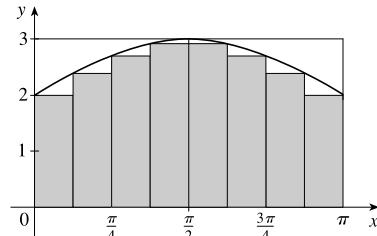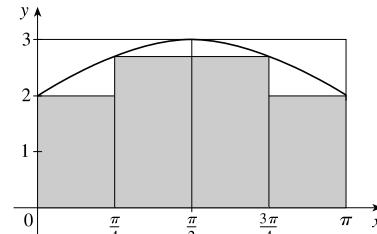
$n = 4$:   *upper sum* $= \left[f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{4}\right)\right]\left(\frac{\pi}{4}\right)$

$\qquad\qquad\qquad = \left[\left(2 + \frac{1}{2}\sqrt{2}\right) + (2 + 1) + (2 + 1) + \left(2 + \frac{1}{2}\sqrt{2}\right)\right]\left(\frac{\pi}{4}\right)$

$\qquad\qquad\qquad = \left(10 + \sqrt{2}\right)\left(\frac{\pi}{4}\right) \approx 8.96$

$\qquad\quad$ *lower sum* $= \left[f(0) + f\left(\frac{\pi}{4}\right) + f\left(\frac{3\pi}{4}\right) + f(\pi)\right]\left(\frac{\pi}{4}\right)$

$\qquad\qquad\qquad = \left[(2 + 0) + \left(2 + \frac{1}{2}\sqrt{2}\right) + \left(2 + \frac{1}{2}\sqrt{2}\right) + (2 + 0)\right]\left(\frac{\pi}{4}\right)$

$\qquad\qquad\qquad = \left(8 + \sqrt{2}\right)\left(\frac{\pi}{4}\right) \approx 7.39$

$n = 8$:   *upper sum* $= \left[f\left(\frac{\pi}{8}\right) + f\left(\frac{\pi}{4}\right) + f\left(\frac{3\pi}{8}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right)\right.$

$\qquad\qquad\qquad\quad \left. + f\left(\frac{5\pi}{8}\right) + f\left(\frac{3\pi}{4}\right) + f\left(\frac{7\pi}{8}\right)\right]\left(\frac{\pi}{8}\right)$

$\qquad\qquad\qquad \approx 8.65$

$\qquad\quad$ *lower sum* $= \left[f(0) + f\left(\frac{\pi}{8}\right) + f\left(\frac{\pi}{4}\right) + f\left(\frac{3\pi}{8}\right) + f\left(\frac{5\pi}{8}\right)\right.$

$\qquad\qquad\qquad\quad \left. + f\left(\frac{3\pi}{4}\right) + f\left(\frac{7\pi}{8}\right) + f(\pi)\right]\left(\frac{\pi}{8}\right)$

$\qquad\qquad\qquad \approx 7.86$
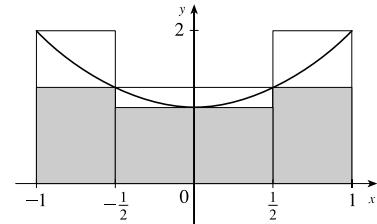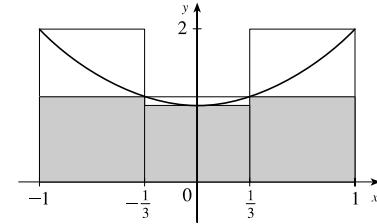
8. $f(x) = 1 + x^2, -1 \le x \le 1, \Delta x = 2/n.$

$n = 3$:   *upper sum* $= \left[f(-1) + f\left(\frac{1}{3}\right) + f(1)\right]\left(\frac{2}{3}\right)$

$\qquad\qquad\qquad = \left(2 + \frac{10}{9} + 2\right)\left(\frac{2}{3}\right)$

$\qquad\qquad\qquad = \frac{92}{27} \approx 3.41$

$\qquad\quad$ *lower sum* $= \left[f\left(-\frac{1}{3}\right) + f(0) + f\left(\frac{1}{3}\right)\right]\left(\frac{2}{3}\right)$

$\qquad\qquad\qquad = \left(\frac{10}{9} + 1 + \frac{10}{9}\right)\left(\frac{2}{3}\right)$

$\qquad\qquad\qquad = \frac{58}{27} \approx 2.15$

$n = 4$:   *upper sum* $= \left[f(-1) + f\left(-\frac{1}{2}\right) + f\left(\frac{1}{2}\right) + f(1)\right]\left(\frac{2}{4}\right)$

$\qquad\qquad\qquad = \left(2 + \frac{5}{4} + \frac{5}{4} + 2\right)\left(\frac{1}{2}\right)$

$\qquad\qquad\qquad = \frac{13}{4} = 3.25$

$\qquad\quad$ *lower sum* $= \left[f\left(-\frac{1}{2}\right) + f(0) + f(0) + f\left(\frac{1}{2}\right)\right]\left(\frac{2}{4}\right)$

$\qquad\qquad\qquad = \left(\frac{5}{4} + 1 + 1 + \frac{5}{4}\right)\left(\frac{1}{2}\right)$

$\qquad\qquad\qquad = \frac{9}{4} = 2.25$

**9.** Here is one possible algorithm (ordered sequence of operations) for calculating the sums:

$\quad$ 1 Let SUM = 0, X_MIN = 0, X_MAX = 1, N = 10 (depending on which sum we are calculating),

$\qquad$ DELTA_X = (X_MAX - X_MIN)/N, and RIGHT_ENDPOINT = X_MIN + DELTA_X.

$\quad$ 2 Repeat steps 2a, 2b in sequence until RIGHT_ENDPOINT > X_MAX.

$\quad\quad$ 2a Add (RIGHT_ENDPOINT)^4 to SUM.

$\qquad$ Add DELTA_X to RIGHT_ENDPOINT.

$\qquad$ At the end of this procedure, (DELTA_X)·(SUM) is equal to the answer we are looking for. We find that

$$R_{10} = \frac{1}{10}\sum_{i=1}^{10}\left(\frac{i}{10}\right)^4 \approx 0.2533, \quad R_{30} = \frac{1}{30}\sum_{i=1}^{30}\left(\frac{i}{30}\right)^4 \approx 0.2170, \quad R_{50} = \frac{1}{50}\sum_{i=1}^{50}\left(\frac{i}{50}\right)^4 \approx 0.2101, \text{ and}$$

$R_{100} = \frac{1}{100} \sum_{i=1}^{100} \left(\frac{i}{100} \right)^4 \approx 0.2050$. It appears that the exact area is 0.2. The following display shows the program

SUMRIGHT and its output from a TI-83/4 Plus calculator. To generalize the program, we have input (rather than

assign) values for Xmin, Xmax, and N. Also, the function, x^4, is assigned to Y_1, enabling us to evaluate any right sum

merely by changing Y_1 and running the program.

```
PROGRAM:SUMRIGHT
:0→S
:Prompt Xmin
:Prompt Xmax
:Prompt N
:(Xmax-Xmin)/N→D
:Xmin+D→R
:For(I,1,N)
:S+Y₁(R)→S
:R+D→R
:End
:D*S→Z
:Disp Z
```

```
prgmSUMRIGHT
Xmin=?0
Xmax=?1
N=?10
            .25333
            Done
```

10. We can use the algorithm from Exercise 9 with X_MIN = 0, X_MAX = $\pi/2$, and cos(RIGHT_ENDPOINT) instead of

(RIGHT_ENDPOINT)^4 in step 2a. We find that $R_{10} = \frac{\pi/2}{10} \sum_{i=1}^{10} \cos\left(\frac{i\pi}{20} \right) \approx 0.9194$, $R_{30} = \frac{\pi/2}{30} \sum_{i=1}^{30} \cos\left(\frac{i\pi}{60} \right) \approx 0.9736$,

and $R_{50} = \frac{\pi/2}{50} \sum_{i=1}^{50} \cos\left(\frac{i\pi}{100} \right) \approx 0.9842$, and $R_{100} = \frac{\pi/2}{100} \sum_{i=1}^{100} \cos\left(\frac{i\pi}{200} \right) \approx 0.9921$. It appears that the exact area is 1.

11. In Maple, we have to perform a number of steps before getting a numerical answer. After loading the student package

[command: `with(student);`] we use the command

`left_sum:=leftsum(1/(x^2+1),x=0..1,10 [or 30, or 50]);` which gives us the expression in summation

notation. To get a numerical approximation to the sum, we use `evalf(left_sum);`. Mathematica does not have a special

command for these sums, so we must type them in manually. For example, the first left sum is given by

`(1/10)*Sum[1/(((i-1)/10)^2+1)],{i,1,10}]`, and we use the `N` command on the resulting output to get a

numerical approximation.

In Derive, we use the `LEFT_RIEMANN` command to get the left sums, but must define the right sums ourselves.

(We can define a new function using `LEFT_RIEMANN` with k ranging from 1 to n instead of from 0 to $n - 1$.)
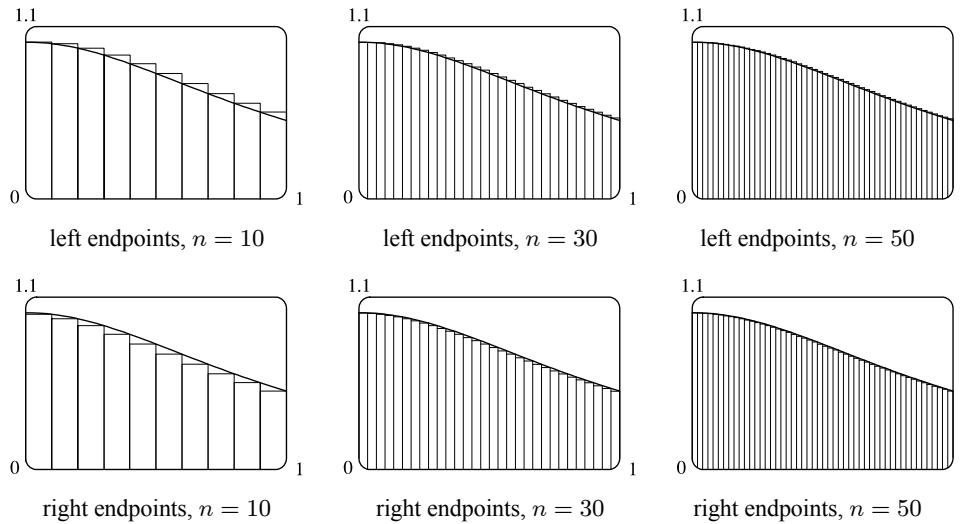
(a) With $f(x) = \frac{1}{x^2 + 1}$, $0 \le x \le 1$, the left sums are of the form $L_n = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\left(\frac{i-1}{n} \right)^2 + 1}$. Specifically, $L_{10} \approx 0.8100$,

$L_{30} \approx 0.7937$, and $L_{50} \approx 0.7904$. The right sums are of the form $R_n = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\left(\frac{i}{n} \right)^2 + 1}$. Specifically, $R_{10} \approx 0.7600$,

$R_{30} \approx 0.7770$, and $R_{50} \approx 0.7804$.

(b) In Maple, we use the `leftbox` (with the same arguments as `left_sum`) and `rightbox` commands to generate the graphs.

left endpoints, $n = 10$ left endpoints, $n = 30$ left endpoints, $n = 50$

right endpoints, $n = 10$ right endpoints, $n = 30$ right endpoints, $n = 50$
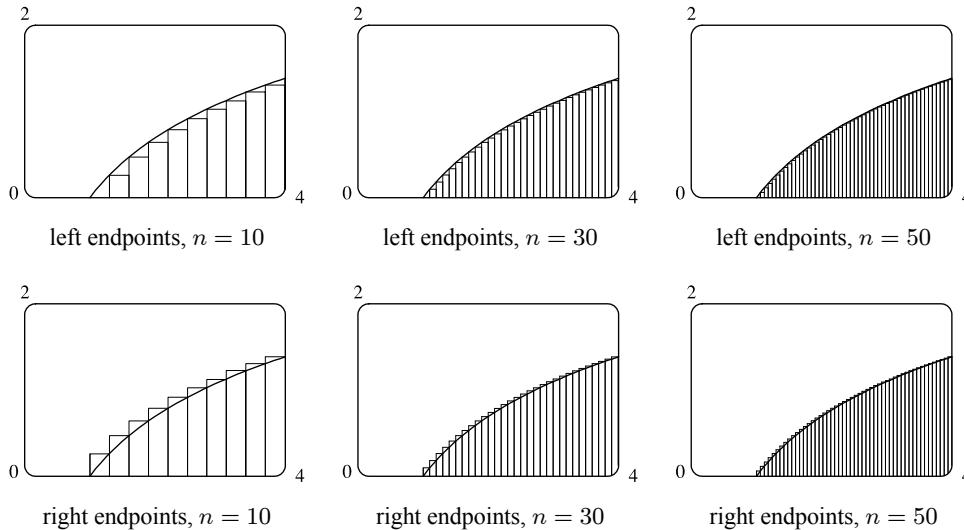
(c) We know that since $y = 1/(x^2 + 1)$ is a decreasing function on $(0, 1)$, all of the left sums are larger than the actual area, and all of the right sums are smaller than the actual area. Since the left sum with $n = 50$ is about $0.7904 < 0.791$ and the right sum with $n = 50$ is about $0.7804 > 0.780$, we conclude that $0.780 < R_{50} < \text{exact area} < L_{50} < 0.791$, so the exact area is between 0.780 and 0.791.

12. See the solution to Exercise 11 for the CAS commands for evaluating the sums.

(a) With $f(x) = \ln x$, $1 \le x \le 4$, the left sums are of the form $L_n = \dfrac{3}{n} \sum\limits_{i=1}^{n} \ln\left(1 + \dfrac{3(i-1)}{n}\right)$. In particular, $L_{10} \approx 2.3316$,

$L_{30} \approx 2.4752$, and $L_{50} \approx 2.5034$. The right sums are of the form $R_n = \dfrac{3}{n} \sum\limits_{i=1}^{n} \ln\left(1 + \dfrac{3i}{n}\right)$. In particular,

$R_{10} \approx 2.7475$, $R_{30} \approx 2.6139$, and $R_{50} \approx 2.5865$.

(b) In Maple, we use the `leftbox` (with the same arguments as `left_sum`) and `rightbox` commands to generate the graphs.

left endpoints, $n = 10$ left endpoints, $n = 30$ left endpoints, $n = 50$

right endpoints, $n = 10$ right endpoints, $n = 30$ right endpoints, $n = 50$

(c) We know that since $y = \ln x$ is an increasing function on $(1, 4)$, all of the left sums are smaller than the actual area, and all of the right sums are larger than the actual area. Since the left sum with $n = 50$ is about $2.503 > 2.50$ and the right sum with $n = 50$ is about $2.587 < 2.59$, we conclude that $2.50 < L_{50} < \text{exact area} < R_{50} < 2.59$, so the exact area is between 2.50 and 2.59.

13. Since v is an increasing function, L_6 will give us a lower estimate and R_6 will give us an upper estimate.

$$L_6 = (0 \text{ ft/s})(0.5 \text{ s}) + (6.2)(0.5) + (10.8)(0.5) + (14.9)(0.5) + (18.1)(0.5) + (19.4)(0.5) = 0.5(69.4) = 34.7 \text{ ft}$$

$$R_6 = 0.5(6.2 + 10.8 + 14.9 + 18.1 + 19.4 + 20.2) = 0.5(89.6) = 44.8 \text{ ft}$$

14. (a) The velocities are given with units mi/h, so we must convert the 10-second intervals to hours:

$$10 \text{ seconds} = \frac{10 \text{ seconds}}{3600 \text{ seconds/h}} = \frac{1}{360} \text{ h}$$

$$\text{distance} \approx L_6 = (182.9 \text{ mi/h})\left(\tfrac{1}{360} \text{ h}\right) + (168.0)\left(\tfrac{1}{360}\right) + (106.6)\left(\tfrac{1}{360}\right) + (99.8)\left(\tfrac{1}{360}\right)$$
$$+ (124.5)\left(\tfrac{1}{360}\right) + (176.1)\left(\tfrac{1}{360}\right)$$

$$= \frac{857.9}{360} \approx 2.383 \text{ miles}$$

(b) Distance $\approx R_6 = \left(\tfrac{1}{360}\right)(168.0 + 106.6 + 99.8 + 124.5 + 176.1 + 175.6) = \dfrac{850.6}{360} \approx 2.363$ miles

(c) The velocity is neither increasing nor decreasing on the given interval, so the estimates in parts (a) and (b) are neither upper nor lower estimates.

15. Lower estimate for oil leakage: $R_5 = (7.6 + 6.8 + 6.2 + 5.7 + 5.3)(2) = (31.6)(2) = 63.2 \text{ L}$.

Upper estimate for oil leakage: $L_5 = (8.7 + 7.6 + 6.8 + 6.2 + 5.7)(2) = (35)(2) = 70 \text{ L}$.

16. We can find an upper estimate by using the final velocity for each time interval. Thus, the distance d traveled after 62 seconds can be approximated by

$$d = \sum_{i=1}^{6} v(t_i) \, \Delta t_i = (185 \text{ ft/s})(10 \text{ s}) + 319 \cdot 5 + 447 \cdot 5 + 742 \cdot 12 + 1325 \cdot 27 + 1445 \cdot 3 = 54{,}694 \text{ ft}$$

17. For a decreasing function, using left endpoints gives us an overestimate and using right endpoints results in an underestimate. We will use M_6 to get an estimate. $\Delta t = 1$, so

$$M_6 = 1[v(0.5) + v(1.5) + v(2.5) + v(3.5) + v(4.5) + v(5.5)] \approx 55 + 40 + 28 + 18 + 10 + 4 = 155 \text{ ft}$$

For a very rough check on the above calculation, we can draw a line from $(0, 70)$ to $(6, 0)$ and calculate the area of the triangle: $\frac{1}{2}(70)(6) = 210$. This is clearly an overestimate, so our midpoint estimate of 155 is reasonable.

18. For an increasing function, using left endpoints gives us an underestimate and using right endpoints results in an overestimate. We will use M_6 to get an estimate. $\Delta t = \frac{30 - 0}{6} = 5 \text{ s} = \frac{5}{3600} \text{ h} = \frac{1}{720} \text{ h}$.

$$M_6 = \tfrac{1}{720}[v(2.5) + v(7.5) + v(12.5) + v(17.5) + v(22.5) + v(27.5)]$$
$$= \tfrac{1}{720}(31.25 + 66 + 88 + 103.5 + 113.75 + 119.25) = \tfrac{1}{720}(521.75) \approx 0.725 \text{ km}$$

For a very rough check on the above calculation, we can draw a line from $(0, 0)$ to $(30, 120)$ and calculate the area of the triangle: $\frac{1}{2}(30)(120) = 1800$. Divide by 3600 to get 0.5, which is clearly an underestimate, making our midpoint estimate of 0.725 seem reasonable. Of course, answers will vary due to different readings of the graph.

19. $f(t) = -t(t - 21)(t + 1)$ and $\Delta t = \frac{12 - 0}{6} = 2$

$$M_6 = 2 \cdot f(1) + 2 \cdot f(3) + 2 \cdot f(5) + 2 \cdot f(7) + 2 \cdot f(9) + 2 \cdot f(11)$$
$$= 2 \cdot 40 + 2 \cdot 216 + 2 \cdot 480 + 2 \cdot 784 + 2 \cdot 1080 + 2 \cdot 1320$$
$$= 7840 \text{ (infected cells/mL)} \cdot \text{days}$$

Thus, the total amount of infection needed to develop symptoms of measles is about 7840 infected cells per mL of blood plasma.

20. (a) Use $\Delta t = 14$ days. The number of people who died of SARS in Singapore between March 1 and May 24, 2003, using left endpoints is

$$L_6 = 14(0.0079 + 0.0638 + 0.1944 + 0.4435 + 0.5620 + 0.4630) = 14(1.7346) = 24.2844 \approx 24 \text{ people}$$

Using right endpoints,

$$R_6 = 14(0.0638 + 0.1944 + 0.4435 + 0.5620 + 0.4630 + 0.2897) = 14(2.0164) = 28.2296 \approx 28 \text{ people}$$

(b) Let t be the number of days since March 1, 2003, $f(t)$ be the number of deaths per day on day t, and the graph of $y = f(t)$ be a reasonable continuous function on the interval $[0, 84]$. Then the number of SARS deaths from $t = a$ to $t = b$ is approximately equal to the area under the curve $y = f(t)$ from $t = a$ to $t = b$.

21. $f(x) = \dfrac{2x}{x^2 + 1}$, $1 \le x \le 3$. $\Delta x = (3 - 1)/n = 2/n$ and $x_i = 1 + i\Delta x = 1 + 2i/n$.

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2(1 + 2i/n)}{(1 + 2i/n)^2 + 1} \cdot \frac{2}{n}.$$

22. $f(x) = x^2 + \sqrt{1 + 2x}$, $4 \le x \le 7$. $\Delta x = (7 - 4)/n = 3/n$ and $x_i = 4 + i\,\Delta x = 4 + 3i/n$.

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} \left[(4 + 3i/n)^2 + \sqrt{1 + 2(4 + 3i/n)} \right] \cdot \frac{3}{n}.$$

23. $f(x) = \sqrt{\sin x}$, $0 \le x \le \pi$. $\Delta x = (\pi - 0)/n = \pi/n$ and $x_i = 0 + i\,\Delta x = \pi i/n$.

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\sin(\pi i/n)} \cdot \frac{\pi}{n}.$$

24. $\displaystyle\lim_{n \to \infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1 + \frac{3i}{n}}$ can be interpreted as the area of the region lying under the graph of $y = \sqrt{1 + x}$ on the interval $[0, 3]$,

since for $y = \sqrt{1 + x}$ on $[0, 3]$ with $\Delta x = \dfrac{3 - 0}{n} = \dfrac{3}{n}$, $x_i = 0 + i\,\Delta x = \dfrac{3i}{n}$, and $x_i^* = x_i$, the expression for the area is

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + \frac{3i}{n}}\,\frac{3}{n}.$$ Note that this answer is not unique. We could use $y = \sqrt{x}$ on $[1, 4]$ or,

in general, $y = \sqrt{x - n}$ on $[n + 1, n + 4]$, where n is any real number.

25. $\displaystyle\lim_{n \to \infty} \sum_{i=1}^{n} \frac{\pi}{4n}\tan\frac{i\pi}{4n}$ can be interpreted as the area of the region lying under the graph of $y = \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$,

since for $y = \tan x$ on $\left[0, \frac{\pi}{4}\right]$ with $\Delta x = \dfrac{\pi/4 - 0}{n} = \dfrac{\pi}{4n}$, $x_i = 0 + i\,\Delta x = \dfrac{i\pi}{4n}$, and $x_i^* = x_i$, the expression for the area is

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} \tan\left(\frac{i\pi}{4n}\right)\frac{\pi}{4n}.$$ Note that this answer is not unique, since the expression for the area is

the same for the function $y = \tan(x - k\pi)$ on the interval $\left[k\pi, k\pi + \frac{\pi}{4}\right]$, where k is any integer.

26. (a) $\Delta x = \dfrac{1 - 0}{n} = \dfrac{1}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{i}{n}$. $A = \lim\limits_{n \to \infty} R_n = \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} f(x_i)\,\Delta x = \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \left(\dfrac{i}{n}\right)^3 \cdot \dfrac{1}{n}$.

(b) $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \dfrac{i^3}{n^3} \cdot \dfrac{1}{n} = \lim\limits_{n \to \infty} \dfrac{1}{n^4} \sum\limits_{i=1}^{n} i^3 = \lim\limits_{n \to \infty} \dfrac{1}{n^4} \left[\dfrac{n(n+1)}{2}\right]^2 = \lim\limits_{n \to \infty} \dfrac{(n+1)^2}{4n^2} = \dfrac{1}{4} \lim\limits_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^2 = \dfrac{1}{4}$

27. (a) Since f is an increasing function, L_n is an underestimate of A [lower sum] and R_n is an overestimate of A [upper sum].

Thus, A, L_n, and R_n are related by the inequality $L_n < A < R_n$.
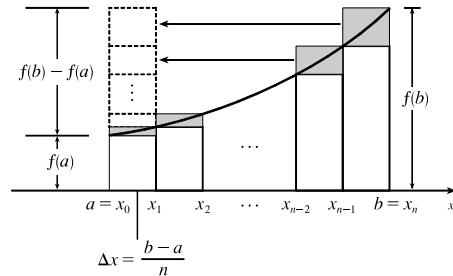
(b) $R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$

$\qquad L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$

$\quad R_n - L_n = f(x_n)\Delta x - f(x_0)\Delta x$

$\qquad\qquad = \Delta x [f(x_n) - f(x_0)]$

$\qquad\qquad = \dfrac{b - a}{n}[f(b) - f(a)]$

In the diagram, $R_n - L_n$ is the sum of the areas of the shaded rectangles. By sliding the shaded rectangles to the left so

that they stack on top of the leftmost shaded rectangle, we form a rectangle of height $f(b) - f(a)$ and width $\dfrac{b - a}{n}$.

(c) $A > L_n$, so $R_n - A < R_n - L_n$; that is, $R_n - A < \dfrac{b - a}{n}[f(b) - f(a)]$.

28. $R_n - A < \dfrac{b - a}{n}[f(b) - f(a)] = \dfrac{3 - 1}{n}[f(3) - f(1)] = \dfrac{2}{n}(e^3 - e)$

Solving $\dfrac{2}{n}(e^3 - e) < 0.0001$ for n gives us $2(e^3 - e) < 0.0001n \quad\Rightarrow\quad n > \dfrac{2(e^3 - e)}{0.0001} \quad\Rightarrow\quad n > 347,345.1$. Thus,

a value of n that assures us that $R_n - A < 0.0001$ is $n = 347,346$. [This is not the *least* value of n.]

29. (a) $y = f(x) = x^5$. $\Delta x = \dfrac{2 - 0}{n} = \dfrac{2}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{2i}{n}$.

$A = \lim\limits_{n \to \infty} R_n = \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} f(x_i)\,\Delta x = \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \left(\dfrac{2i}{n}\right)^5 \cdot \dfrac{2}{n} = \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \dfrac{32i^5}{n^5} \cdot \dfrac{2}{n} = \lim\limits_{n \to \infty} \dfrac{64}{n^6} \sum\limits_{i=1}^{n} i^5$.

(b) $\sum\limits_{i=1}^{n} i^5 \overset{\mathrm{CAS}}{=} \dfrac{n^2(n+1)^2\left(2n^2 + 2n - 1\right)}{12}$

(c) $\lim\limits_{n \to \infty} \dfrac{64}{n^6} \cdot \dfrac{n^2(n+1)^2\left(2n^2 + 2n - 1\right)}{12} = \dfrac{64}{12} \lim\limits_{n \to \infty} \dfrac{\left(n^2 + 2n + 1\right)\left(2n^2 + 2n - 1\right)}{n^2 \cdot n^2}$

$\qquad\qquad = \dfrac{16}{3} \lim\limits_{n \to \infty} \left(1 + \dfrac{2}{n} + \dfrac{1}{n^2}\right)\left(2 + \dfrac{2}{n} - \dfrac{1}{n^2}\right) = \tfrac{16}{3} \cdot 1 \cdot 2 = \tfrac{32}{3}$

30. From Example 3(a), we have $A = \lim\limits_{n \to \infty} \dfrac{2}{n} \sum\limits_{i=1}^{n} e^{-2i/n}$. Using a CAS, $\sum\limits_{i=1}^{n} e^{-2i/n} = \dfrac{e^{-2}\left(e^2 - 1\right)}{e^{2/n} - 1}$ and

$\lim\limits_{n \to \infty} \dfrac{2}{n} \cdot \dfrac{e^{-2}\left(e^2 - 1\right)}{e^{2/n} - 1} = e^{-2}\left(e^2 - 1\right) \approx 0.8647$, whereas the estimate from Example 3(b) using M_{10} was 0.8632.
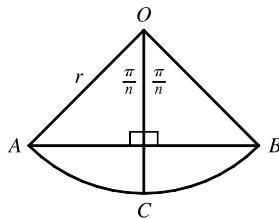
31. $y = f(x) = \cos x$. $\Delta x = \dfrac{b - 0}{n} = \dfrac{b}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{bi}{n}$.

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} \cos\left(\frac{bi}{n}\right) \cdot \frac{b}{n}$$

$$\overset{\text{CAS}}{=} \lim_{n \to \infty} \left[\frac{b \sin\left(b\left(\dfrac{1}{2n} + 1\right)\right)}{2n \sin\left(\dfrac{b}{2n}\right)} - \frac{b}{2n} \right] \overset{\text{CAS}}{=} \sin b$$

If $b = \frac{\pi}{2}$, then $A = \sin\frac{\pi}{2} = 1$.

32. (a)

The diagram shows one of the n congruent triangles, $\triangle AOB$, with central angle $2\pi/n$. O is the center of the circle and AB is one of the sides of the polygon. Radius OC is drawn so as to bisect $\angle AOB$. It follows that OC intersects AB at right angles and bisects AB. Thus, $\triangle AOB$ is divided into two right triangles with legs of length $\frac{1}{2}(AB) = r\sin(\pi/n)$ and $r\cos(\pi/n)$. $\triangle AOB$ has area

$2 \cdot \frac{1}{2}[r\sin(\pi/n)][r\cos(\pi/n)] = r^2 \sin(\pi/n)\cos(\pi/n) = \frac{1}{2}r^2 \sin(2\pi/n)$,

so $A_n = n \cdot \text{area}(\triangle AOB) = \frac{1}{2}nr^2 \sin(2\pi/n)$.

(b) To use Equation 3.3.2, $\displaystyle\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$, we need to have the same expression in the denominator as we have in the argument of the sine function—in this case, $2\pi/n$.

$$\lim_{n \to \infty} A_n = \lim_{n \to \infty} \frac{1}{2}nr^2 \sin(2\pi/n) = \lim_{n \to \infty} \frac{1}{2}nr^2 \frac{\sin(2\pi/n)}{2\pi/n} \cdot \frac{2\pi}{n} = \lim_{n \to \infty} \frac{\sin(2\pi/n)}{2\pi/n}\pi r^2. \text{ Let } \theta = \frac{2\pi}{n}.$$

Then as $n \to \infty$, $\theta \to 0$, so $\displaystyle\lim_{n \to \infty} \frac{\sin(2\pi/n)}{2\pi/n}\pi r^2 = \lim_{\theta \to 0} \frac{\sin\theta}{\theta}\pi r^2 = (1)\,\pi r^2 = \pi r^2$.
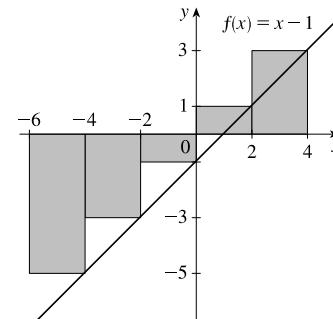
5.2 The Definite Integral

1. $f(x) = x - 1$, $-6 \le x \le 4$. $\Delta x = \dfrac{b - a}{n} = \dfrac{4 - (-6)}{5} = 2$.

Since we are using right endpoints, $x_i^* = x_i$.

$$R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x$$

$$= (\Delta x)[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)]$$

$$= 2[f(-4) + f(-2) + f(0) + f(2) + f(4)]$$

$$= 2[-5 + (-3) + (-1) + 1 + 3]$$

$$= 2(-5) = -10$$

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the three rectangles below the x-axis; that is, the *net area* of the rectangles with respect to the x-axis.
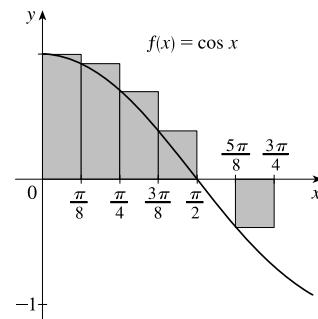
2. $f(x) = \cos x$, $0 \le x \le \frac{3\pi}{4}$. $\Delta x = \dfrac{b-a}{n} = \dfrac{3\pi/4 - 0}{6} = \dfrac{\pi}{8}$.

Since we are using left endpoints, $x_i^* = x_{i-1}$.

$$L_6 = \sum_{i=1}^{6} f(x_{i-1})\,\Delta x$$
$$= (\Delta x)[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$$
$$= \tfrac{\pi}{8}\left[f(0) + f\!\left(\tfrac{\pi}{8}\right) + f\!\left(\tfrac{2\pi}{8}\right) + f\!\left(\tfrac{3\pi}{8}\right) + f\!\left(\tfrac{4\pi}{8}\right) + f\!\left(\tfrac{5\pi}{8}\right)\right]$$
$$\approx 1.033186$$
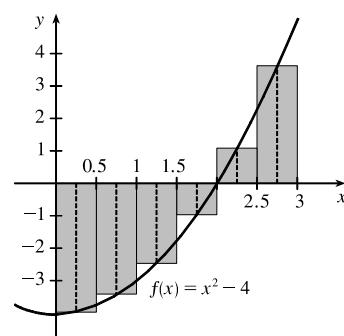
The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the area of the rectangle below the x-axis; that is, the *net area* of the rectangles with respect to the x-axis. A sixth rectangle is degenerate, with height 0, and has no area.

3. $f(x) = x^2 - 4$, $0 \le x \le 3$. $\Delta x = \dfrac{b-a}{n} = \dfrac{3-0}{6} = \dfrac{1}{2}$.

Since we are using midpoints, $x_i^* = \overline{x}_i = \tfrac{1}{2}(x_{i-1} + x_i)$.

$$M_6 = \sum_{i=1}^{6} f(\overline{x}_i)\,\Delta x$$
$$= (\Delta x)[f(\overline{x}_1) + f(\overline{x}_2) + f(\overline{x}_3) + f(\overline{x}_4) + f(\overline{x}_5) + f(\overline{x}_6)]$$
$$= \tfrac{1}{2}\left[f\!\left(\tfrac{1}{4}\right) + f\!\left(\tfrac{3}{4}\right) + f\!\left(\tfrac{5}{4}\right) + f\!\left(\tfrac{7}{4}\right) + f\!\left(\tfrac{9}{4}\right) + f\!\left(\tfrac{11}{4}\right)\right]$$
$$= \tfrac{1}{2}\left(-\tfrac{63}{16} - \tfrac{55}{16} - \tfrac{39}{16} - \tfrac{15}{16} + \tfrac{17}{16} + \tfrac{57}{16}\right) = \tfrac{1}{2}\left(-\tfrac{98}{16}\right) = -\tfrac{49}{16}$$
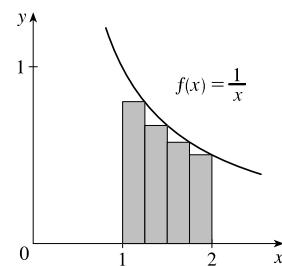
The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the four rectangles below the x-axis; that is, the *net area* of the rectangles with respect to the x-axis.

4. (a) $f(x) = \dfrac{1}{x}$, $1 \le x \le 2$. $\Delta x = \dfrac{b-a}{n} = \dfrac{2-1}{4} = \dfrac{1}{4}$.

Since we are using right endpoints, $x_i^* = x_i$.
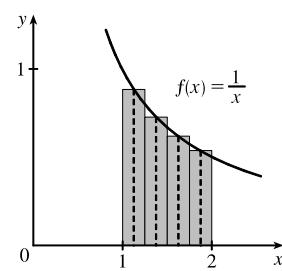
$$R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x$$
$$= (\Delta x)[f(x_1) + f(x_2) + f(x_3) + f(x_4)]$$
$$= \tfrac{1}{4}\left[f\!\left(\tfrac{5}{4}\right) + f\!\left(\tfrac{6}{4}\right) + f\!\left(\tfrac{7}{4}\right) + f\!\left(\tfrac{8}{4}\right)\right]$$
$$= \tfrac{1}{4}\left[\tfrac{4}{5} + \tfrac{2}{3} + \tfrac{4}{7} + \tfrac{1}{2}\right]$$
$$\approx 0.634524$$

The Riemann sum represents the sum of the areas of the four rectangles.

(b) Since we are using midpoints, $x_i^* = \overline{x}_i = \tfrac{1}{2}(x_{i-1} + x_i)$.

$$M_4 = \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x$$
$$= (\Delta x)[f(\overline{x}_1) + f(\overline{x}_2) + f(\overline{x}_3) + f(\overline{x}_4)]$$
$$= \tfrac{1}{4}\left[f\!\left(\tfrac{9}{8}\right) + f\!\left(\tfrac{11}{8}\right) + f\!\left(\tfrac{13}{8}\right) + f\!\left(\tfrac{15}{8}\right)\right]$$
$$= \tfrac{1}{4}\left(\tfrac{8}{9} + \tfrac{8}{11} + \tfrac{8}{13} + \tfrac{8}{15}\right) \approx 0.691220$$

The Riemann sum represents the sum of the areas of the four rectangles.

5. (a) $\displaystyle\int_0^{10} f(x)\,dx \approx R_5 = [f(2) + f(4) + f(6) + f(8) + f(10)]\,\Delta x$

$$= [-1 + 0 + (-2) + 2 + 4](2) = 3(2) = 6$$

(b) $\displaystyle\int_0^{10} f(x)\,dx \approx L_5 = [f(0) + f(2) + f(4) + f(6) + f(8)]\,\Delta x$

$$= [3 + (-1) + 0 + (-2) + 2](2) = 2(2) = 4$$

(c) $\displaystyle\int_0^{10} f(x)\,dx \approx M_5 = [f(1) + f(3) + f(5) + f(7) + f(9)]\,\Delta x$

$$= [0 + (-1) + (-1) + 0 + 3](2) = 1(2) = 2$$

6. (a) $\displaystyle\int_{-2}^{4} g(x)\,dx \approx R_6 = [g(-1) + g(0) + g(1) + g(2) + g(3) + g(4)]\,\Delta x$

$$= \left[-\tfrac{3}{2} + 0 + \tfrac{3}{2} + \tfrac{1}{2} + (-1) + \tfrac{1}{2}\right](1) = 0$$

(b) $\displaystyle\int_{-2}^{4} g(x)\,dx \approx L_6 = [g(-2) + g(-1) + g(0) + g(1) + g(2) + g(3)]\,\Delta x$

$$= \left[0 + \left(-\tfrac{3}{2}\right) + 0 + \tfrac{3}{2} + \tfrac{1}{2} + (-1)\right](1) = -\tfrac{1}{2}$$

(c) $\displaystyle\int_{-2}^{4} g(x)\,dx \approx M_6 = \left[g\left(-\tfrac{3}{2}\right) + g\left(-\tfrac{1}{2}\right) + g\left(\tfrac{1}{2}\right) + g\left(\tfrac{3}{2}\right) + g\left(\tfrac{5}{2}\right) + g\left(\tfrac{7}{2}\right)\right]\Delta x$

$$= \left[-1 + (-1) + 1 + 1 + 0 + \left(-\tfrac{1}{2}\right)\right](1) = -\tfrac{1}{2}$$

7. Since f is increasing, $L_5 \leq \int_{10}^{30} f(x)\,dx \leq R_5$.

Lower estimate $= L_5 = \displaystyle\sum_{i=1}^{5} f(x_{i-1})\Delta x = 4[f(10) + f(14) + f(18) + f(22) + f(26)]$

$$= 4[-12 + (-6) + (-2) + 1 + 3] = 4(-16) = -64$$

Upper estimate $= R_5 = \displaystyle\sum_{i=1}^{5} f(x)\Delta x = 4[f(14) + f(18) + f(22) + f(26) + f(30)]$

$$= 4[-6 + (-2) + 1 + 3 + 8] = 4(4) = 16$$

8. (a) Using the right endpoints to approximate $\int_3^9 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(x_i)\,\Delta x = 2[f(5) + f(7) + f(9)] = 2\,(-0.6 + 0.9 + 1.8) = 4.2.$$

Since f is *increasing*, using *right* endpoints gives an *overestimate*.

(b) Using the left endpoints to approximate $\int_3^9 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(x_{i-1})\,\Delta x = 2[f(3) + f(5) + f(7)] = 2\,(-3.4 - 0.6 + 0.9) = -6.2.$$

Since f is *increasing*, using *left* endpoints gives an *underestimate*.

(c) Using the midpoint of each interval to approximate $\int_3^9 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(\overline{x}_i)\,\Delta x = 2[f(4) + f(6) + f(8)] = 2\,(-2.1 + 0.3 + 1.4) = -0.8.$$

We cannot say anything about the midpoint estimate compared to the exact value of the integral.

9. $\Delta x = (8 - 0)/4 = 2$, so the endpoints are 0, 2, 4, 6, and 8, and the midpoints are 1, 3, 5, and 7. The Midpoint Rule gives

$$\int_0^8 \sin \sqrt{x}\, dx \approx \sum_{i=1}^4 f(\bar{x}_i)\, \Delta x = 2\left(\sin \sqrt{1} + \sin \sqrt{3} + \sin \sqrt{5} + \sin \sqrt{7}\,\right) \approx 2(3.0910) = 6.1820.$$

10. $\Delta x = (1 - 0)/5 = \frac{1}{5}$, so the endpoints are 0, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, and 1, and the midpoints are $\frac{1}{10}$, $\frac{3}{10}$, $\frac{5}{10}$, $\frac{7}{10}$, and $\frac{9}{10}$. The Midpoint Rule gives

$$\int_0^1 \sqrt{x^3 + 1}\, dx \approx \sum_{i=1}^5 f(\bar{x}_i)\, \Delta x = \tfrac{1}{5}\left(\sqrt{\left(\tfrac{1}{10}\right)^3 + 1} + \sqrt{\left(\tfrac{3}{10}\right)^3 + 1} + \sqrt{\left(\tfrac{5}{10}\right)^3 + 1} + \sqrt{\left(\tfrac{7}{10}\right)^3 + 1} + \sqrt{\left(\tfrac{9}{10}\right)^3 + 1}\,\right)$$
$$\approx 1.1097$$

11. $\Delta x = (2 - 0)/5 = \frac{2}{5}$, so the endpoints are 0, $\frac{2}{5}$, $\frac{4}{5}$, $\frac{6}{5}$, $\frac{8}{5}$, and 2, and the midpoints are $\frac{1}{5}$, $\frac{3}{5}$, $\frac{5}{5}$, $\frac{7}{5}$ and $\frac{9}{5}$. The Midpoint Rule gives
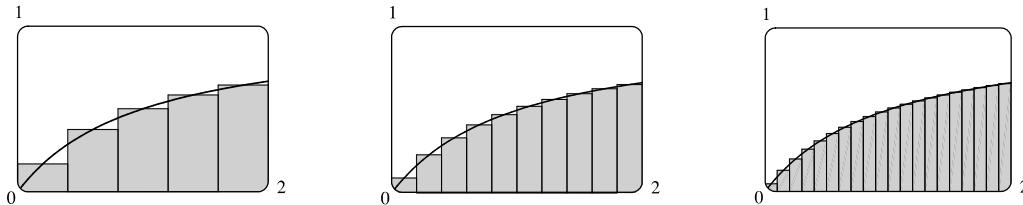
$$\int_0^2 \frac{x}{x + 1}\, dx \approx \sum_{i=1}^5 f(\bar{x}_i)\, \Delta x = \frac{2}{5}\left(\frac{\frac{1}{5}}{\frac{1}{5} + 1} + \frac{\frac{3}{5}}{\frac{3}{5} + 1} + \frac{\frac{5}{5}}{\frac{5}{5} + 1} + \frac{\frac{7}{5}}{\frac{7}{5} + 1} + \frac{\frac{9}{5}}{\frac{9}{5} + 1} \right) = \frac{2}{5}\left(\frac{127}{56}\right) = \frac{127}{140} \approx 0.9071.$$

12. $\Delta x = (\pi - 0)/4 = \frac{\pi}{4}$, so the endpoints are $\frac{\pi}{4}$, $\frac{2\pi}{4}$, $\frac{3\pi}{4}$, and $\frac{4\pi}{4}$, and the midpoints are $\frac{\pi}{8}$, $\frac{3\pi}{8}$, $\frac{5\pi}{8}$, and $\frac{7\pi}{8}$. The Midpoint Rule gives

$$\int_0^\pi x \sin^2 x\, dx \approx \sum_{i=1}^5 f(\bar{x}_i)\, \Delta x = \frac{\pi}{4}\left(\frac{\pi}{8} \sin^2 \frac{\pi}{8} + \frac{3\pi}{8} \sin^2 \frac{3\pi}{8} + \frac{5\pi}{8} \sin^2 \frac{5\pi}{8} + \frac{7\pi}{8} \sin^2 \frac{7\pi}{8} \right) \approx 2.4674$$

13. In Maple 14, use the commands `with(Student[Calculus1])` and

`ReimannSum(x/(x+1),0..2,partition=5,method=midpoint,output=plot)`. In some older versions of

Maple, use `with(student)` to load the sum and box commands, then `m:=middlesum(x/(x+1),x=0..2)`, which

gives us the sum in summation notation, then `M:=evalf(m)` to get the numerical approximation, and finally

`middlebox(x/(x+1),x=0..2)` to generate the graph. The values obtained for $n = 5, 10$, and 20 are 0.9071, 0.9029, and

0.9018, respectively.

14. For $f(x) = x/(x + 1)$ on $[0, 2]$, we calculate $L_{100} \approx 0.89469$ and $R_{100} \approx 0.90802$. Since f is increasing on $[0, 2]$, L_{100} is

an underestimate of $\displaystyle\int_0^2 \frac{x}{x + 1}\, dx$ and R_{100} is an overestimate. Thus, $0.8946 < \displaystyle\int_0^2 \frac{x}{x + 1}\, dx < 0.9081$.

15. We'll create the table of values to approximate $\int_0^\pi \sin x\, dx$ by using the

program in the solution to Exercise 5.1.9 with $Y_1 = \sin x$, $\text{Xmin} = 0$,

$\text{Xmax} = \pi$, and $n = 5, 10, 50$, and 100.

The values of R_n appear to be approaching 2.

n	R_n
5	1.933766
10	1.983524
50	1.999342
100	1.999836

16. $\int_0^2 e^{-x^2}\, dx$ with $n = 5, 10, 50,$ and 100.

n	L_n	R_n
5	1.077467	0.684794
10	0.980007	0.783670
50	0.901705	0.862438
100	0.891896	0.872262

The value of the integral lies between 0.872 and 0.892. Note that

$f(x) = e^{-x^2}$ is decreasing on $(0, 2)$. We cannot make a similar statement

for $\int_{-1}^2 e^{-x^2}\, dx$ since f is increasing on $(-1, 0)$.

17. On $[0, 1]$, $\displaystyle\lim_{n\to\infty} \sum_{i=1}^n \frac{e^{x_i}}{1 + x_i}\, \Delta x = \int_0^1 \frac{e^x}{1 + x}\, dx$.

18. On $[2, 5]$, $\displaystyle\lim_{n\to\infty} \sum_{i=1}^n x_i\sqrt{1 + x_i^3}\, \Delta x = \int_2^5 x\sqrt{1 + x^3}\, dx$.

19. On $[2, 7]$, $\displaystyle\lim_{n\to\infty} \sum_{i=1}^n [5(x_i^*)^3 - 4x_i^*]\, \Delta x = \int_2^7 (5x^3 - 4x)\, dx$.

20. On $[1, 3]$, $\displaystyle\lim_{n\to\infty} \sum_{i=1}^n \frac{x_i^*}{(x_i^*)^2 + 4}\, \Delta x = \int_1^3 \frac{x}{x^2 + 4}\, dx$.

21. Note that $\Delta x = \dfrac{5 - 2}{n} = \dfrac{3}{n}$ and $x_i = 2 + i\,\Delta x = 2 + \dfrac{3i}{n}$.

$$\int_2^5 (4 - 2x)\, dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^n f\left(2 + \frac{3i}{n}\right)\frac{3}{n} = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^n \left[4 - 2\left(2 + \frac{3i}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^n \left[-\frac{6i}{n}\right] = \lim_{n\to\infty} \frac{3}{n}\left(-\frac{6}{n}\right) \sum_{i=1}^n i = \lim_{n\to\infty} \left(-\frac{18}{n^2}\right)\left[\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left(-\frac{18}{2}\right)\left(\frac{n+1}{n}\right) = -9 \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = -9(1) = -9$$

22. Note that $\Delta x = \dfrac{4 - 1}{n} = \dfrac{3}{n}$ and $x_i = 1 + i\,\Delta x = 1 + \dfrac{3i}{n}$.

$$\int_1^4 (x^2 - 4x + 2)\, dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^n f\left(1 + \frac{3i}{n}\right)\frac{3}{n} = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^n \left[\left(1 + \frac{3i}{n}\right)^2 - 4\left(1 + \frac{3i}{n}\right) + 2\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^n \left[1 + \frac{6i}{n} + \frac{9i^2}{n^2} - 4 - \frac{12i}{n} + 2\right] = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^n \left[\frac{9i^2}{n^2} - \frac{6i}{n} - 1\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[\frac{9}{n^2} \sum_{i=1}^n i^2 - \frac{6}{n} \sum_{i=1}^n i - \sum_{i=1}^n 1\right]$$

$$= \lim_{n\to\infty} \left[\frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{18}{n^2}\frac{n(n+1)}{2} - \frac{3}{n}\cdot n(1)\right]$$

$$= \lim_{n\to\infty} \left[\frac{9}{2}\frac{(n+1)(2n+1)}{n^2} - 9\frac{n+1}{n} - 3\right] = \lim_{n\to\infty} \left[\frac{9}{2}\frac{n+1}{n}\frac{2n+1}{n} - 9\left(1 + \frac{1}{n}\right) - 3\right]$$

$$= \lim_{n\to\infty} \left[\frac{9}{2}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - 9\left(1 + \frac{1}{n}\right) - 3\right] = \frac{9}{2}(1)(2) - 9(1) - 3 = -3$$

23. Note that $\Delta x = \dfrac{0-(-2)}{n} = \dfrac{2}{n}$ and $x_i = -2 + i\,\Delta x = -2 + \dfrac{2i}{n}$.

$$\int_{-2}^{0} (x^2 + x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\!\left(-2+\frac{2i}{n}\right)\frac{2}{n} = \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[\left(-2+\frac{2i}{n}\right)^2 + \left(-2+\frac{2i}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[4 - \frac{8i}{n} + \frac{4i^2}{n^2} - 2 + \frac{2i}{n}\right] = \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left(\frac{4i^2}{n^2} - \frac{6i}{n} + 2\right)$$

$$= \lim_{n\to\infty}\frac{2}{n}\left[\frac{4}{n^2}\sum_{i=1}^{n} i^2 - \frac{6}{n}\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2\right] = \lim_{n\to\infty}\left[\frac{8}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{12}{n^2}\frac{n(n+1)}{2} + \frac{2}{n}\cdot n(2)\right]$$

$$= \lim_{n\to\infty}\left[\frac{4}{3}\frac{(n+1)(2n+1)}{n^2} - 6\frac{n+1}{n} + 4\right] = \lim_{n\to\infty}\left[\frac{4}{3}\frac{n+1}{n}\frac{2n+1}{n} - 6\left(1+\frac{1}{n}\right) + 4\right]$$

$$= \lim_{n\to\infty}\left[\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 6\left(1+\frac{1}{n}\right) + 4\right] = \frac{4}{3}(1)(2) - 6(1) + 4 = \frac{2}{3}$$

**24.** Note that $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{2i}{n}$.
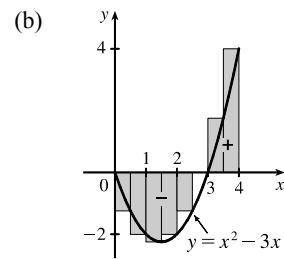
$$\int_{0}^{2} (2x - x^3)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\!\left(\frac{2i}{n}\right)\frac{2}{n} = \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[2\left(\frac{2i}{n}\right) - \left(\frac{2i}{n}\right)^3\right]$$

$$= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[\frac{4i}{n} - \frac{8i^3}{n^3}\right] = \lim_{n\to\infty}\frac{2}{n}\left[\frac{4}{n}\sum_{i=1}^{n} i - \frac{8}{n^3}\sum_{i=1}^{n} i^3\right]$$

$$= \lim_{n\to\infty}\left\{\frac{8}{n^2}\frac{n(n+1)}{2} - \frac{16}{n^4}\left[\frac{n(n+1)}{2}\right]^2\right\} = \lim_{n\to\infty}\left[4\frac{n+1}{n} - 4\frac{(n+1)^2}{n^2}\right]$$

$$= \lim_{n\to\infty}\left[4\left(1+\frac{1}{n}\right) - 4\frac{n+1}{n}\frac{n+1}{n}\right] = \lim_{n\to\infty}\left[4\left(1+\frac{1}{n}\right) - 4\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\right]$$

$$= 4(1) - 4(1)(1) = 0$$

25. Note that $\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{i}{n}$.

$$\int_{0}^{1} (x^3 - 3x^2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\!\left(\frac{i}{n}\right)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^3 - 3\left(\frac{i}{n}\right)^2\right]\frac{1}{n}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{i^3}{n^3} - \frac{3i^2}{n^2}\right] = \lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{n^3}\sum_{i=1}^{n} i^3 - \frac{3}{n^2}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty}\left\{\frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{3}{n^3}\frac{n(n+1)(2n+1)}{6}\right\} = \lim_{n\to\infty}\left[\frac{1}{4}\frac{n+1}{n}\frac{n+1}{n} - \frac{1}{2}\frac{n+1}{n}\frac{2n+1}{n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{4}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right) - \frac{1}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = \frac{1}{4}(1)(1) - \frac{1}{2}(1)(2) = -\frac{3}{4}$$

26. (a) $\Delta x = (4-0)/8 = 0.5$ and $x_i^* = x_i = 0.5i$.
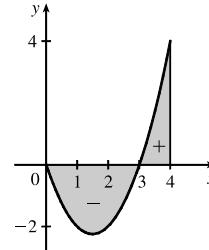
$$\int_{0}^{4} (x^2 - 3x)\,dx \approx \sum_{i=1}^{8} f(x_i^*)\,\Delta x$$

$$= 0.5\big\{\big[0.5^2 - 3(0.5)\big] + \big[1.0^2 - 3(1.0)\big] + \cdots$$
$$+ \big[3.5^2 - 3(3.5)\big] + \big[4.0^2 - 3(4.0)\big]\big\}$$
$$= \tfrac{1}{2}\left(-\tfrac{5}{4} - 2 - \tfrac{9}{4} - 2 - \tfrac{5}{4} + 0 + \tfrac{7}{4} + 4\right) = -1.5$$

(b)

$y = x^2 - 3x$

(c) $\displaystyle\int_0^4 (x^2 - 3x)\, dx = \lim_{n\to\infty} \sum_{i=1}^n \left[\left(\frac{4i}{n}\right)^2 - 3\left(\frac{4i}{n}\right) \right]\left(\frac{4}{n}\right)$

$$= \lim_{n\to\infty} \frac{4}{n}\left[\frac{16}{n^2}\sum_{i=1}^n i^2 - \frac{12}{n}\sum_{i=1}^n i \right]$$

$$= \lim_{n\to\infty}\left[\frac{64}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{48}{n^2}\cdot\frac{n(n+1)}{2} \right]$$

$$= \lim_{n\to\infty}\left[\frac{32}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 24\left(1+\frac{1}{n}\right) \right]$$

$$= \tfrac{32}{3}\cdot 2 - 24 = -\tfrac{8}{3}$$

(d) $\int_0^4 (x^2 - 3x)\, dx = A_1 - A_2$, where A_1 is the area marked $+$ and A_2 is the area marked $-$.

27. $\displaystyle\int_a^b x\, dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^n\left[a + \frac{b-a}{n}i \right] = \lim_{n\to\infty}\left[\frac{a(b-a)}{n}\sum_{i=1}^n 1 + \frac{(b-a)^2}{n^2}\sum_{i=1}^n i \right]$

$$= \lim_{n\to\infty}\left[\frac{a(b-a)}{n}n + \frac{(b-a)^2}{n^2}\cdot\frac{n(n+1)}{2} \right] = a(b-a) + \lim_{n\to\infty}\frac{(b-a)^2}{2}\left(1+\frac{1}{n}\right)$$

$$= a(b-a) + \tfrac{1}{2}(b-a)^2 = (b-a)\left(a + \tfrac{1}{2}b - \tfrac{1}{2}a\right) = (b-a)\tfrac{1}{2}(b+a) = \tfrac{1}{2}\left(b^2 - a^2\right)$$

28. $\displaystyle\int_a^b x^2\, dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^n\left[a + \frac{b-a}{n}i \right]^2 = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^n\left[a^2 + 2a\frac{b-a}{n}i + \frac{(b-a)^2}{n^2}i^2 \right]$

$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{n^3}\sum_{i=1}^n i^2 + \frac{2a(b-a)^2}{n^2}\sum_{i=1}^n i + \frac{a^2(b-a)}{n}\sum_{i=1}^n 1 \right]$$

$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{2a(b-a)^2}{n^2}\frac{n(n+1)}{2} + \frac{a^2(b-a)}{n}n \right]$$

$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{6}\cdot 1\cdot\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + a(b-a)^2\cdot 1\cdot\left(1+\frac{1}{n}\right) + a^2(b-a) \right]$$

$$= \frac{(b-a)^3}{3} + a(b-a)^2 + a^2(b-a) = \frac{b^3 - 3ab^2 + 3a^2b - a^3}{3} + ab^2 - 2a^2b + a^3 + a^2b - a^3$$

$$= \frac{b^3}{3} - \frac{a^3}{3} - ab^2 + a^2b + ab^2 - a^2b = \frac{b^3 - a^3}{3}$$

29. $f(x) = \sqrt{4 + x^2}$, $a = 1$, $b = 3$, and $\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$. Using Theorem 4, we get $x_i^* = x_i = 1 + i\,\Delta x = 1 + \dfrac{2i}{n}$, so

$$\int_1^3 \sqrt{4+x^2}\, dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^n \sqrt{4 + \left(1 + \frac{2i}{n}\right)^2}\cdot\frac{2}{n}.$$

30. $f(x) = x^2 + \dfrac{1}{x}$, $a = 2$, $b = 5$, and $\Delta x = \dfrac{5-2}{n} = \dfrac{3}{n}$. Using Theorem 4, we get $x_i^* = x_i = 2 + i\,\Delta x = 2 + \dfrac{3i}{n}$, so

$$\int_2^5\left(x^2 + \frac{1}{x}\right) dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^n\left[\left(2 + \frac{3i}{n}\right)^2 + \frac{1}{2 + \dfrac{3i}{n}} \right]\cdot\frac{3}{n}.$$

31. $\Delta x = (\pi - 0)/n = \pi/n$ and $x_i^* = x_i = \pi i/n$.

$$\int_0^\pi \sin 5x\, dx = \lim_{n\to\infty}\sum_{i=1}^n (\sin 5x_i)\left(\frac{\pi}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^n\left(\sin\frac{5\pi i}{n}\right)\frac{\pi}{n} \overset{\text{CAS}}{=} \pi\lim_{n\to\infty}\frac{1}{n}\cot\left(\frac{5\pi}{2n}\right) \overset{\text{CAS}}{=} \pi\left(\frac{2}{5\pi}\right) = \frac{2}{5}$$

32. $\Delta x = (10 - 2)/n = 8/n$ and $x_i^* = x_i = 2 + 8i/n$.

$$\int_2^{10} x^6\, dx = \lim_{n \to \infty} \sum_{i=1}^n \left(2 + \frac{8i}{n}\right)^6 \left(\frac{8}{n}\right) = 8 \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \left(2 + \frac{8i}{n}\right)^6$$

$$\overset{\text{CAS}}{=} 8 \lim_{n \to \infty} \frac{1}{n} \cdot \frac{64\left(58{,}593n^6 + 164{,}052n^5 + 131{,}208n^4 - 27{,}776n^2 + 2048\right)}{21n^5}$$

$$\overset{\text{CAS}}{=} 8\left(\frac{1{,}249{,}984}{7}\right) = \frac{9{,}999{,}872}{7} \approx 1{,}428{,}553.1$$

33. (a) Think of $\int_0^2 f(x)\, dx$ as the area of a trapezoid with bases 1 and 3 and height 2. The area of a trapezoid is $A = \frac{1}{2}(b + B)h$,

so $\int_0^2 f(x)\, dx = \frac{1}{2}(1 + 3)2 = 4$.

(b) $\int_0^5 f(x)\, dx = \int_0^2 f(x)\, dx + \int_2^3 f(x)\, dx + \int_3^5 f(x)\, dx$

                 trapezoid   rectangle   triangle

          $= \frac{1}{2}(1 + 3)2 +$ $3 \cdot 1$ $+ \quad \frac{1}{2} \cdot 2 \cdot 3 \quad = 4 + 3 + 3 = 10$

(c) $\int_5^7 f(x)\, dx$ is the negative of the area of the triangle with base 2 and height 3. $\int_5^7 f(x)\, dx = -\frac{1}{2} \cdot 2 \cdot 3 = -3$.

(d) $\int_7^9 f(x)\, dx$ is the negative of the area of a trapezoid with bases 3 and 2 and height 2, so it equals

    $-\frac{1}{2}(B + b)h = -\frac{1}{2}(3 + 2)2 = -5$. Thus,

    $\int_0^9 f(x)\, dx = \int_0^5 f(x)\, dx + \int_5^7 f(x)\, dx + \int_7^9 f(x)\, dx = 10 + (-3) + (-5) = 2$.

34. (a) $\int_0^2 g(x)\, dx = \frac{1}{2} \cdot 4 \cdot 2 = 4$ [area of a triangle]
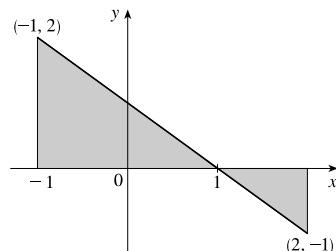
(b) $\int_2^6 g(x)\, dx = -\frac{1}{2}\pi(2)^2 = -2\pi$ [negative of the area of a semicircle]

(c) $\int_6^7 g(x)\, dx = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$ [area of a triangle]

    $\int_0^7 g(x)\, dx = \int_0^2 g(x)\, dx + \int_2^6 g(x)\, dx + \int_6^7 g(x)\, dx = 4 - 2\pi + \frac{1}{2} = 4.5 - 2\pi$
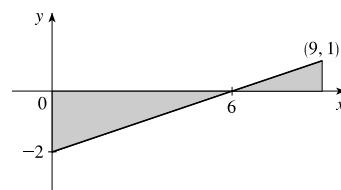
35. $\int_{-1}^2 (1 - x)\, dx$ can be interpreted as the difference of the areas of the two

shaded triangles; that is, $\frac{1}{2}(2)(2) - \frac{1}{2}(1)(1) = 2 - \frac{1}{2} = \frac{3}{2}$.

36. $\int_0^9 \left(\frac{1}{3}x - 2\right) dx$ can be interpreted as the difference of the areas of the two

shaded triangles; that is, $-\frac{1}{2}(6)(2) + \frac{1}{2}(3)(1) = -6 + \frac{3}{2} = -\frac{9}{2}$.
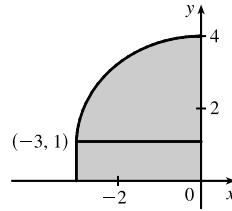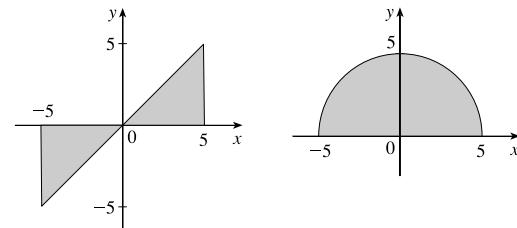
37. $\int_{-3}^{0}\left(1+\sqrt{9-x^2}\,\right)dx$ can be interpreted as the area under the graph of

$f(x) = 1 + \sqrt{9-x^2}$ between $x = -3$ and $x = 0$. This is equal to one-quarter

the area of the circle with radius 3, plus the area of the rectangle, so
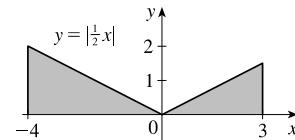
$\int_{-3}^{0}\left(1+\sqrt{9-x^2}\,\right)dx = \frac{1}{4}\pi \cdot 3^2 + 1 \cdot 3 = 3 + \frac{9}{4}\pi.$
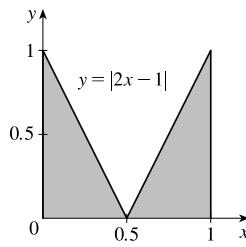
38. $\int_{-5}^{5}\left(x - \sqrt{25-x^2}\,\right)dx = \int_{-5}^{5} x\, dx - \int_{-5}^{5}\sqrt{25-x^2}\, dx.$ By

symmetry, the value of the first integral is 0 since the shaded

area above the x-axis equals the shaded area below the x-axis.

The second integral can be interpreted as one half the area of a

circle with radius 5; that is, $\frac{1}{2}\pi(5)^2 = \frac{25}{2}\pi.$ Thus, the value of

the original integral is $0 - \frac{25}{2}\pi = -\frac{25}{2}\pi.$

39. $\int_{-4}^{3}\left|\frac{1}{2}x\right|\,dx$ can be interpreted as the sum of the areas of the two shaded

triangles; that is, $\frac{1}{2}(4)(2) + \frac{1}{2}(3)\left(\frac{3}{2}\right) = 4 + \frac{9}{4} = \frac{25}{4}.$

40. $\int_{0}^{1}|2x - 1|\,dx$ can be interpreted as the sum of the areas of the two shaded

triangles; that is, $2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(1) = \frac{1}{2}.$

41. $\int_{1}^{1}\sqrt{1+x^4}\,dx = 0$ since the limits of integration are equal.

42. $\int_{\pi}^{0}\sin^4\theta\,d\theta = -\int_{0}^{\pi}\sin^4\theta\,d\theta$ [because we reversed the limits of integration]

$\qquad\qquad = -\int_{0}^{\pi}\sin^4 x\,dx$ [we can use any letter without changing the value of the integral]

$\qquad\qquad = -\frac{3}{8}\pi$ [given value]

43. $\int_{0}^{1}\left(5 - 6x^2\right)dx = \int_{0}^{1} 5\,dx - 6\int_{0}^{1} x^2\,dx = 5(1-0) - 6\left(\frac{1}{3}\right) = 5 - 2 = 3$

**44.** $\int_{1}^{3}\left(2e^x - 1\right)dx = 2\int_{1}^{3} e^x\,dx - \int_{1}^{3} 1\,dx = 2(e^3 - e) - 1(3-1) = 2e^3 - 2e - 2$

45. $\int_{1}^{3} e^{x+2}\,dx = \int_{1}^{3} e^x \cdot e^2\,dx = e^2 \int_{1}^{3} e^x\,dx = e^2(e^3 - e) = e^5 - e^3$

46. $\int_{0}^{\pi/2}(2\cos x - 5x)\,dx = \int_{0}^{\pi/2} 2\cos x\,dx - \int_{0}^{\pi/2} 5x\,dx = 2\int_{0}^{\pi/2}\cos x\,dx - 5\int_{0}^{\pi/2} x\,dx$

$\qquad\qquad\qquad = 2(1) - 5\dfrac{(\pi/2)^2 - 0^2}{2} = 2 - \dfrac{5\pi^2}{8}$

47. $\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx = \int_{-2}^{5} f(x)\,dx + \int_{-1}^{-2} f(x)\,dx$ [by Property 5 and reversing limits]

$\qquad\qquad\qquad\qquad\qquad = \int_{-1}^{5} f(x)\,dx$ [Property 5]
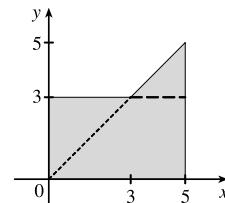
48. $\int_2^4 f(x)\,dx + \int_4^8 f(x)\,dx = \int_2^8 f(x)\,dx$, so $\int_4^8 f(x)\,dx = \int_2^8 f(x)\,dx - \int_2^4 f(x)\,dx = 7.3 - 5.9 = 1.4$.

49. $\int_0^9 [2f(x) + 3g(x)]\,dx = 2\int_0^9 f(x)\,dx + 3\int_0^9 g(x)\,dx = 2(37) + 3(16) = 122$

50. If $f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geq 3 \end{cases}$, then $\int_0^5 f(x)\,dx$ can be interpreted as the area of the shaded

region, which consists of a 5-by-3 rectangle surmounted by an isosceles right triangle

whose legs have length 2. Thus, $\int_0^5 f(x)\,dx = 5(3) + \frac{1}{2}(2)(2) = 17$.

51. $\int_0^3 f(x)\,dx$ is clearly less than -1 and has the smallest value. The slope of the tangent line of f at $x = 1$, $f'(1)$, has a value

between -1 and 0, so it has the next smallest value. The largest value is $\int_3^8 f(x)\,dx$, followed by $\int_4^8 f(x)\,dx$, which has a

value about 1 unit less than $\int_3^8 f(x)\,dx$. Still positive, but with a smaller value than $\int_4^8 f(x)\,dx$, is $\int_0^8 f(x)\,dx$. Ordering these

quantities from smallest to largest gives us

$$\int_0^3 f(x)\,dx < f'(1) < \int_0^8 f(x)\,dx < \int_4^8 f(x)\,dx < \int_3^8 f(x)\,dx \text{ or } \text{B} < \text{E} < \text{A} < \text{D} < \text{C}$$
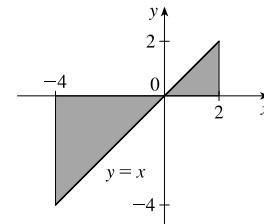
52. $F(0) = \int_2^0 f(t)\,dt = -\int_0^2 f(t)\,dt$, so $F(0)$ is negative, and similarly, so is $F(1)$. $F(3)$ and $F(4)$ are negative since they

represent negatives of areas below the x-axis. Since $F(2) = \int_2^2 f(t)\,dt = 0$ is the only non-negative value, choice C is the

largest.

53. $I = \int_{-4}^2 [f(x) + 2x + 5]\,dx = \int_{-4}^2 f(x)\,dx + 2\int_{-4}^2 x\,dx + \int_{-4}^2 5\,dx = I_1 + 2I_2 + I_3$

$I_1 = -3$ [area below x-axis] $+3 - 3 = -3$

$I_2 = -\frac{1}{2}(4)(4)$ [area of triangle, see figure] $+\frac{1}{2}(2)(2)$

$\quad = -8 + 2 = -6$

$I_3 = 5[2 - (-4)] = 5(6) = 30$

Thus, $I = -3 + 2(-6) + 30 = 15$.

54. Using Integral Comparison Property 8, $m \leq f(x) \leq M \quad \Rightarrow \quad m(2 - 0) \leq \int_0^2 f(x)\,dx \leq M(2 - 0) \quad \Rightarrow$

$2m \leq \int_0^2 f(x)\,dx \leq 2M$.

55. $x^2 - 4x + 4 = (x - 2)^2 \geq 0$ on $[0, 4]$, so $\int_0^4 (x^2 - 4x + 4)\,dx \geq 0$ [Property 6].

56. $x^2 \leq x$ on $[0, 1]$, so $\sqrt{1 + x^2} \leq \sqrt{1 + x}$ on $[0, 1]$. Hence, $\int_0^1 \sqrt{1 + x^2}\,dx \leq \int_0^1 \sqrt{1 + x}\,dx$ [Property 7].

57. If $-1 \leq x \leq 1$, then $0 \leq x^2 \leq 1$ and $1 \leq 1 + x^2 \leq 2$, so $1 \leq \sqrt{1 + x^2} \leq \sqrt{2}$ and

$1[1 - (-1)] \leq \int_{-1}^1 \sqrt{1 + x^2}\,dx \leq \sqrt{2}\,[1 - (-1)]$ [Property 8]; that is, $2 \leq \int_{-1}^1 \sqrt{1 + x^2}\,dx \leq 2\sqrt{2}$.

58. If $\frac{\pi}{6} \leq x \leq \frac{\pi}{3}$, then $\frac{1}{2} \leq \sin x \leq \frac{\sqrt{3}}{2}$ ($\sin x$ is increasing on $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$), so

$\frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) \leq \int_{\pi/6}^{\pi/3} \sin x\,dx \leq \frac{\sqrt{3}}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$ [Property 8]; that is, $\frac{\pi}{12} \leq \int_{\pi/6}^{\pi/3} \sin x\,dx \leq \frac{\sqrt{3}\,\pi}{12}$.

59. If $0 \le x \le 1$, then $0 \le x^3 \le 1$, so $0(1 - 0) \le \int_0^1 x^3 \, dx \le 1(1 - 0)$ [Property 8]; that is, $0 \le \int_0^1 x^3 \, dx \le 1$.

60. If $0 \le x \le 3$, then $4 \le x + 4 \le 7$ and $\frac{1}{7} \le \frac{1}{x + 4} \le \frac{1}{4}$, so $\frac{1}{7}(3 - 0) \le \int_0^3 \frac{1}{x + 4} \, dx \le \frac{1}{4}(3 - 0)$ [Property 8]; that is,

$\frac{3}{7} \le \int_0^3 \frac{1}{x + 4} \, dx \le \frac{3}{4}$.

61. If $\frac{\pi}{4} \le x \le \frac{\pi}{3}$, then $1 \le \tan x \le \sqrt{3}$, so $1\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \le \int_{\pi/4}^{\pi/3} \tan x \, dx \le \sqrt{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ or $\frac{\pi}{12} \le \int_{\pi/4}^{\pi/3} \tan x \, dx \le \frac{\pi}{12}\sqrt{3}$.

62. Let $f(x) = x^3 - 3x + 3$ for $0 \le x \le 2$. Then $f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$, so f is decreasing on $(0, 1)$ and

increasing on $(1, 2)$. f has the absolute minimum value $f(1) = 1$. Since $f(0) = 3$ and $f(2) = 5$, the absolute maximum

value of f is $f(2) = 5$. Thus, $1 \le x^3 - 3x + 3 \le 5$ for x in $[0, 2]$. It follows from Property 8 that

$1 \cdot (2 - 0) \le \int_0^2 \left(x^3 - 3x + 3\right) dx \le 5 \cdot (2 - 0)$; that is, $2 \le \int_0^2 \left(x^3 - 3x + 3\right) dx \le 10$.

63. The only critical number of $f(x) = xe^{-x}$ on $[0, 2]$ is $x = 1$. Since $f(0) = 0$, $f(1) = e^{-1} \approx 0.368$, and

$f(2) = 2e^{-2} \approx 0.271$, we know that the absolute minimum value of f on $[0, 2]$ is 0, and the absolute maximum is e^{-1}. By

Property 8, $0 \le xe^{-x} \le e^{-1}$ for $0 \le x \le 2$ $\Rightarrow$ $0(2 - 0) \le \int_0^2 xe^{-x} \, dx \le e^{-1}(2 - 0)$ $\Rightarrow$ $0 \le \int_0^2 xe^{-x} \, dx \le 2/e$.

64. Let $f(x) = x - 2 \sin x$ for $\pi \le x \le 2\pi$. Then $f'(x) = 1 - 2 \cos x$ and $f'(x) = 0$ $\Rightarrow$ $\cos x = \frac{1}{2}$ $\Rightarrow$ $x = \frac{5\pi}{3}$.

f has the absolute maximum value $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2 \sin \frac{5\pi}{3} = \frac{5\pi}{3} + \sqrt{3} \approx 6.97$ since $f(\pi) = \pi$ and $f(2\pi) = 2\pi$ are both

smaller than 6.97. Thus, $\pi \le f(x) \le \frac{5\pi}{3} + \sqrt{3}$ $\Rightarrow$ $\pi(2\pi - \pi) \le \int_\pi^{2\pi} f(x) \, dx \le \left(\frac{5\pi}{3} + \sqrt{3}\right)(2\pi - \pi)$; that is,

$\pi^2 \le \int_\pi^{2\pi} (x - 2 \sin x) \, dx \le \frac{5}{3}\pi^2 + \sqrt{3}\,\pi$.

65. $\sqrt{x^4 + 1} \ge \sqrt{x^4} = x^2$, so $\int_1^3 \sqrt{x^4 + 1} \, dx \ge \int_1^3 x^2 \, dx = \frac{1}{3}\left(3^3 - 1^3\right) = \frac{26}{3}$.

66. $0 \le \sin x \le 1$ for $0 \le x \le \frac{\pi}{2}$, so $x \sin x \le x$ $\Rightarrow$ $\int_0^{\pi/2} x \sin x \, dx \le \int_0^{\pi/2} x \, dx = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - 0^2\right] = \frac{\pi^2}{8}$.

67. $\sin x < \sqrt{x} < x$ for $1 \le x \le 2$ and arctan is an increasing function, so $\arctan(\sin x) < \arctan \sqrt{x} < \arctan x$, and hence,

$\int_1^2 \arctan(\sin x) \, dx < \int_1^2 \arctan \sqrt{x} \, dx < \int_1^2 \arctan x \, dx$. Thus, $\int_1^2 \arctan x \, dx$ has the largest value.

68. $x^2 < \sqrt{x}$ for $0 < x \le 0.5$ and cosine is a decreasing function on $[0, 0.5]$, so $\cos(x^2) > \cos \sqrt{x}$, and hence,

$\int_0^{0.5} \cos(x^2) \, dx > \int_0^{0.5} \cos \sqrt{x} \, dx$. Thus, $\int_0^{0.5} \cos(x^2) \, dx$ is larger.

69. Using right endpoints as in the proof of Property 2, we calculate

$\int_a^b cf(x) \, dx = \lim\limits_{n \to \infty} \sum\limits_{i=1}^n cf(x_i) \, \Delta x = \lim\limits_{n \to \infty} c \sum\limits_{i=1}^n f(x_i) \, \Delta x = c \lim\limits_{n \to \infty} \sum\limits_{i=1}^n f(x_i) \, \Delta x = c \int_a^b f(x) \, dx$.

70. (a) Since $-|f(x)| \le f(x) \le |f(x)|$, it follows from Property 7 that

$$-\int_a^b |f(x)| \, dx \le \int_a^b f(x) \, dx \le \int_a^b |f(x)| \, dx \quad \Rightarrow \quad \left|\int_a^b f(x) \, dx\right| \le \int_a^b |f(x)| \, dx$$

Note that the definite integral is a real number, and so the following property applies: $-a \le b \le a$ $\Rightarrow$ $|b| \le a$ for all

real numbers b and nonnegative numbers a.

(b) $\left| \int_0^{2\pi} f(x) \sin 2x \, dx \right| \leq \int_0^{2\pi} |f(x) \sin 2x| \, dx$ [by part (a)] $= \int_0^{2\pi} |f(x)| \, |\sin 2x| \, dx \leq \int_0^{2\pi} |f(x)| \, dx$ by Property 7,

since $|\sin 2x| \leq 1 \implies |f(x)| \, |\sin 2x| \leq |f(x)|$.

71. Suppose that f is integrable on $[0, 1]$, that is, $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} f(x_i^*) \, \Delta x$ exists for any choice of x_i^* in $[x_{i-1}, x_i]$. Let n denote a

positive integer and divide the interval $[0, 1]$ into n equal subintervals $\left[0, \dfrac{1}{n}\right]$, $\left[\dfrac{1}{n}, \dfrac{2}{n}\right]$, ..., $\left[\dfrac{n-1}{n}, 1\right]$. If we choose x_i^* to be

a rational number in the ith subinterval, then we obtain the Riemann sum $\sum\limits_{i=1}^{n} f(x_i^*) \cdot \dfrac{1}{n} = 0$, so

$\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} f(x_i^*) \cdot \dfrac{1}{n} = \lim\limits_{n \to \infty} 0 = 0$. Now suppose we choose x_i^* to be an irrational number. Then we get

$\sum\limits_{i=1}^{n} f(x_i^*) \cdot \dfrac{1}{n} = \sum\limits_{i=1}^{n} 1 \cdot \dfrac{1}{n} = n \cdot \dfrac{1}{n} = 1$ for each n, so $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} f(x_i^*) \cdot \dfrac{1}{n} = \lim\limits_{n \to \infty} 1 = 1$. Since the value of

$\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} f(x_i^*) \, \Delta x$ depends on the choice of the sample points x_i^*, the limit does not exist, and f is not integrable on $[0, 1]$.

72. Partition the interval $[0, 1]$ into n equal subintervals and choose $x_1^* = \dfrac{1}{n^2}$. Then with $f(x) = \dfrac{1}{x}$,

$\sum\limits_{i=1}^{n} f(x_i^*) \, \Delta x \geq f(x_1^*) \Delta x = \dfrac{1}{1/n^2} \cdot \dfrac{1}{n} = n$. Thus, $\sum\limits_{i=1}^{n} f(x_i^*) \, \Delta x$ can be made arbitrarily large and hence, f is not integrable

on $[0, 1]$.

73. $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \dfrac{i^4}{n^5} = \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \dfrac{i^4}{n^4} \cdot \dfrac{1}{n} = \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \left(\dfrac{i}{n}\right)^4 \dfrac{1}{n}$. At this point, we need to recognize the limit as being of the form

$\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} f(x_i) \, \Delta x$, where $\Delta x = (1 - 0)/n = 1/n$, $x_i = 0 + i \, \Delta x = i/n$, and $f(x) = x^4$. Thus, the definite integral

is $\int_0^1 x^4 \, dx$.
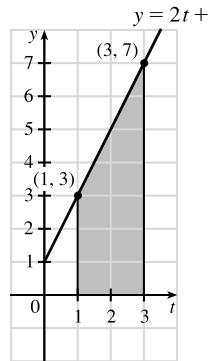
74. $\lim\limits_{n \to \infty} \dfrac{1}{n} \sum\limits_{i=1}^{n} \dfrac{1}{1 + (i/n)^2} = \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \dfrac{1}{1 + (i/n)^2} \cdot \dfrac{1}{n} = \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} f(x_i) \, \Delta x$, where $\Delta x = (1 - 0)/n = 1/n$,

$x_i = 0 + i \, \Delta x = i/n$, and $f(x) = \dfrac{1}{1 + x^2}$. Thus, the definite integral is $\displaystyle\int_0^1 \dfrac{dx}{1 + x^2}$.

75. Choose $x_i = 1 + \dfrac{i}{n}$ and $x_i^* = \sqrt{x_{i-1} x_i} = \sqrt{\left(1 + \dfrac{i-1}{n}\right)\left(1 + \dfrac{i}{n}\right)}$. Then

$$\int_1^2 x^{-2} \, dx = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\left(1 + \frac{i-1}{n}\right)\left(1 + \frac{i}{n}\right)} = \lim_{n \to \infty} n \sum_{i=1}^{n} \frac{1}{(n+i-1)(n+i)}$$

$$= \lim_{n \to \infty} n \sum_{i=1}^{n} \left(\frac{1}{n+i-1} - \frac{1}{n+i}\right) \quad \text{[by the hint]} = \lim_{n \to \infty} n \left(\sum_{i=0}^{n-1} \frac{1}{n+i} - \sum_{i=1}^{n} \frac{1}{n+i}\right)$$

$$= \lim_{n \to \infty} n \left(\left[\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}\right] - \left[\frac{1}{n+1} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right]\right)$$

$$= \lim_{n \to \infty} n \left(\frac{1}{n} - \frac{1}{2n}\right) = \lim_{n \to \infty} \left(1 - \tfrac{1}{2}\right) = \tfrac{1}{2}$$

DISCOVERY PROJECT   Area Functions

1. (a)

$$y = 2t + 1$$

$(3, 7)$

$(1, 3)$
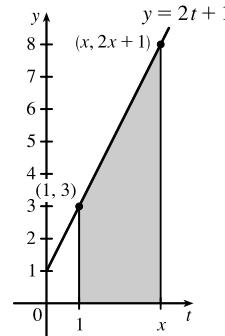
Area of trapezoid $= \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}(3 + 7)2$
$$= 10 \text{ square units}$$

Or:

Area of rectangle $+$ area of triangle

$$= b_r h_r + \frac{1}{2} b_t h_t = (2)(3) + \frac{1}{2}(2)(4) = 10 \text{ square units}$$

(b)

$$y = 2t + 1$$

$(x, 2x + 1)$

$(1, 3)$

As in part (a),

$$A(x) = \frac{1}{2}[3 + (2x + 1)](x - 1) = \frac{1}{2}(2x + 4)(x - 1)$$
$$= (x + 2)(x - 1) = x^2 + x - 2 \text{ square units}$$

(c) $A'(x) = 2x + 1$. This is the y-coordinate of the point $(x, 2x + 1)$ on the given line.

2. (a)

$y = 1 + t^2$

$(x, 1 + x^2)$

$A(x)$
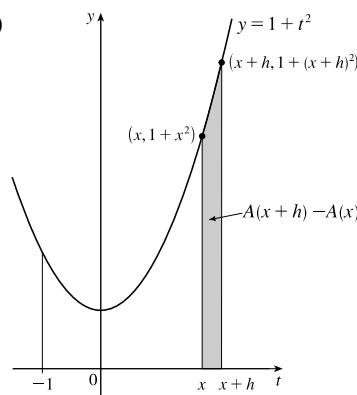
(b) $A(x) = \int_{-1}^{x} \left(1 + t^2\right) dt = \int_{-1}^{x} 1 \, dt + \int_{-1}^{x} t^2 \, dt$ [Property 2]

$$= 1[x - (-1)] + \frac{x^3 - (-1)^3}{3} \quad \begin{bmatrix} \text{Property 1 and} \\ \text{Exercise 5.2.28} \end{bmatrix}$$

$$= x + 1 + \frac{1}{3}x^3 + \frac{1}{3}$$

$$= \frac{1}{3}x^3 + x + \frac{4}{3}$$
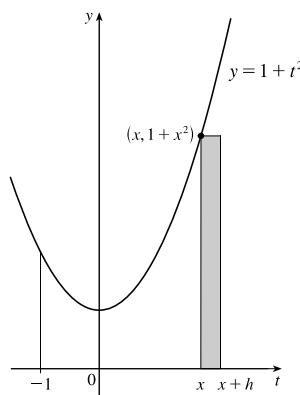
(c) $A'(x) = x^2 + 1$. This is the y-coordinate of the point $\left(x, 1 + x^2\right)$ on the given curve.

(d)

$y = 1 + t^2$

$(x + h, 1 + (x + h)^2)$

$(x, 1 + x^2)$

$A(x + h) - A(x)$

$A(x + h) - A(x)$ is the area under the curve $y = 1 + t^2$ from $t = x$ to $t = x + h$.

(e)

$y = 1 + t^2$

$(x, 1 + x^2)$

An approximating rectangle is shown in the figure.
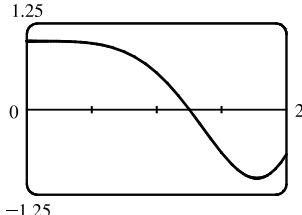
It has height $1 + x^2$, width h, and area $h(1 + x^2)$, so

$$A(x + h) - A(x) \approx h(1 + x^2) \quad \Rightarrow \quad \frac{A(x + h) - A(x)}{h} \approx 1 + x^2.$$

(f) Part (e) says that the average rate of change of A is approximately $1 + x^2$. As h approaches 0, the quotient approaches the

instantaneous rate of change—namely, $A'(x)$. So the result of part (c), $A'(x) = x^2 + 1$, is geometrically plausible.
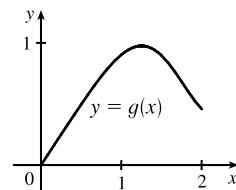
3. (a) $f(x) = \cos(x^2)$

(b) $g(x)$ starts to decrease at that value of x where $\cos(t^2)$ changes from

positive to negative; that is, at about $x = 1.25$.
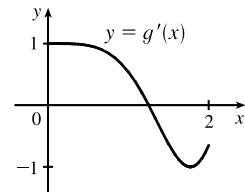
(c) $g(x) = \int_0^x \cos(t^2)\, dt$. Using an integration command, we find that

$g(0) = 0$, $g(0.2) \approx 0.200$, $g(0.4) \approx 0.399$, $g(0.6) \approx 0.592$,

$g(0.8) \approx 0.768$, $g(1.0) \approx 0.905$, $g(1.2) \approx 0.974$, $g(1.4) \approx 0.950$,

$g(1.6) \approx 0.826$, $g(1.8) \approx 0.635$, and $g(2.0) \approx 0.461$.

(d) We sketch the graph of g' using the method of Example 1 in Section 2.8.

The graphs of $g'(x)$ and $f(x)$ look alike, so we guess that $g'(x) = f(x)$.

4. In Problems 1 and 2, we showed that if $g(x) = \int_a^x f(t)\, dt$, then $g'(x) = f(x)$, for the functions $f(t) = 2t + 1$ and

$f(t) = 1 + t^2$. In Problem 3 we guessed that the same is true for $f(t) = \cos(t^2)$, based on visual evidence. So we conjecture

that $g'(x) = f(x)$ for any continuous function f. This turns out to be true and is proved in Section 5.3 (the Fundamental

Theorem of Calculus).

5.3 The Fundamental Theorem of Calculus

1. One process undoes what the other one does. The precise version of this statement is given by the Fundamental Theorem of

Calculus. See the statement of this theorem and the paragraph that follows it on page 398.

2. (a) $g(x) = \int_0^x f(t)\, dt$, so $g(0) = \int_0^0 f(t)\, dt = 0$.

$g(1) = \int_0^1 f(t)\, dt = \frac{1}{2} \cdot 1 \cdot 1$ [area of triangle] $= \frac{1}{2}$.

$g(2) = \int_0^2 f(t)\, dt = \int_0^1 f(t)\, dt + \int_1^2 f(t)\, dt$ [below the t-axis]

$\quad = \frac{1}{2} - \frac{1}{2} \cdot 1 \cdot 1 = 0$.

$g(3) = g(2) + \int_2^3 f(t)\, dt = 0 - \frac{1}{2} \cdot 1 \cdot 1 = -\frac{1}{2}$.

$g(4) = g(3) + \int_3^4 f(t)\, dt = -\frac{1}{2} + \frac{1}{2} \cdot 1 \cdot 1 = 0$.

$g(5) = g(4) + \int_4^5 f(t)\, dt = 0 + 1.5 = 1.5$.
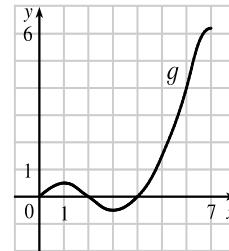
$g(6) = g(5) + \int_5^6 f(t)\, dt = 1.5 + 2.5 = 4$.

(b) $g(7) = g(6) + \int_6^7 f(t)\,dt \approx 4 + 2.2$ [estimate from the graph] $= 6.2$.

(c) The answers from part (a) and part (b) indicate that g has a minimum at

$x = 3$ and a maximum at $x = 7$. This makes sense from the graph of f

since we are subtracting area on $1 < x < 3$ and adding area on $3 < x < 7$.

(d)

3. (a) $g(x) = \int_0^x f(t)\,dt$.

$g(0) = \int_0^0 f(t)\,dt = 0$

$g(1) = \int_0^1 f(t)\,dt = 1 \cdot 2 = 2$ [rectangle],
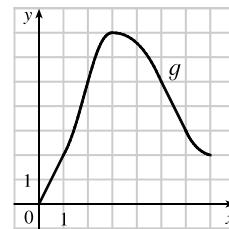
$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = g(1) + \int_1^2 f(t)\,dt$
    $= 2 + 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 = 5$ [rectangle plus triangle],

$g(3) = \int_0^3 f(t)\,dt = g(2) + \int_2^3 f(t)\,dt = 5 + \frac{1}{2} \cdot 1 \cdot 4 = 7$,

$g(6) = g(3) + \int_3^6 f(t)\,dt$ [the integral is negative since f lies under the t-axis]
    $= 7 + \left[-\left(\frac{1}{2} \cdot 2 \cdot 2 + 1 \cdot 2 \right) \right] = 7 - 4 = 3$

(b) g is increasing on $(0, 3)$ because as x increases from 0 to 3, we keep

adding more area.

(c) g has a maximum value when we start subtracting area; that is,

at $x = 3$.

(d)
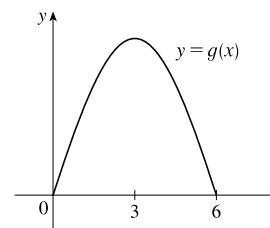
4. (a) $g(x) = \int_0^x f(t)\,dt$, so $g(0) = 0$ since the limits of integration are equal and $g(6) = 0$ since the areas above and below the

t-axis are equal.

(b) $g(1)$ is the area under the curve from 0 to 1, which includes two unit squares and about 80% to 90% of a third unit square,

so $g(1) \approx 2.8$. Similarly, $g(2) \approx 4.9$ and $g(3) \approx 5.7$. Now $g(3) - g(2) \approx 0.8$, so $g(4) \approx g(3) - 0.8 \approx 4.9$ by the

symmetry of f about $x = 3$. Likewise, $g(5) \approx 2.8$.

(c) As we go from $x = 0$ to $x = 3$, we are adding area, so g increases on the interval $(0, 3)$.

(d) g increases on $(0, 3)$ and decreases on $(3, 6)$ [where we are subtracting area], so g has a maximum value at $x = 3$.

(e) A graph of g must have a maximum at $x = 3$, be symmetric about $x = 3$,

and have zeros at $x = 0$ and $x = 6$.
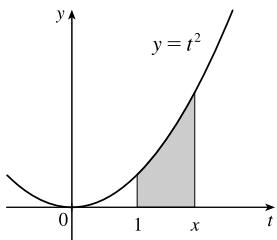
(f) If we sketch the graph of g' by estimating slopes on the graph of g (as in Section 2.8), we get a graph that looks like f (as

indicated by FTC1).
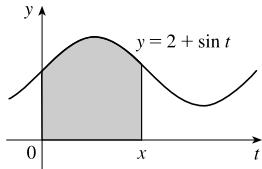
5.

(a) By FTC1 with $f(t) = t^2$ and $a = 1$, $g(x) = \int_1^x t^2\, dt$ $\Rightarrow$

$$g'(x) = f(x) = x^2.$$

(b) Using FTC2, $g(x) = \int_1^x t^2\, dt = \left[\frac{1}{3}t^3\right]_1^x = \frac{1}{3}x^3 - \frac{1}{3}$ $\Rightarrow$ $g'(x) = x^2.$

6.

(a) By FTC1 with $f(t) = 2 + \sin t$ and $a = 0$, $g(x) = \int_0^x (2 + \sin t)\, dt$ $\Rightarrow$

$$g'(x) = f(x) = 2 + \sin x.$$

(b) Using FTC2,

$$g(x) = \int_0^x (2 + \sin t)\, dt = [2t - \cos t]_0^x = (2x - \cos x) - (0 - 1)$$
$$= 2x - \cos x + 1 \quad \Rightarrow$$
$$g'(x) = 2 - (-\sin x) + 0 = 2 + \sin x$$

7. $f(t) = \sqrt{t + t^3}$ and $g(x) = \int_0^x \sqrt{t + t^3}\, dt$, so by FTC1, $g'(x) = f(x) = \sqrt{x + x^3}$.

8. $f(t) = \ln(1 + t^2)$ and $g(x) = \int_1^x \ln(1 + t^2)\, dt$, so by FTC1, $g'(x) = f(x) = \ln(1 + x^2)$.

9. $f(t) = (t - t^2)^8$ and $g(s) = \int_5^s (t - t^2)^8\, dt$, so by FTC1, $g'(s) = f(s) = (s - s^2)^8$.

10. $f(t) = \dfrac{\sqrt{t}}{t + 1}$ and $h(u) = \displaystyle\int_0^u \dfrac{\sqrt{t}}{t + 1}\, dt$, so by FTC1, $h'(u) = f(u) = \dfrac{\sqrt{u}}{u + 1}$.

11. $F(x) = \displaystyle\int_x^0 \sqrt{1 + \sec t}\, dt = -\int_0^x \sqrt{1 + \sec t}\, dt$ $\Rightarrow$ $F'(x) = -\dfrac{d}{dx}\displaystyle\int_0^x \sqrt{1 + \sec t}\, dt = -\sqrt{1 + \sec x}$

12. $R(y) = \displaystyle\int_y^2 t^3 \sin t\, dt = -\int_2^y t^3 \sin t\, dt$ $\Rightarrow$ $R'(y) = -\dfrac{d}{dy}\displaystyle\int_2^y t^3 \sin t\, dt = -y^3 \sin y$

13. Let $u = e^x$. Then $\dfrac{du}{dx} = e^x$. Also, $\dfrac{dh}{dx} = \dfrac{dh}{du}\dfrac{du}{dx}$, so

$$h'(x) = \frac{d}{dx}\int_1^{e^x} \ln t\, dt = \frac{d}{du}\int_1^u \ln t\, dt \cdot \frac{du}{dx} = \ln u \frac{du}{dx} = (\ln e^x) \cdot e^x = xe^x.$$

14. Let $u = \sqrt{x}$. Then $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$. Also, $\dfrac{dh}{dx} = \dfrac{dh}{du}\dfrac{du}{dx}$, so

$$h'(x) = \frac{d}{dx}\int_1^{\sqrt{x}} \frac{z^2}{z^4 + 1}\, dz = \frac{d}{du}\int_1^u \frac{z^2}{z^4 + 1}\, dz \cdot \frac{du}{dx} = \frac{u^2}{u^4 + 1}\frac{du}{dx} = \frac{x}{x^2 + 1}\frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2(x^2 + 1)}.$$

15. Let $u = 3x + 2$. Then $\dfrac{du}{dx} = 3$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_1^{3x+2} \frac{t}{1 + t^3}\, dt = \frac{d}{du}\int_1^u \frac{t}{1 + t^3}\, dt \cdot \frac{du}{dx} = \frac{u}{1 + u^3}\frac{du}{dx} = \frac{3x + 2}{1 + (3x + 2)^3} \cdot 3 = \frac{3(3x + 2)}{1 + (3x + 2)^3}$$

16. Let $u = x^4$. Then $\dfrac{du}{dx} = 4x^3$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_0^{x^4} \cos^2 \theta\, d\theta = \frac{d}{du}\int_0^u \cos^2 \theta\, d\theta \cdot \frac{du}{dx} = \cos^2 u \frac{du}{dx} = \cos^2(x^4) \cdot 4x^3.$$

17. Let $u = \sqrt{x}$. Then $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_{\sqrt{x}}^{\pi/4} \theta\tan\theta\, d\theta = -\frac{d}{du}\int_{\pi/4}^{\sqrt{x}} \theta\tan\theta\, d\theta \cdot \frac{du}{dx} = -u\tan u\,\frac{du}{dx} = -\sqrt{x}\tan\sqrt{x}\cdot\frac{1}{2\sqrt{x}} = -\frac{1}{2}\tan\sqrt{x}$$

18. Let $u = \sin x$. Then $\dfrac{du}{dx} = \cos x$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_{\sin x}^{1} \sqrt{1+t^2}\, dt = \frac{d}{du}\int_{u}^{1} \sqrt{1+t^2}\, dt\cdot\frac{du}{dx} = -\frac{d}{du}\int_{1}^{u} \sqrt{1+t^2}\, dt\cdot\frac{du}{dx}$$

$$= -\sqrt{1+u^2}\ \cos x = -\sqrt{1+\sin^2 x}\ \cos x$$

19. $\int_{1}^{3}(x^2 + 2x - 4)\, dx = \left[\frac{1}{3}x^3 + x^2 - 4x\right]_{1}^{3} = (9 + 9 - 12) - \left(\frac{1}{3} + 1 - 4\right) = 6 + \frac{8}{3} = \frac{26}{3}$

20. $\int_{-1}^{1} x^{100}\, dx = \left[\frac{1}{101}x^{101}\right]_{-1}^{1} = \frac{1}{101} - \left(-\frac{1}{101}\right) = \frac{2}{101}$

21. $\int_{0}^{2}\left(\frac{4}{5}t^3 - \frac{3}{4}t^2 + \frac{2}{5}t\right) dt = \left[\frac{1}{5}t^4 - \frac{1}{4}t^3 + \frac{1}{5}t^2\right]_{0}^{2} = \left(\frac{16}{5} - 2 + \frac{4}{5}\right) - 0 = 2$

22. $\int_{0}^{1}(1 - 8v^3 + 16v^7)\, dv = \left[v - 2v^4 + 2v^8\right]_{0}^{1} = (1 - 2 + 2) - 0 = 1$

23. $\displaystyle\int_{1}^{9}\sqrt{x}\, dx = \int_{1}^{9} x^{1/2}\, dx = \left[\frac{x^{3/2}}{3/2}\right]_{1}^{9} = \frac{2}{3}\left[x^{3/2}\right]_{1}^{9} = \frac{2}{3}(9^{3/2} - 1^{3/2}) = \frac{2}{3}(27 - 1) = \frac{52}{3}$

24. $\displaystyle\int_{1}^{8} x^{-2/3}\, dx = \left[\frac{x^{1/3}}{1/3}\right]_{1}^{8} = 3\left[x^{1/3}\right]_{1}^{8} = 3(8^{1/3} - 1^{1/3}) = 3(2 - 1) = 3$

25. $\displaystyle\int_{\pi/6}^{\pi} \sin\theta\, d\theta = \left[-\cos\theta\right]_{\pi/6}^{\pi} = -\cos\pi - \left(-\cos\frac{\pi}{6}\right) = -(-1) - \left(-\sqrt{3}/2\right) = 1 + \sqrt{3}/2$

26. $\int_{-5}^{5} e\, dx = \left[ex\right]_{-5}^{5} = 5e - (-5e) = 10e$

27. $\displaystyle\int_{0}^{1}(u + 2)(u - 3)\, du = \int_{0}^{1}(u^2 - u - 6)\, du = \left[\frac{1}{3}u^3 - \frac{1}{2}u^2 - 6u\right]_{0}^{1} = \left(\frac{1}{3} - \frac{1}{2} - 6\right) - 0 = -\frac{37}{6}$

28. $\displaystyle\int_{0}^{4}(4 - t)\sqrt{t}\, dt = \int_{0}^{4}(4 - t)t^{1/2}\, dt = \int_{0}^{4}(4t^{1/2} - t^{3/2})\, dt = \left[\frac{8}{3}t^{3/2} - \frac{2}{5}t^{5/2}\right]_{0}^{4} = \frac{8}{3}(8) - \frac{2}{5}(32) = \frac{320-192}{15} = \frac{128}{15}$

29. $\displaystyle\int_{1}^{4}\frac{2 + x^2}{\sqrt{x}}\, dx = \int_{1}^{4}\left(\frac{2}{\sqrt{x}} + \frac{x^2}{\sqrt{x}}\right) dx = \int_{1}^{4}(2x^{-1/2} + x^{3/2})\, dx$

$$= \left[4x^{1/2} + \frac{2}{5}x^{5/2}\right]_{1}^{4} = \left[4(2) + \frac{2}{5}(32)\right] - \left(4 + \frac{2}{5}\right) = 8 + \frac{64}{5} - 4 - \frac{2}{5} = \frac{82}{5}$$

30. $\int_{-1}^{2}(3u - 2)(u + 1)\, du = \int_{-1}^{2}(3u^2 + u - 2)\, du = \left[u^3 + \frac{1}{2}u^2 - 2u\right]_{-1}^{2} = (8 + 2 - 4) - \left(-1 + \frac{1}{2} + 2\right) = 6 - \frac{3}{2} = \frac{9}{2}$

31. $\displaystyle\int_{\pi/6}^{\pi/2} \csc t\cot t\, dt = \left[-\csc t\right]_{\pi/6}^{\pi/2} = \left(-\csc\frac{\pi}{2}\right) - \left(-\csc\frac{\pi}{6}\right) = -1 - (-2) = 1$

32. $\displaystyle\int_{\pi/4}^{\pi/3} \csc^2\theta\, d\theta = \left[-\cot\theta\right]_{\pi/4}^{\pi/3} = \left(-\cot\frac{\pi}{3}\right) - \left(-\cot\frac{\pi}{4}\right) = -\frac{1}{\sqrt{3}} - (-1) = 1 - \frac{1}{\sqrt{3}}$

33. $\int_0^1 (1+r)^3 \, dr = \int_0^1 (1 + 3r + 3r^2 + r^3) \, dr = \left[r + \frac{3}{2} r^2 + r^3 + \frac{1}{4} r^4 \right]_0^1 = \left(1 + \frac{3}{2} + 1 + \frac{1}{4} \right) - 0 = \frac{15}{4}$

34. $\int_0^3 (2 \sin x - e^x) \, dx = \left[-2 \cos x - e^x \right]_0^3 = (-2 \cos 3 - e^3) - (-2 - 1) = 3 - 2 \cos 3 - e^3$

35. $\int_1^2 \frac{v^3 + 3v^6}{v^4} = \int_1^2 \left(\frac{1}{v} + 3v^2 \right) dv = \left[\ln |v| + v^3 \right]_1^2 = (\ln 2 + 8) - (\ln 1 + 1) = \ln 2 + 7$

36. $\int_1^{18} \sqrt{\frac{3}{z}} \, dz = \int_1^{18} \sqrt{3} z^{-1/2} \, dz = \sqrt{3} \left[2z^{1/2} \right]_1^{18} = 2\sqrt{3}(18^{1/2} - 1^{1/2}) = 2\sqrt{3}(3\sqrt{2} - 1)$

37. $\int_0^1 (x^e + e^x) \, dx = \left[\frac{x^{e+1}}{e+1} + e^x \right]_0^1 = \left(\frac{1}{e+1} + e \right) - (0 + 1) = \frac{1}{e+1} + e - 1$

38. $\int_0^1 \cosh t \, dt = \left[\sinh t \right]_0^1 = \sinh 1 - \sinh 0 = \sinh 1 \quad \left[\text{or } \frac{1}{2}(e - e^{-1}) \right]$

39. $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1 + x^2} \, dx = \left[8 \arctan x \right]_{1/\sqrt{3}}^{\sqrt{3}} = 8 \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = 8 \left(\frac{\pi}{6} \right) = \frac{4\pi}{3}$

40. $\int_1^3 \frac{y^3 - 2y^2 - y}{y^2} \, dy = \int_1^3 \left(y - 2 - \frac{1}{y} \right) dy = \left[\frac{1}{2} y^2 - 2y - \ln |y| \right]_1^3 = \left(\frac{9}{2} - 6 - \ln 3 \right) - \left(\frac{1}{2} - 2 - 0 \right) = -\ln 3$

41. $\int_0^4 2^s \, ds = \left[\frac{1}{\ln 2} 2^s \right]_0^4 = \frac{16}{\ln 2} - \frac{1}{\ln 2} = \frac{15}{\ln 2}$

42. $\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1 - x^2}} \, dx = \left[4 \arcsin x \right]_{1/2}^{1/\sqrt{2}} = 4 \left(\frac{\pi}{4} - \frac{\pi}{6} \right) = 4 \left(\frac{\pi}{12} \right) = \frac{\pi}{3}$

43. If $f(x) = \begin{cases} \sin x & \text{if } 0 \le x < \pi/2 \\ \cos x & \text{if } \pi/2 \le x \le \pi \end{cases}$ then

$\int_0^\pi f(x) \, dx = \int_0^{\pi/2} \sin x \, dx + \int_{\pi/2}^\pi \cos x \, dx = \left[-\cos x \right]_0^{\pi/2} + \left[\sin x \right]_{\pi/2}^\pi = -\cos \frac{\pi}{2} + \cos 0 + \sin \pi - \sin \frac{\pi}{2}$

$= -0 + 1 + 0 - 1 = 0$
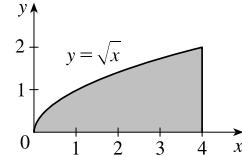
Note that f is integrable by Theorem 3 in Section 5.2.

44. If $f(x) = \begin{cases} 2 & \text{if } -2 \le x \le 0 \\ 4 - x^2 & \text{if } 0 < x \le 2 \end{cases}$ then

$\int_{-2}^2 f(x) \, dx = \int_{-2}^0 2 \, dx + \int_0^2 (4 - x^2) \, dx = \left[2x \right]_{-2}^0 + \left[4x - \frac{1}{3} x^3 \right]_0^2 = [0 - (-4)] + \left(\frac{16}{3} - 0 \right) = \frac{28}{3}$

Note that f is integrable by Theorem 3 in Section 5.2.

45. Area $= \int_0^4 \sqrt{x} \, dx = \int_0^4 x^{1/2} \, dx = \left[\frac{2}{3} x^{3/2} \right]_0^4 = \frac{2}{3} (8) - 0 = \frac{16}{3}$
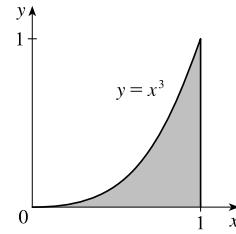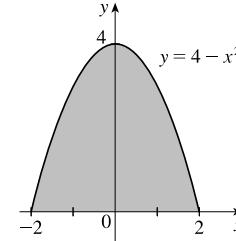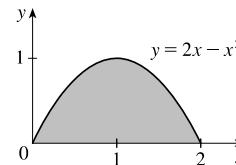
46. Area $= \displaystyle\int_0^1 x^3 \, dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$

47. Area $= \displaystyle\int_{-2}^2 (4 - x^2) \, dx = \left[4x - \frac{1}{3}x^3\right]_{-2}^2 = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = \frac{32}{3}$
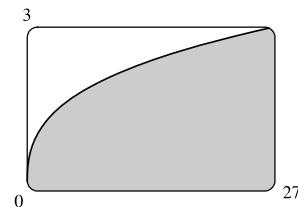
48. Area $= \displaystyle\int_0^2 (2x - x^2) \, dx = \left[x^2 - \frac{1}{3}x^3\right]_0^2 = \left(4 - \frac{8}{3}\right) - 0 = \frac{4}{3}$
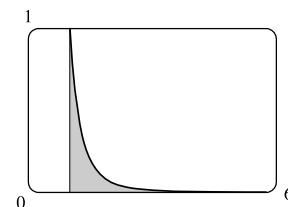
49. From the graph, it appears that the area is about 60. The actual area is

$\displaystyle\int_0^{27} x^{1/3} \, dx = \left[\frac{3}{4}x^{4/3}\right]_0^{27} = \frac{3}{4} \cdot 81 - 0 = \frac{243}{4} = 60.75$. This is $\frac{3}{4}$ of the

area of the viewing rectangle.
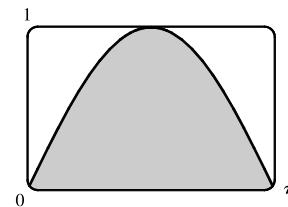
50. From the graph, it appears that the area is about $\frac{1}{3}$. The actual area is

$\displaystyle\int_1^6 x^{-4} \, dx = \left[\frac{x^{-3}}{-3}\right]_1^6 = \left[\frac{-1}{3x^3}\right]_1^6 = -\frac{1}{3 \cdot 216} + \frac{1}{3} = \frac{215}{648} \approx 0.3318$.
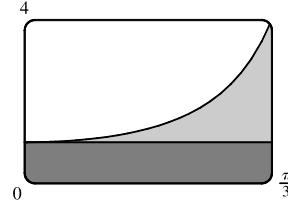
51. It appears that the area under the graph is about $\frac{2}{3}$ of the area of the viewing

rectangle, or about $\frac{2}{3}\pi \approx 2.1$. The actual area is

$\displaystyle\int_0^\pi \sin x \, dx = [-\cos x]_0^\pi = (-\cos \pi) - (-\cos 0) = -(-1) + 1 = 2$.

52. Splitting up the region as shown, we estimate that the area under the graph

is $\frac{\pi}{3} + \frac{1}{4}\left(3 \cdot \frac{\pi}{3}\right) \approx 1.8$. The actual area is
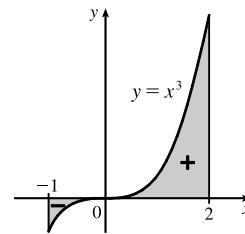
$\displaystyle\int_0^{\pi/3} \sec^2 x \, dx = [\tan x]_0^{\pi/3} = \sqrt{3} - 0 = \sqrt{3} \approx 1.73$.
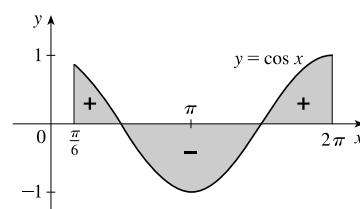
53. $\int_{-1}^{2} x^3 \, dx = \left[\frac{1}{4}x^4\right]_{-1}^{2} = 4 - \frac{1}{4} = \frac{15}{4} = 3.75$

54. $\int_{\pi/6}^{2\pi} \cos x \, dx = \left[\sin x\right]_{\pi/6}^{2\pi} = 0 - \frac{1}{2} = -\frac{1}{2}$

55. $f(x) = x^{-4}$ is not continuous on the interval $[-2, 1]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = 0$, so $\int_{-2}^{1} x^{-4} \, dx$ does not exist.

56. $f(x) = \dfrac{4}{x^3}$ is not continuous on the interval $[-1, 2]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = 0$, so $\displaystyle\int_{-1}^{2} \frac{4}{x^3} \, dx$ does not exist.

57. $f(\theta) = \sec\theta \, \tan\theta$ is not continuous on the interval $[\pi/3, \pi]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = \pi/2$, so $\int_{\pi/3}^{\pi} \sec\theta \, \tan\theta \, d\theta$ does not exist.

58. $f(x) = \sec^2 x$ is not continuous on the interval $[0, \pi]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = \pi/2$, so $\int_{0}^{\pi} \sec^2 x \, dx$ does not exist.

59. $g(x) = \displaystyle\int_{2x}^{3x} \frac{u^2 - 1}{u^2 + 1} \, du = \int_{2x}^{0} \frac{u^2 - 1}{u^2 + 1} \, du + \int_{0}^{3x} \frac{u^2 - 1}{u^2 + 1} \, du = -\int_{0}^{2x} \frac{u^2 - 1}{u^2 + 1} \, du + \int_{0}^{3x} \frac{u^2 - 1}{u^2 + 1} \, du \quad \Rightarrow$

$g'(x) = -\dfrac{(2x)^2 - 1}{(2x)^2 + 1} \cdot \dfrac{d}{dx}(2x) + \dfrac{(3x)^2 - 1}{(3x)^2 + 1} \cdot \dfrac{d}{dx}(3x) = -2 \cdot \dfrac{4x^2 - 1}{4x^2 + 1} + 3 \cdot \dfrac{9x^2 - 1}{9x^2 + 1}$

60. $g(x) = \displaystyle\int_{1-2x}^{1+2x} t \sin t \, dt = \int_{1-2x}^{0} t \sin t \, dt + \int_{0}^{1+2x} t \sin t \, dt = -\int_{0}^{1-2x} t \sin t \, dt + \int_{0}^{1+2x} t \sin t \, dt \quad \Rightarrow$

$g'(x) = -(1 - 2x) \sin(1 - 2x) \cdot \dfrac{d}{dx}(1 - 2x) + (1 + 2x) \sin(1 + 2x) \cdot \dfrac{d}{dx}(1 + 2x)$

$\quad\quad = 2(1 - 2x) \sin(1 - 2x) + 2(1 + 2x) \sin(1 + 2x)$

61. $F(x) = \displaystyle\int_{x}^{x^2} e^{t^2} \, dt = \int_{x}^{0} e^{t^2} \, dt + \int_{0}^{x^2} e^{t^2} \, dt = -\int_{0}^{x} e^{t^2} \, dt + \int_{0}^{x^2} e^{t^2} \, dt \quad \Rightarrow$

$F'(x) = -e^{x^2} + e^{(x^2)^2} \cdot \dfrac{d}{dx}(x^2) = -e^{x^2} + 2xe^{x^4}$

62. $F(x) = \displaystyle\int_{\sqrt{x}}^{2x} \arctan t \, dt = \int_{\sqrt{x}}^{0} \arctan t \, dt + \int_{0}^{2x} \arctan t \, dt = -\int_{0}^{\sqrt{x}} \arctan t \, dt + \int_{0}^{2x} \arctan t \, dt \quad \Rightarrow$

$F'(x) = -\arctan\sqrt{x} \cdot \dfrac{d}{dx}(\sqrt{x}) + \arctan 2x \cdot \dfrac{d}{dx}(2x) = -\dfrac{1}{2\sqrt{x}}\arctan\sqrt{x} + 2\arctan 2x$

63. $y = \displaystyle\int_{\cos x}^{\sin x} \ln(1 + 2v)\, dv = \int_{\cos x}^{0} \ln(1 + 2v)\, dv + \int_{0}^{\sin x} \ln(1 + 2v)\, dv$

$\qquad\qquad = -\displaystyle\int_{0}^{\cos x} \ln(1 + 2v)\, dv + \int_{0}^{\sin x} \ln(1 + 2v)\, dv \quad \Rightarrow$

$y' = -\ln(1 + 2\cos x) \cdot \dfrac{d}{dx}\cos x + \ln(1 + 2\sin x)\cdot \dfrac{d}{dx}\sin x = \sin x\,\ln(1 + 2\cos x) + \cos x\,\ln(1 + 2\sin x)$

64. $f(x) = \displaystyle\int_{0}^{x}(1 - t^2)e^{t^2}\, dt$ is increasing when $f'(x) = (1 - x^2)e^{x^2}$ is positive.

Since $e^{x^2} > 0$, $f'(x) > 0 \quad\Leftrightarrow\quad 1 - x^2 > 0 \quad\Leftrightarrow\quad |x| < 1$, so f is increasing on $(-1, 1)$.

65. $y = \displaystyle\int_{0}^{x} \dfrac{t^2}{t^2 + t + 2}\, dt \quad\Rightarrow\quad y' = \dfrac{x^2}{x^2 + x + 2} \quad\Rightarrow$

$y'' = \dfrac{(x^2 + x + 2)(2x) - x^2(2x + 1)}{(x^2 + x + 2)^2} = \dfrac{2x^3 + 2x^2 + 4x - 2x^3 - x^2}{(x^2 + x + 2)^2} = \dfrac{x^2 + 4x}{(x^2 + x + 2)^2} = \dfrac{x(x + 4)}{(x^2 + x + 2)^2}.$

The curve y is concave downward when $y'' < 0$; that is, on the interval $(-4, 0)$.

66. If $F(x) = \int_{1}^{x} f(t)\, dt$, then by FTC1, $F'(x) = f(x)$, and also, $F''(x) = f'(x)$. F is concave downward where F'' is

negative; that is, where f' is negative. The given graph shows that f is decreasing $(f' < 0)$ on the interval $(-1, 1)$.

67. $F(x) = \int_{2}^{x} e^{t^2}\, dt \quad\Rightarrow\quad F'(x) = e^{x^2}$, so the slope at $x = 2$ is $e^{2^2} = e^4$. The y-coordinate of the point on F at $x = 2$ is

$F(2) = \int_{2}^{2} e^{t^2}\, dt = 0$ since the limits are equal. An equation of the tangent line is $y - 0 = e^4(x - 2)$, or $y = e^4 x - 2e^4$.

68. $g(y) = \int_{3}^{y} f(x)\, dx \quad\Rightarrow\quad g'(y) = f(y)$. Since $f(x) = \int_{0}^{\sin x} \sqrt{1 + t^2}\, dt$, $g''(y) = f'(y) = \sqrt{1 + \sin^2 y}\cdot \cos y$,

so $g''\!\left(\frac{\pi}{6}\right) = \sqrt{1 + \sin^2\!\left(\frac{\pi}{6}\right)}\cdot\cos\frac{\pi}{6} = \sqrt{1 + (\frac{1}{2})^2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{5}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{15}}{4}.$

69. By FTC2, $\int_{1}^{4} f'(x)\, dx = f(4) - f(1)$, so $17 = f(4) - 12 \quad\Rightarrow\quad f(4) = 17 + 12 = 29.$

70. (a) $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_{0}^{x} e^{-t^2}\, dt \quad\Rightarrow\quad \int_{0}^{x} e^{-t^2}\, dt = \dfrac{\sqrt{\pi}}{2}\operatorname{erf}(x).$ By Property 5 of definite integrals in Section 5.2,

$\int_{0}^{b} e^{-t^2}\, dt = \int_{0}^{a} e^{-t^2}\, dt + \int_{a}^{b} e^{-t^2}\, dt$, so

$\displaystyle\int_{a}^{b} e^{-t^2}\, dt = \int_{0}^{b} e^{-t^2}\, dt - \int_{0}^{a} e^{-t^2}\, dt = \dfrac{\sqrt{\pi}}{2}\operatorname{erf}(b) - \dfrac{\sqrt{\pi}}{2}\operatorname{erf}(a) = \tfrac{1}{2}\sqrt{\pi}\,[\operatorname{erf}(b) - \operatorname{erf}(a)].$

(b) $y = e^{x^2}\operatorname{erf}(x) \quad\Rightarrow\quad y' = 2xe^{x^2}\operatorname{erf}(x) + e^{x^2}\operatorname{erf}'(x) = 2xy + e^{x^2}\cdot\dfrac{2}{\sqrt{\pi}}e^{-x^2} \quad\text{[by FTC1]}\quad = 2xy + \dfrac{2}{\sqrt{\pi}}.$

71. (a) The Fresnel function $S(x) = \int_{0}^{x}\sin\!\left(\frac{\pi}{2}t^2\right)dt$ has local maximum values where $0 = S'(x) = \sin\!\left(\frac{\pi}{2}t^2\right)$ and

S' changes from positive to negative. For $x > 0$, this happens when $\frac{\pi}{2}x^2 = (2n - 1)\pi$ [odd multiples of π] $\Leftrightarrow$

$x^2 = 2(2n-1)$ ⇔ $x = \sqrt{4n-2}$, n any positive integer. For $x < 0$, S' changes from positive to negative where

$\frac{\pi}{2}x^2 = 2n\pi$ [even multiples of π] ⇔ $x^2 = 4n$ ⇔ $x = -2\sqrt{n}$. S' does not change sign at $x = 0$.

(b) S is concave upward on those intervals where $S''(x) > 0$. Differentiating our expression for $S'(x)$, we get
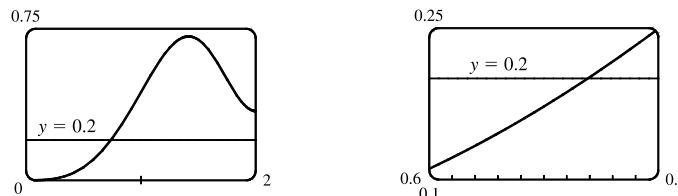
$S''(x) = \cos\left(\frac{\pi}{2}x^2\right)\left(2\frac{\pi}{2}x\right) = \pi x \cos\left(\frac{\pi}{2}x^2\right)$. For $x > 0$, $S''(x) > 0$ where $\cos\left(\frac{\pi}{2}x^2\right) > 0$ ⇔ $0 < \frac{\pi}{2}x^2 < \frac{\pi}{2}$ or

$\left(2n - \frac{1}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n + \frac{1}{2}\right)\pi$, n any integer ⇔ $0 < x < 1$ or $\sqrt{4n-1} < x < \sqrt{4n+1}$, n any positive integer.

For $x < 0$, $S''(x) > 0$ where $\cos\left(\frac{\pi}{2}x^2\right) < 0$ ⇔ $\left(2n - \frac{3}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n - \frac{1}{2}\right)\pi$, n any integer ⇔

$4n - 3 < x^2 < 4n - 1$ ⇔ $\sqrt{4n-3} < |x| < \sqrt{4n-1}$ ⇒ $\sqrt{4n-3} < -x < \sqrt{4n-1}$ ⇒

$-\sqrt{4n-3} > x > -\sqrt{4n-1}$, so the intervals of upward concavity for $x < 0$ are $\left(-\sqrt{4n-1}, -\sqrt{4n-3}\right)$, n any

positive integer. To summarize: S is concave upward on the intervals $(0, 1)$, $\left(-\sqrt{3}, -1\right)$, $\left(\sqrt{3}, \sqrt{5}\right)$, $\left(-\sqrt{7}, -\sqrt{5}\right)$,

$\left(\sqrt{7}, 3\right), \ldots$.

(c) In Maple, we use `plot({int(sin(Pi*t^2/2),t=0..x),0.2},x=0..2);`. Note that

Maple recognizes the Fresnel function, calling it `FresnelS(x)`. In Mathematica, we use

`Plot[{Integrate[Sin[Pi*t^2/2],{t,0,x}],0.2},{x,0,2}]`. In Derive, we load the utility file

`FRESNEL` and plot `FRESNEL_SIN(x)`. From the graphs, we see that $\int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt = 0.2$ at $x \approx 0.74$.

72. (a) In Maple, we should start by setting `si:=int(sin(t)/t,t=0..x);`. In
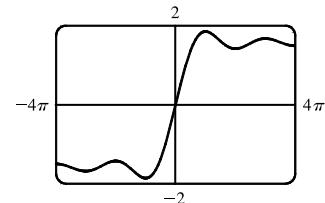
Mathematica, the command is `si=Integrate[Sin[t]/t,{t,0,x}]`.

Note that both systems recognize this function; Maple calls it `Si(x)` and

Mathematica calls it `SinIntegral[x]`. In Maple, the command to generate

the graph is `plot(si,x=-4*Pi..4*Pi);`. In Mathematica, it is

`Plot[si,{x,-4*Pi,4*Pi}]`. In Derive, we load the utility file `EXP_INT` and plot `SI(x)`.

(b) $\text{Si}(x)$ has local maximum values where $\text{Si}'(x)$ changes from positive to negative, passing through 0. From the

Fundamental Theorem we know that $\text{Si}'(x) = \dfrac{d}{dx}\displaystyle\int_0^x \dfrac{\sin t}{t}\, dt = \dfrac{\sin x}{x}$, so we must have $\sin x = 0$ for a maximum, and

for $x > 0$ we must have $x = (2n-1)\pi$, n any positive integer, for Si' to be changing from positive to negative at x.

For $x < 0$, we must have $x = 2n\pi$, n any positive integer, for a maximum, since the denominator of $\text{Si}'(x)$ is negative

for $x < 0$. Thus, the local maxima occur at $x = \pi, -2\pi, 3\pi, -4\pi, 5\pi, -6\pi, \ldots$.

(c) To find the first inflection point, we solve $\text{Si}''(x) = \dfrac{\cos x}{x} - \dfrac{\sin x}{x^2} = 0$. We can see from the graph that the first inflection

point lies somewhere between $x = 3$ and $x = 5$. Using a rootfinder gives the value $x \approx 4.4934$. To find the y-coordinate

of the inflection point, we evaluate $\text{Si}(4.4934) \approx 1.6556$. So the coordinates of the first inflection point to the right of the

origin are about $(4.4934, 1.6556)$. Alternatively, we could graph $S''(x)$ and estimate the first positive x-value at which it

changes sign.

(d) It seems from the graph that the function has horizontal asymptotes at $y \approx 1.5$, with $\lim\limits_{x \to \pm\infty} \text{Si}(x) \approx \pm 1.5$ respectively.

Using the limit command, we get $\lim\limits_{x \to \infty} \text{Si}(x) = \frac{\pi}{2}$. Since $\text{Si}(x)$ is an odd function, $\lim\limits_{x \to -\infty} \text{Si}(x) = -\frac{\pi}{2}$. So $\text{Si}(x)$ has the

horizontal asymptotes $y = \pm\frac{\pi}{2}$.
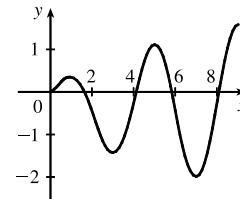
(e) We use the `fsolve` command in Maple (or `FindRoot` in Mathematica) to find that the solution is $x \approx 1.1$. Or, as in

Exercise 65(c), we graph $y = \text{Si}(x)$ and $y = 1$ on the same screen to see where they intersect.

73. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 1, 3, 5, 7$, and 9. g has local maxima at $x = 1$ and 5 (since $f = g'$

changes from positive to negative there) and local minima at $x = 3$ and 7. There is no local maximum or minimum at

$x = 9$, since f is not defined for $x > 9$.

(b) We can see from the graph that $\left|\int_0^1 f\,dt\right| < \left|\int_1^3 f\,dt\right| < \left|\int_3^5 f\,dt\right| < \left|\int_5^7 f\,dt\right| < \left|\int_7^9 f\,dt\right|$. So $g(1) = \left|\int_0^1 f\,dt\right|$,

$g(5) = \int_0^5 f\,dt = g(1) - \left|\int_1^3 f\,dt\right| + \left|\int_3^5 f\,dt\right|$, and $g(9) = \int_0^9 f\,dt = g(5) - \left|\int_5^7 f\,dt\right| + \left|\int_7^9 f\,dt\right|$. Thus,

$g(1) < g(5) < g(9)$, and so the absolute maximum of $g(x)$ occurs at $x = 9$.

(c) g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$,

so $g''(x) = f'(x)$, which is negative on (approximately) $\left(\frac{1}{2}, 2\right)$, $(4, 6)$ and

$(8, 9)$. So g is concave downward on these intervals.

(d)
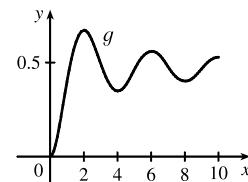
74. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 2, 4, 6, 8$, and 10. g has local maxima at $x = 2$ and 6 (since $f = g'$

changes from positive to negative there) and local minima at $x = 4$ and 8. There is no local maximum or minimum at

$x = 10$, since f is not defined for $x > 10$.

(b) We can see from the graph that $\left|\int_0^2 f\,dt\right| > \left|\int_2^4 f\,dt\right| > \left|\int_4^6 f\,dt\right| > \left|\int_6^8 f\,dt\right| > \left|\int_8^{10} f\,dt\right|$. So $g(2) = \left|\int_0^2 f\,dt\right|$,

$g(6) = \int_0^6 f\,dt = g(2) - \left|\int_2^4 f\,dt\right| + \left|\int_4^6 f\,dt\right|$, and $g(10) = \int_0^{10} f\,dt = g(6) - \left|\int_6^8 f\,dt\right| + \left|\int_8^{10} f\,dt\right|$. Thus,

$g(2) > g(6) > g(10)$, and so the absolute maximum of $g(x)$ occurs at $x = 2$.

(c) g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$,

so $g''(x) = f'(x)$, which is negative on $(1, 3)$, $(5, 7)$ and $(9, 10)$. So g is

concave downward on these intervals.

(d)

75. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^4}{n^5}+\frac{i}{n^2}\right)=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^4}{n^4}+\frac{i}{n}\right)\frac{1}{n}=\lim_{n\to\infty}\frac{1-0}{n}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^4+\frac{i}{n}\right]=\int_0^1(x^4+x)\,dx$

$$=\left[\tfrac{1}{5}x^5+\tfrac{1}{2}x^2\right]_0^1=\left(\tfrac{1}{5}+\tfrac{1}{2}\right)-0=\tfrac{7}{10}$$

76. $\displaystyle\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\cdots+\sqrt{\frac{n}{n}}\right)=\lim_{n\to\infty}\frac{1-0}{n}\sum_{i=1}^{n}\sqrt{\frac{i}{n}}=\int_0^1\sqrt{x}\,dx=\left[\frac{2x^{3/2}}{3}\right]_0^1=\frac{2}{3}-0=\frac{2}{3}$

77. Suppose $h<0$. Since f is continuous on $[x+h,x]$, the Extreme Value Theorem says that there are numbers u and v in

$[x+h,x]$ such that $f(u)=m$ and $f(v)=M$, where m and M are the absolute minimum and maximum values of f on

$[x+h,x]$. By Property 8 of integrals, $m(-h)\le\int_{x+h}^x f(t)\,dt\le M(-h)$; that is, $f(u)(-h)\le-\int_x^{x+h}f(t)\,dt\le f(v)(-h)$.

Since $-h>0$, we can divide this inequality by $-h$: $f(u)\le\dfrac{1}{h}\displaystyle\int_x^{x+h}f(t)\,dt\le f(v)$. By Equation 2,

$\dfrac{g(x+h)-g(x)}{h}=\dfrac{1}{h}\displaystyle\int_x^{x+h}f(t)\,dt$ for $h\ne 0$, and hence $f(u)\le\dfrac{g(x+h)-g(x)}{h}\le f(v)$, which is Equation 3 in the

case where $h<0$.

78. $\dfrac{d}{dx}\displaystyle\int_{g(x)}^{h(x)}f(t)\,dt=\dfrac{d}{dx}\left[\int_{g(x)}^{a}f(t)\,dt+\int_a^{h(x)}f(t)\,dt\right]$ [where a is in the domain of f]

$$=\dfrac{d}{dx}\left[-\int_a^{g(x)}f(t)\,dt\right]+\dfrac{d}{dx}\left[\int_a^{h(x)}f(t)\,dt\right]=-f(g(x))\,g'(x)+f(h(x))\,h'(x)$$

$$=f(h(x))\,h'(x)-f(g(x))\,g'(x)$$

79. (a) Let $f(x)=\sqrt{x}$ $\Rightarrow$ $f'(x)=1/(2\sqrt{x})>0$ for $x>0$ $\Rightarrow$ f is increasing on $(0,\infty)$. If $x\ge 0$, then $x^3\ge 0$, so

$1+x^3\ge 1$ and since f is increasing, this means that $f(1+x^3)\ge f(1)$ $\Rightarrow$ $\sqrt{1+x^3}\ge 1$ for $x\ge 0$. Next let

$g(t)=t^2-t$ $\Rightarrow$ $g'(t)=2t-1$ $\Rightarrow$ $g'(t)>0$ when $t\ge 1$. Thus, g is increasing on $(1,\infty)$. And since $g(1)=0$,

$g(t)\ge 0$ when $t\ge 1$. Now let $t=\sqrt{1+x^3}$, where $x\ge 0$. $\sqrt{1+x^3}\ge 1$ (from above) $\Rightarrow$ $t\ge 1$ $\Rightarrow$ $g(t)\ge 0$ $\Rightarrow$

$(1+x^3)-\sqrt{1+x^3}\ge 0$ for $x\ge 0$. Therefore, $1\le\sqrt{1+x^3}\le 1+x^3$ for $x\ge 0$.

(b) From part (a) and Property 7: $\int_0^1 1\,dx\le\int_0^1\sqrt{1+x^3}\,dx\le\int_0^1(1+x^3)\,dx$ $\Leftrightarrow$

$\left[x\right]_0^1\le\int_0^1\sqrt{1+x^3}\,dx\le\left[x+\tfrac{1}{4}x^4\right]_0^1$ $\Leftrightarrow$ $1\le\int_0^1\sqrt{1+x^3}\,dx\le 1+\tfrac{1}{4}=1.25$.

80. (a) For $0\le x\le 1$, we have $x^2\le x$. Since $f(x)=\cos x$ is a decreasing function on $[0,1]$, $\cos(x^2)\ge\cos x$.

(b) $\pi/6<1$, so by part (a), $\cos(x^2)\ge\cos x$ on $[0,\pi/6]$. Thus,

$\int_0^{\pi/6}\cos(x^2)\,dx\ge\int_0^{\pi/6}\cos x\,dx=\left[\sin x\right]_0^{\pi/6}=\sin(\pi/6)-\sin 0=\tfrac{1}{2}-0=\tfrac{1}{2}$.

81. $0<\dfrac{x^2}{x^4+x^2+1}<\dfrac{x^2}{x^4}=\dfrac{1}{x^2}$ on $[5,10]$, so

$0\le\displaystyle\int_5^{10}\frac{x^2}{x^4+x^2+1}\,dx<\int_5^{10}\frac{1}{x^2}\,dx=\left[-\frac{1}{x}\right]_5^{10}=-\frac{1}{10}-\left(-\frac{1}{5}\right)=\frac{1}{10}=0.1$.

82. (a) If $x < 0$, then $g(x) = \int_0^x f(t)\,dt = \int_0^x 0\,dt = 0$.

If $0 \le x \le 1$, then $g(x) = \int_0^x f(t)\,dt = \int_0^x t\,dt = \left[\frac{1}{2}t^2\right]_0^x = \frac{1}{2}x^2$.

If $1 < x \le 2$, then

$g(x) = \int_0^x f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^x f(t)\,dt = g(1) + \int_1^x (2-t)\,dt$

$\quad = \frac{1}{2}(1)^2 + \left[2t - \frac{1}{2}t^2\right]_1^x = \frac{1}{2} + \left(2x - \frac{1}{2}x^2\right) - \left(2 - \frac{1}{2}\right) = 2x - \frac{1}{2}x^2 - 1$.

If $x > 2$, then $g(x) = \int_0^x f(t)\,dt = g(2) + \int_2^x 0\,dt = 1 + 0 = 1$. So

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2}x^2 & \text{if } 0 \le x \le 1 \\ 2x - \frac{1}{2}x^2 - 1 & \text{if } 1 < x \le 2 \\ 1 & \text{if } x > 2 \end{cases}$$

(b)

(c) f is not differentiable at its corners at $x = 0$, 1, and 2. f is differentiable on $(-\infty, 0)$, $(0, 1)$, $(1, 2)$ and $(2, \infty)$.

g is differentiable on $(-\infty, \infty)$.

83. Using FTC1, we differentiate both sides of $6 + \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}$ to get $\frac{f(x)}{x^2} = 2\frac{1}{2\sqrt{x}}$ $\Rightarrow$ $f(x) = x^{3/2}$.

To find a, we substitute $x = a$ in the original equation to obtain $6 + \int_a^a \frac{f(t)}{t^2}\,dt = 2\sqrt{a}$ $\Rightarrow$ $6 + 0 = 2\sqrt{a}$ $\Rightarrow$

$3 = \sqrt{a}$ $\Rightarrow$ $a = 9$.

84. $B = 3A$ $\Rightarrow$ $\int_0^b e^x\,dx = 3\int_0^a e^x\,dx$ $\Rightarrow$ $[e^x]_0^b = 3\,[e^x]_0^a$ $\Rightarrow$ $e^b - 1 = 3(e^a - 1)$ $\Rightarrow$ $e^b = 3e^a - 2$ $\Rightarrow$

$b = \ln(3e^a - 2)$

85. (a) Let $F(t) = \int_0^t f(s)\,ds$. Then, by FTC1, $F'(t) = f(t) = $ rate of depreciation, so $F(t)$ represents the loss in value over the

interval $[0, t]$.

(b) $C(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right] = \frac{A + F(t)}{t}$ represents the average expenditure per unit of t during the interval $[0, t]$,

assuming that there has been only one overhaul during that time period. The company wants to minimize average

expenditure.
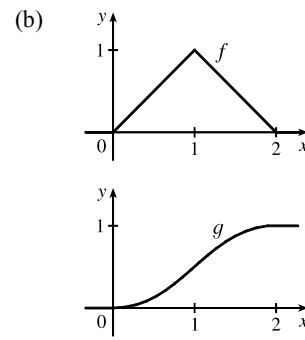
(c) $C(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right]$. Using FTC1, we have $C'(t) = -\frac{1}{t^2}\left[A + \int_0^t f(s)\,ds\right] + \frac{1}{t}f(t)$.

$C'(t) = 0$ $\Rightarrow$ $t\,f(t) = A + \int_0^t f(s)\,ds$ $\Rightarrow$ $f(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right] = C(t)$.

86. (a) $C(t) = \frac{1}{t}\int_0^t [f(s) + g(s)]\,ds$. Using FTC1 and the Product Rule, we have

$C'(t) = \frac{1}{t}[f(t) + g(t)] - \frac{1}{t^2}\int_0^t [f(s) + g(s)]\,ds$. Set $C'(t) = 0$: $\frac{1}{t}[f(t) + g(t)] - \frac{1}{t^2}\int_0^t [f(s) + g(s)]\,ds = 0$ $\Rightarrow$

$[f(t) + g(t)] - \frac{1}{t}\int_0^t [f(s) + g(s)]\,ds = 0$ $\Rightarrow$ $[f(t) + g(t)] - C(t) = 0$ $\Rightarrow$ $C(t) = f(t) + g(t)$.

(b) For $0 \le t \le 30$, we have $D(t) = \int_0^t \left(\dfrac{V}{15} - \dfrac{V}{450}s \right) ds = \left[\dfrac{V}{15}s - \dfrac{V}{900}s^2 \right]_0^t = \dfrac{V}{15}t - \dfrac{V}{900}t^2$.

So $D(t) = V \Rightarrow \dfrac{V}{15}t - \dfrac{V}{900}t^2 = V \Rightarrow 60t - t^2 = 900 \Rightarrow t^2 - 60t + 900 = 0 \Rightarrow$

$(t - 30)^2 = 0 \Rightarrow t = 30$. So the length of time T is 30 months.

(c) $C(t) = \dfrac{1}{t} \int_0^t \left(\dfrac{V}{15} - \dfrac{V}{450}s + \dfrac{V}{12{,}900}s^2 \right) ds = \dfrac{1}{t} \left[\dfrac{V}{15}s - \dfrac{V}{900}s^2 + \dfrac{V}{38{,}700}s^3 \right]_0^t$

$\qquad = \dfrac{1}{t} \left(\dfrac{V}{15}t - \dfrac{V}{900}t^2 + \dfrac{V}{38{,}700}t^3 \right) = \dfrac{V}{15} - \dfrac{V}{900}t + \dfrac{V}{38{,}700}t^2 \Rightarrow$

$C'(t) = -\dfrac{V}{900} + \dfrac{V}{19{,}350}t = 0$ when $\dfrac{1}{19{,}350}t = \dfrac{1}{900} \Rightarrow t = 21.5$.
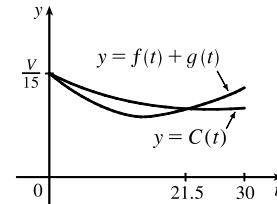
$C(21.5) = \dfrac{V}{15} - \dfrac{V}{900}(21.5) + \dfrac{V}{38{,}700}(21.5)^2 \approx 0.05472V$, $C(0) = \dfrac{V}{15} \approx 0.06667V$, and

$C(30) = \dfrac{V}{15} - \dfrac{V}{900}(30) + \dfrac{V}{38{,}700}(30)^2 \approx 0.05659V$, so the absolute minimum is $C(21.5) \approx 0.05472V$.

(d) As in part (c), we have $C(t) = \dfrac{V}{15} - \dfrac{V}{900}t + \dfrac{V}{38{,}700}t^2$, so $C(t) = f(t) + g(t) \Leftrightarrow$

$\dfrac{V}{15} - \dfrac{V}{900}t + \dfrac{V}{38{,}700}t^2 = \dfrac{V}{15} - \dfrac{V}{450}t + \dfrac{V}{12{,}900}t^2 \Leftrightarrow$

$t^2 \left(\dfrac{1}{12{,}900} - \dfrac{1}{38{,}700} \right) = t \left(\dfrac{1}{450} - \dfrac{1}{900} \right) \Leftrightarrow t = \dfrac{1/900}{2/38{,}700} = \dfrac{43}{2} = 21.5$.

This is the value of t that we obtained as the critical number of C in part (c), so we

have verified the result of (a) in this case.

5.4 Indefinite Integrals and the Net Change Theorem

1. $\dfrac{d}{dx}\left[-\dfrac{\sqrt{1+x^2}}{x} + C \right] = \dfrac{d}{dx}\left[-\dfrac{(1+x^2)^{1/2}}{x} + C \right] = -\dfrac{x \cdot \frac{1}{2}(1+x^2)^{-1/2}(2x) - (1+x^2)^{1/2} \cdot 1}{(x)^2} + 0$

$\qquad = -\dfrac{(1+x^2)^{-1/2}\left[x^2 - (1+x^2) \right]}{x^2} = -\dfrac{-1}{(1+x^2)^{1/2}x^2} = \dfrac{1}{x^2\sqrt{1+x^2}}$

2. $\dfrac{d}{dx}\left(\dfrac{1}{2}x + \dfrac{1}{4}\sin 2x + C \right) = \dfrac{1}{2} + \dfrac{1}{4}\cos 2x \cdot 2 + 0 = \dfrac{1}{2} + \dfrac{1}{2}\cos 2x$

$\qquad = \frac{1}{2} + \frac{1}{2}\left(2\cos^2 x - 1 \right) = \frac{1}{2} + \cos^2 x - \frac{1}{2} = \cos^2 x$

3. $\dfrac{d}{dx}(\tan x - x + C) = \sec^2 x - 1 + 0 = \tan^2 x$

4. $\dfrac{d}{dx}\left[\dfrac{2}{15b^2}(3bx - 2a)(a + bx)^{3/2} + C \right] = \dfrac{2}{15b^2}\left[(3bx - 2a)\frac{3}{2}(a + bx)^{1/2}(b) + (a + bx)^{3/2}(3b) + 0 \right]$

$\qquad = \dfrac{2}{15b^2}(3b)(a + bx)^{1/2}\left[(3bx - 2a)\frac{1}{2} + (a + bx) \right]$

$\qquad = \dfrac{2}{5b}(a + bx)^{1/2}\left(\dfrac{5}{2}bx \right) = x\sqrt{a + bx}$

5. $\displaystyle\int (x^{1.3} + 7x^{2.5})\, dx = \frac{1}{2.3}x^{2.3} + \frac{7}{3.5}x^{3.5} + C = \frac{1}{2.3}x^{2.3} + 2x^{3.5} + C$

6. $\int \sqrt[4]{x^5}\, dx = \int x^{5/4}\, dx = \frac{4}{9}x^{9/4} + C$

7. $\int \left(5 + \frac{2}{3}x^2 + \frac{3}{4}x^3\right) dx = 5x + \frac{2}{3}\cdot\frac{1}{3}x^3 + \frac{3}{4}\cdot\frac{1}{4}x^4 + C = 5x + \frac{2}{9}x^3 + \frac{3}{16}x^4 + C$

8. $\int (u^6 - 2u^5 - u^3 + \frac{2}{7})\, du = \frac{1}{7}u^7 - 2\cdot\frac{1}{6}u^6 - \frac{1}{4}u^4 + \frac{2}{7}u + C = \frac{1}{7}u^7 - \frac{1}{3}u^6 - \frac{1}{4}u^4 + \frac{2}{7}u + C$

9. $\displaystyle\int (u+4)(2u+1)\, du = \int (2u^2 + 9u + 4)\, du = 2\frac{u^3}{3} + 9\frac{u^2}{2} + 4u + C = \frac{2}{3}u^3 + \frac{9}{2}u^2 + 4u + C$

10. $\int \sqrt{t}(t^2 + 3t + 2)\, dt = \int t^{1/2}(t^2 + 3t + 2)\, dt = \int (t^{5/2} + 3t^{3/2} + 2t^{1/2})\, dt$

$$= \frac{2}{7}t^{7/2} + 3\cdot\frac{2}{5}t^{5/2} + 2\cdot\frac{2}{3}t^{3/2} + C = \frac{2}{7}t^{7/2} + \frac{6}{5}t^{5/2} + \frac{4}{3}t^{3/2} + C$$

11. $\displaystyle\int \frac{1 + \sqrt{x} + x}{x}\, dx = \int \left(\frac{1}{x} + \frac{\sqrt{x}}{x} + \frac{x}{x}\right) dx = \int \left(\frac{1}{x} + x^{-1/2} + 1\right) dx$

$$= \ln|x| + 2x^{1/2} + x + C = \ln|x| + 2\sqrt{x} + x + C$$

12. $\displaystyle\int \left(x^2 + 1 + \frac{1}{x^2 + 1}\right) dx = \frac{x^3}{3} + x + \tan^{-1}x + C$

13. $\int (\sin x + \sinh x)\, dx = -\cos x + \cosh x + C$

14. $\displaystyle\int \left(\frac{1+r}{r}\right)^2 dr = \int \frac{1 + 2r + r^2}{r^2}\, dr = \int (r^{-2} + 2r^{-1} + 1)\, dr = -r^{-1} + 2\ln|r| + r + C = -\frac{1}{r} + 2\ln|r| + r + C$
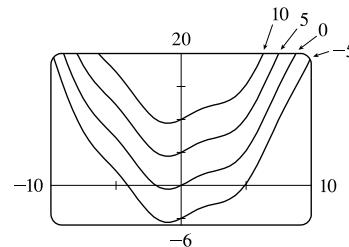
15. $\int (2 + \tan^2\theta)\, d\theta = \int [2 + (\sec^2\theta - 1)]\, d\theta = \int (1 + \sec^2\theta)\, d\theta = \theta + \tan\theta + C$

16. $\int \sec t\,(\sec t + \tan t)\, dt = \int (\sec^2 t + \sec t\,\tan t)\, dt = \tan t + \sec t + C$
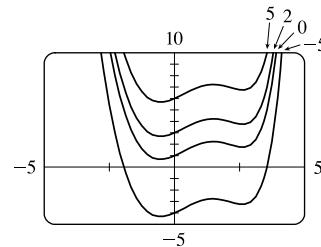
17. $\displaystyle\int 2^t(1 + 5^t)\, dt = \int (2^t + 2^t\cdot 5^t)\, dt = \int (2^t + 10^t)\, dt = \frac{2^t}{\ln 2} + \frac{10^t}{\ln 10} + C$

18. $\displaystyle\int \frac{\sin 2x}{\sin x}\, dx = \int \frac{2\sin x\,\cos x}{\sin x}\, dx = \int 2\cos x\, dx = 2\sin x + C$

19. $\int \left(\cos x + \frac{1}{2}x\right) dx = \sin x + \frac{1}{4}x^2 + C$. The members of the family

in the figure correspond to $C = -5, 0, 5$, and 10.

20. $\int (e^x - 2x^2)\, dx = e^x - \frac{2}{3}x^3 + C$. The members of the family in the

figure correspond to $C = -5, 0, 2$, and 5.

21. $\int_{-2}^{3} (x^2 - 3)\, dx = \left[\frac{1}{3}x^3 - 3x\right]_{-2}^{3} = (9 - 9) - \left(-\frac{8}{3} + 6\right) = \frac{8}{3} - \frac{18}{3} = -\frac{10}{3}$

22. $\int_{1}^{2} (4x^3 - 3x^2 + 2x)\, dx = \left[x^4 - x^3 + x^2\right]_{1}^{2} = (16 - 8 + 4) - (1 - 1 + 1) = 12 - 1 = 11$

23. $\int_{-2}^{0} \left(\frac{1}{2}t^4 + \frac{1}{4}t^3 - t\right) dt = \left[\frac{1}{10}t^5 + \frac{1}{16}t^4 - \frac{1}{2}t^2\right]_{-2}^{0} = 0 - \left[\frac{1}{10}(-32) + \frac{1}{16}(16) - \frac{1}{2}(4)\right] = -\left(-\frac{16}{5} + 1 - 2\right) = \frac{21}{5}$

24. $\int_{0}^{3} (1 + 6w^2 - 10w^4)\, dw = \left[w + 2w^3 - 2w^5\right]_{0}^{3} = (3 + 54 - 486) - 0 = -429$

25. $\int_{0}^{2} (2x - 3)(4x^2 + 1)\, dx = \int_{0}^{2} (8x^3 - 12x^2 + 2x - 3)\, dx = \left[2x^4 - 4x^3 + x^2 - 3x\right]_{0}^{2} = (32 - 32 + 4 - 6) - 0 = -2$

26. $\int_{-1}^{1} t(1 - t)^2\, dt = \int_{-1}^{1} t(1 - 2t + t^2)\, dt = \int_{-1}^{1} (t - 2t^2 + t^3)\, dt = \left[\frac{1}{2}t^2 - \frac{2}{3}t^3 + \frac{1}{4}t^4\right]_{-1}^{1}$

$\qquad = \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) - \left(\frac{1}{2} + \frac{2}{3} + \frac{1}{4}\right) = -\frac{4}{3}$

27. $\int_{0}^{\pi} (5e^x + 3\sin x)\, dx = \left[5e^x - 3\cos x\right]_{0}^{\pi} = \left[5e^{\pi} - 3(-1)\right] - \left[5(1) - 3(1)\right] = 5e^{\pi} + 1$

28. $\int_{1}^{2} \left(\frac{1}{x^2} - \frac{4}{x^3}\right) dx = \int_{1}^{2} (x^{-2} - 4x^{-3})\, dx = \left[\frac{x^{-1}}{-1} - \frac{4x^{-2}}{-2}\right]_{1}^{2} = \left[-\frac{1}{x} + \frac{2}{x^2}\right]_{1}^{2} = \left(-\frac{1}{2} + \frac{1}{2}\right) - (-1 + 2) = -1$

29. $\int_{1}^{4} \left(\frac{4 + 6u}{\sqrt{u}}\right) du = \int_{1}^{4} \left(\frac{4}{\sqrt{u}} + \frac{6u}{\sqrt{u}}\right) du = \int_{1}^{4} (4u^{-1/2} + 6u^{1/2})\, du = \left[8u^{1/2} + 4u^{3/2}\right]_{1}^{4} = (16 + 32) - (8 + 4) = 36$

30. $\int_{0}^{1} \frac{4}{1 + p^2}\, dp = \left[4\arctan p\right]_{0}^{1} = 4\arctan 1 - 4\arctan 0 = 4\left(\frac{\pi}{4}\right) - 4(0) = \pi$

31. $\int_{0}^{1} x\left(\sqrt[3]{x} + \sqrt[4]{x}\right) dx = \int_{0}^{1} (x^{4/3} + x^{5/4})\, dx = \left[\frac{3}{7}x^{7/3} + \frac{4}{9}x^{9/4}\right]_{0}^{1} = \left(\frac{3}{7} + \frac{4}{9}\right) - 0 = \frac{55}{63}$

32. $\int_{1}^{4} \frac{\sqrt{y} - y}{y^2}\, dy = \int_{1}^{4} \left(\frac{\sqrt{y}}{y^2} - \frac{y}{y^2}\right) dy = \int_{1}^{4} (y^{-3/2} - y^{-1})\, dy = \left[-2y^{-1/2} - \ln|y|\right]_{1}^{4} = \left[-\frac{2}{\sqrt{y}} - \ln|y|\right]_{1}^{4}$

$\qquad = (-1 - \ln 4) - (-2 - \ln 1) = 1 - \ln 4$

33. $\int_{1}^{2} \left(\frac{x}{2} - \frac{2}{x}\right) dx = \left[\frac{1}{4}x^2 - 2\ln|x|\right]_{1}^{2} = (1 - 2\ln 2) - \left(\frac{1}{4} - 2\ln 1\right) = \frac{3}{4} - 2\ln 2$

34. $\int_{0}^{1} (5x - 5^x)\, dx = \left[\frac{5}{2}x^2 - \frac{5^x}{\ln 5}\right]_{0}^{1} = \left(\frac{5}{2} - \frac{5}{\ln 5}\right) - \left(0 - \frac{1}{\ln 5}\right) = \frac{5}{2} - \frac{4}{\ln 5}$

35. $\int_{0}^{1} (x^{10} + 10^x)\, dx = \left[\frac{x^{11}}{11} + \frac{10^x}{\ln 10}\right]_{0}^{1} = \left(\frac{1}{11} + \frac{10}{\ln 10}\right) - \left(0 + \frac{1}{\ln 10}\right) = \frac{1}{11} + \frac{9}{\ln 10}$

36. $\int_{0}^{\pi/4} \sec\theta\, \tan\theta\, d\theta = \left[\sec\theta\right]_{0}^{\pi/4} = \sec\frac{\pi}{4} - \sec 0 = \sqrt{2} - 1$

37. $\int_{0}^{\pi/4} \frac{1 + \cos^2\theta}{\cos^2\theta}\, d\theta = \int_{0}^{\pi/4} \left(\frac{1}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta}\right) d\theta = \int_{0}^{\pi/4} (\sec^2\theta + 1)\, d\theta$

$\qquad = \left[\tan\theta + \theta\right]_{0}^{\pi/4} = \left(\tan\frac{\pi}{4} + \frac{\pi}{4}\right) - (0 + 0) = 1 + \frac{\pi}{4}$

38. $\int_0^{\pi/3} \dfrac{\sin\theta + \sin\theta \tan^2\theta}{\sec^2\theta}\, d\theta = \int_0^{\pi/3} \dfrac{\sin\theta\,(1+\tan^2\theta)}{\sec^2\theta}\, d\theta = \int_0^{\pi/3} \dfrac{\sin\theta\,\sec^2\theta}{\sec^2\theta}\, d\theta = \int_0^{\pi/3} \sin\theta\, d\theta$

$\qquad\qquad = \big[-\cos\theta\big]_0^{\pi/3} = -\tfrac{1}{2} - (-1) = \tfrac{1}{2}$

39. $\int_1^8 \dfrac{2+t}{\sqrt[3]{t^2}}\, dt = \int_1^8 \left(\dfrac{2}{t^{2/3}} + \dfrac{t}{t^{2/3}}\right) dt = \int_1^8 (2t^{-2/3} + t^{1/3})\, dt = \left[2\cdot 3t^{1/3} + \tfrac{3}{4}t^{4/3}\right]_1^8 = (12+12) - \left(6+\tfrac{3}{4}\right) = \tfrac{69}{4}$

40. $\int_{-10}^{10} \dfrac{2e^x}{\sinh x + \cosh x}\, dx = \int_{-10}^{10} \dfrac{2e^x}{\dfrac{e^x - e^{-x}}{2} + \dfrac{e^x + e^{-x}}{2}}\, dx = \int_{-10}^{10} \dfrac{2e^x}{e^x}\, dx = \int_{-10}^{10} 2\, dx = \big[2x\big]_{-10}^{10} = 20 - (-20) = 40$

41. $\int_0^{\sqrt{3}/2} \dfrac{dr}{\sqrt{1-r^2}} = \Big[\arcsin r\Big]_0^{\sqrt{3}/2} = \arcsin\!\left(\sqrt{3}/2\right) - \arcsin 0 = \dfrac{\pi}{3} - 0 = \dfrac{\pi}{3}$

42. $\int_1^2 \dfrac{(x-1)^3}{x^2}\, dx = \int_1^2 \dfrac{x^3 - 3x^2 + 3x - 1}{x^2}\, dx = \int_1^2 \left(x - 3 + \dfrac{3}{x} - \dfrac{1}{x^2}\right) dx = \left[\tfrac{1}{2}x^2 - 3x + 3\ln|x| + \dfrac{1}{x}\right]_1^2$

$\qquad\qquad = \left(2 - 6 + 3\ln 2 + \tfrac{1}{2}\right) - \left(\tfrac{1}{2} - 3 + 0 + 1\right) = 3\ln 2 - 2$

43. $\int_0^{1/\sqrt{3}} \dfrac{t^2 - 1}{t^4 - 1}\, dt = \int_0^{1/\sqrt{3}} \dfrac{t^2 - 1}{(t^2 + 1)(t^2 - 1)}\, dt = \int_0^{1/\sqrt{3}} \dfrac{1}{t^2 + 1}\, dt = \Big[\arctan t\Big]_0^{1/\sqrt{3}} = \arctan\!\left(1/\sqrt{3}\right) - \arctan 0$

$\qquad\qquad = \tfrac{\pi}{6} - 0 = \tfrac{\pi}{6}$

44. $|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \geq 0 \\ -(2x-1) & \text{if } 2x-1 < 0 \end{cases} = \begin{cases} 2x-1 & \text{if } x \geq \tfrac{1}{2} \\ 1-2x & \text{if } x < \tfrac{1}{2} \end{cases}$

Thus, $\int_0^2 |2x-1|\, dx = \int_0^{1/2}(1-2x)\, dx + \int_{1/2}^2 (2x-1)\, dx = \big[x - x^2\big]_0^{1/2} + \big[x^2 - x\big]_{1/2}^2$

$\qquad\qquad = \left(\tfrac{1}{2} - \tfrac{1}{4}\right) - 0 + (4-2) - \left(\tfrac{1}{4} - \tfrac{1}{2}\right) = \tfrac{1}{4} + 2 - \left(-\tfrac{1}{4}\right) = \tfrac{5}{2}$

45. $\int_{-1}^2 (x - 2|x|)\, dx = \int_{-1}^0 [x - 2(-x)]\, dx + \int_0^2 [x - 2(x)]\, dx = \int_{-1}^0 3x\, dx + \int_0^2 (-x)\, dx = 3\big[\tfrac{1}{2}x^2\big]_{-1}^0 - \big[\tfrac{1}{2}x^2\big]_0^2$

$\qquad\qquad = 3\left(0 - \tfrac{1}{2}\right) - (2-0) = -\tfrac{7}{2} = -3.5$
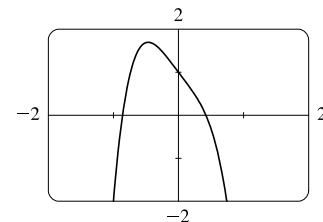
46. $\int_0^{3\pi/2} |\sin x|\, dx = \int_0^\pi \sin x\, dx + \int_\pi^{3\pi/2}(-\sin x)\, dx = \big[-\cos x\big]_0^\pi + \big[\cos x\big]_\pi^{3\pi/2} = [1-(-1)] + [0-(-1)] = 2+1 = 3$

47. The graph shows that $y = 1 - 2x - 5x^4$ has x-intercepts at

$x = a \approx -0.86$ and at $x = b \approx 0.42$. So the area of the region that lies

under the curve and above the x-axis is

$\int_a^b (1 - 2x - 5x^4)\, dx = \big[x - x^2 - x^5\big]_a^b$

$\qquad\qquad = (b - b^2 - b^5) - (a - a^2 - a^5) \approx 1.36$
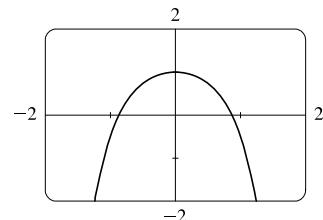
48. The graph shows that $y = (x^2 + 1)^{-1} - x^4$ has x-intercepts at

$x = a \approx -0.87$ and at $x = b \approx 0.87$. So the area of the region that lies

under the curve and above the x-axis is

$\int_a^b \big[(x^2 + 1)^{-1} - x^4\big]\, dx = \big[\tan^{-1} x - \tfrac{1}{5}x^5\big]_a^b$

$\qquad\qquad = \left(\tan^{-1} b - \tfrac{1}{5}b^5\right) - \left(\tan^{-1} a - \tfrac{1}{5}a^5\right)$

$\qquad\qquad \approx 1.23$

49. $A = \int_0^2 (2y - y^2)\, dy = \left[y^2 - \frac{1}{3}y^3\right]_0^2 = \left(4 - \frac{8}{3}\right) - 0 = \frac{4}{3}$

50. $y = \sqrt[4]{x} \;\Rightarrow\; x = y^4$, so $A = \int_0^1 y^4\, dy = \left[\frac{1}{5}y^5\right]_0^1 = \frac{1}{5}$.

51. If $w'(t)$ is the rate of change of weight in pounds per year, then $w(t)$ represents the weight in pounds of the child at age t. We know from the Net Change Theorem that $\int_5^{10} w'(t)\, dt = w(10) - w(5)$, so the integral represents the increase in the child's weight (in pounds) between the ages of 5 and 10.

**52.** $\int_a^b I(t)\, dt = \int_a^b Q'(t)\, dt = Q(b) - Q(a)$ by the Net Change Theorem, so it represents the change in the charge Q from time $t = a$ to $t = b$.

53. Since $r(t)$ is the rate at which oil leaks, we can write $r(t) = -V'(t)$, where $V(t)$ is the volume of oil at time t. [Note that the minus sign is needed because V is decreasing, so $V'(t)$ is negative, but $r(t)$ is positive.] Thus, by the Net Change Theorem, $\int_0^{120} r(t)\, dt = -\int_0^{120} V'(t)\, dt = -[V(120) - V(0)] = V(0) - V(120)$, which is the number of gallons of oil that leaked from the tank in the first two hours (120 minutes).

54. By the Net Change Theorem, $\int_0^{15} n'(t)\, dt = n(15) - n(0) = n(15) - 100$ represents the increase in the bee population in 15 weeks. So $100 + \int_0^{15} n'(t)\, dt = n(15)$ represents the total bee population after 15 weeks.

55. By the Net Change Theorem, $\int_{1000}^{5000} R'(x)\, dx = R(5000) - R(1000)$, so it represents the increase in revenue when production is increased from 1000 units to 5000 units.

56. The slope of the trail is the rate of change of the elevation E, so $f(x) = E'(x)$. By the Net Change Theorem, $\int_3^5 f(x)\, dx = \int_3^5 E'(x)\, dx = E(5) - E(3)$ is the change in the elevation E between $x = 3$ miles and $x = 5$ miles from the start of the trail.

57. In general, the unit of measurement for $\int_a^b f(x)\, dx$ is the product of the unit for $f(x)$ and the unit for x. Since $f(x)$ is measured in newtons and x is measured in meters, the units for $\int_0^{100} f(x)\, dx$ are newton-meters (or joules). (A newton-meter is abbreviated N·m.)

58. The units for $a(x)$ are pounds per foot and the units for x are feet, so the units for da/dx are pounds per foot per foot, denoted (lb/ft)/ft. The unit of measurement for $\int_2^8 a(x)\, dx$ is the product of pounds per foot and feet; that is, pounds.

59. (a) Displacement $= \int_0^3 (3t - 5)\, dt = \left[\frac{3}{2}t^2 - 5t\right]_0^3 = \frac{27}{2} - 15 = -\frac{3}{2}$ m

    (b) Distance traveled $= \int_0^3 |3t - 5|\, dt = \int_0^{5/3} (5 - 3t)\, dt + \int_{5/3}^3 (3t - 5)\, dt$

$$= \left[5t - \frac{3}{2}t^2\right]_0^{5/3} + \left[\frac{3}{2}t^2 - 5t\right]_{5/3}^3 = \frac{25}{3} - \frac{3}{2} \cdot \frac{25}{9} + \frac{27}{2} - 15 - \left(\frac{3}{2} \cdot \frac{25}{9} - \frac{25}{3}\right) = \frac{41}{6} \text{ m}$$

60. (a) Displacement $= \int_2^4 (t^2 - 2t - 3)\, dt = \left[\frac{1}{3}t^3 - t^2 - 3t\right]_2^4 = \left(\frac{64}{3} - 16 - 12\right) - \left(\frac{8}{3} - 4 - 6\right) = \frac{2}{3}$ m

    (b) $v(t) = t^2 - 2t - 3 = (t + 1)(t - 3)$, so $v(t) < 0$ for $-1 < t < 3$, but on the interval $[2, 4]$, $v(t) < 0$ for $2 \le t < 3$.

    Distance traveled $= \int_2^4 |t^2 - 2t - 3|\, dt = \int_2^3 -(t^2 - 2t - 3)\, dt + \int_3^4 (t^2 - 2t - 3)\, dt$

$$= \left[-\frac{1}{3}t^3 + t^2 + 3t\right]_2^3 + \left[\frac{1}{3}t^3 - t^2 - 3t\right]_3^4$$

$$= (-9 + 9 + 9) - \left(-\frac{8}{3} + 4 + 6\right) + \left(\frac{64}{3} - 16 - 12\right) - (9 - 9 - 9) = 4 \text{ m}$$

61. (a) $v'(t) = a(t) = t + 4$ $\Rightarrow$ $v(t) = \frac{1}{2}t^2 + 4t + C$ $\Rightarrow$ $v(0) = C = 5$ $\Rightarrow$ $v(t) = \frac{1}{2}t^2 + 4t + 5$ m/s

(b) Distance traveled $= \int_0^{10} |v(t)|\, dt = \int_0^{10} \left|\frac{1}{2}t^2 + 4t + 5\right| dt = \int_0^{10}\left(\frac{1}{2}t^2 + 4t + 5\right) dt = \left[\frac{1}{6}t^3 + 2t^2 + 5t\right]_0^{10}$

$\qquad\qquad\qquad = \frac{500}{3} + 200 + 50 = 416\frac{2}{3}$ m

62. (a) $v'(t) = a(t) = 2t + 3$ $\Rightarrow$ $v(t) = t^2 + 3t + C$ $\Rightarrow$ $v(0) = C = -4$ $\Rightarrow$ $v(t) = t^2 + 3t - 4$

(b) Distance traveled $= \int_0^3 |t^2 + 3t - 4|\, dt = \int_0^3 |(t+4)(t-1)|\, dt = \int_0^1 \left(-t^2 - 3t + 4\right) dt + \int_1^3 \left(t^2 + 3t - 4\right) dt$

$\qquad\qquad\qquad = \left[-\frac{1}{3}t^3 - \frac{3}{2}t^2 + 4t\right]_0^1 + \left[\frac{1}{3}t^3 + \frac{3}{2}t^2 - 4t\right]_1^3$

$\qquad\qquad\qquad = \left(-\frac{1}{3} - \frac{3}{2} + 4\right) + \left(9 + \frac{27}{2} - 12\right) - \left(\frac{1}{3} + \frac{3}{2} - 4\right) = \frac{89}{6}$ m

63. Since $m'(x) = \rho(x)$, $m = \int_0^4 \rho(x)\, dx = \int_0^4 \left(9 + 2\sqrt{x}\right) dx = \left[9x + \frac{4}{3}x^{3/2}\right]_0^4 = 36 + \frac{32}{3} - 0 = \frac{140}{3} = 46\frac{2}{3}$ kg.

64. By the Net Change Theorem, the amount of water that flows from the tank during the first 10 minutes is

$\int_0^{10} r(t)\, dt = \int_0^{10}(200 - 4t)\, dt = \left[200t - 2t^2\right]_0^{10} = (2000 - 200) - 0 = 1800$ liters.

65. Let s be the position of the car. We know from Equation 2 that $s(100) - s(0) = \int_0^{100} v(t)\, dt$. We use the Midpoint Rule for

$0 \le t \le 100$ with $n = 5$. Note that the length of each of the five time intervals is 20 seconds $= \frac{20}{3600}$ hour $= \frac{1}{180}$ hour.

So the distance traveled is

$\int_0^{100} v(t)\, dt \approx \frac{1}{180}[v(10) + v(30) + v(50) + v(70) + v(90)] = \frac{1}{180}(38 + 58 + 51 + 53 + 47) = \frac{247}{180} \approx 1.4$ miles.

66. (a) By the Net Change Theorem, the total amount spewed into the atmosphere is $Q(6) - Q(0) = \int_0^6 r(t)\, dt = Q(6)$ since

$Q(0) = 0$. The rate $r(t)$ is positive, so Q is an increasing function. Thus, an upper estimate for $Q(6)$ is R_6 and a lower

estimate for $Q(6)$ is L_6. $\Delta t = \dfrac{b - a}{n} = \dfrac{6 - 0}{6} = 1$.

$R_6 = \sum_{i=1}^{6} r(t_i)\, \Delta t = 10 + 24 + 36 + 46 + 54 + 60 = 230$ tonnes.

$L_6 = \sum_{i=1}^{6} r(t_{i-1})\, \Delta t = R_6 + r(0) - r(6) = 230 + 2 - 60 = 172$ tonnes.

(b) $\Delta t = \dfrac{b - a}{n} = \dfrac{6 - 0}{3} = 2$. $Q(6) \approx M_3 = 2[r(1) + r(3) + r(5)] = 2(10 + 36 + 54) = 2(100) = 200$ tonnes.

67. From the Net Change Theorem, the increase in cost if the production level is raised from 2000 yards to 4000 yards is

$C(4000) - C(2000) = \int_{2000}^{4000} C'(x)\, dx$.

$$\int_{2000}^{4000} C'(x)\, dx = \int_{2000}^{4000}(3 - 0.01x + 0.000006x^2)\, dx = \left[3x - 0.005x^2 + 0.000002x^3\right]_{2000}^{4000}$$

$$= 60{,}000 - 2{,}000 = \$58{,}000$$

68. By the Net Change Theorem, the amount of water after four days is

$$25{,}000 + \int_0^4 r(t)\, dt \approx 25{,}000 + M_4 = 25{,}000 + \frac{4-0}{4}\left[r(0.5) + r(1.5) + r(2.5) + r(3.5)\right]$$

$$\approx 25{,}000 + [1500 + 1770 + 740 + (-690)] = 28{,}320 \text{ liters}$$

69. To use the Midpoint Rule, we'll use the midpoint of each of three 2-second intervals.

$$v(6) - v(0) = \int_0^6 a(t)\, dt \approx [a(1) + a(3) + a(5)]\frac{6-0}{3} \approx (0.6 + 10 + 9.3)(2) = 39.8 \text{ ft/s}$$

70. Use the midpoint of each of four 2-day intervals. Let $t = 0$ correspond to July 18 and note that the inflow rate, $r(t)$, is in ft^3/s.

$$\text{Amount of water} = \int_0^8 r(t)\, dt \approx [r(1) + r(3) + r(5) + r(7)]\frac{8-0}{4} \approx [6401 + 4249 + 3821 + 2628](2) = 34{,}198.$$

Now multiply by the number of seconds in a day, $24 \cdot 60^2$, to get 2,954,707,200 ft^3.

71. Let $P(t)$ denote the bacteria population at time t (in hours). By the Net Change Theorem,

$$P(1) - P(0) = \int_0^1 P'(t)\, dt = \int_0^1 (1000 \cdot 2^t)\, dt = \left[1000\,\frac{2^t}{\ln 2}\right]_0^1 = \frac{1000}{\ln 2}(2^1 - 2^0) = \frac{1000}{\ln 2} \approx 1443.$$

Thus, the population after one hour is $4000 + 1443 = 5443$.

72. Let $M(t)$ denote the number of megabits transmitted at time t (in hours) [note that $D(t)$ is measured in megabits/second]. By the Net Change Theorem and the Midpoint Rule,

$$M(8) - M(0) = \int_0^8 3600 D(t)\, dt \approx 3600 \cdot \tfrac{8-0}{4}[D(1) + D(3) + D(5) + D(7)]$$
$$\approx 7200(0.32 + 0.50 + 0.56 + 0.83) = 7200(2.21) = 15{,}912 \text{ megabits}$$

73. Power is the rate of change of energy with respect to time; that is, $P(t) = E'(t)$. By the Net Change Theorem and the Midpoint Rule,

$$E(24) - E(0) = \int_0^{24} P(t)\, dt \approx \frac{24-0}{12}[P(1) + P(3) + P(5) + \cdots + P(21) + P(23)]$$
$$\approx 2(16{,}900 + 16{,}400 + 17{,}000 + 19{,}800 + 20{,}700 + 21{,}200$$
$$+\ 20{,}500 + 20{,}500 + 21{,}700 + 22{,}300 + 21{,}700 + 18{,}900)$$
$$= 2(237{,}600) = 475{,}200$$

Thus, the energy used on that day was approximately 4.75×10^5 megawatt-hours.

74. (a) From Exercise 4.1.74(a), $v(t) = 0.00146t^3 - 0.11553t^2 + 24.98169t - 21.26872$.

(b) $h(125) - h(0) = \int_0^{125} v(t)\, dt = \left[0.000365t^4 - 0.03851t^3 + 12.490845t^2 - 21.26872t\right]_0^{125} \approx 206{,}407$ ft

5.5 The Substitution Rule

1. Let $u = 2x$. Then $du = 2\,dx$ and $dx = \frac{1}{2}\,du$, so $\int \cos 2x\, dx = \int \cos u \left(\frac{1}{2}\,du\right) = \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2x + C$.

2. Let $u = -x^2$. Then $du = -2x\,dx$ and $x\,dx = -\frac{1}{2}\,du$, so $\int xe^{-x^2}\, dx = \int e^u\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$.

3. Let $u = x^3 + 1$. Then $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$, so

$$\int x^2\sqrt{x^3+1}\, dx = \int \sqrt{u}\left(\tfrac{1}{3}\,du\right) = \frac{1}{3}\frac{u^{3/2}}{3/2} + C = \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + C = \tfrac{2}{9}(x^3+1)^{3/2} + C.$$

4. Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$, so $\int \sin^2\theta\,\cos\theta\,d\theta = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3\theta + C$.

5. Let $u = x^4 - 5$. Then $du = 4x^3\,dx$ and $x^3\,dx = \frac{1}{4}\,du$, so

$$\int \frac{x^3}{x^4 - 5}\,dx = \int \frac{1}{u}\left(\frac{1}{4}\,du\right) = \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln\left|x^4 - 5\right| + C.$$

6. Let $u = 2t + 1$. Then $du = 2\,dt$ and $dt = \frac{1}{2}\,du$, so $\int \sqrt{2t + 1}\,dt = \int \sqrt{u}\,\left(\frac{1}{2}\,du\right) = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}(2t+1)^{3/2} + C.$

7. Let $u = 1 - x^2$. Then $du = -2x\,dx$ and $x\,dx = -\frac{1}{2}\,du$, so

$$\int x\sqrt{1 - x^2}\,dx = \int \sqrt{u}\,\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = -\frac{1}{3}(1 - x^2)^{3/2} + C.$$

8. Let $u = x^3$. Then $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$, so $\int x^2 e^{x^3}\,dx = \int e^u\left(\frac{1}{3}\,du\right) = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C.$

9. Let $u = 1 - 2x$. Then $du = -2\,dx$ and $dx = -\frac{1}{2}\,du$, so

$$\int(1 - 2x)^9\,dx = \int u^9\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\cdot\frac{1}{10}u^{10} + C = -\frac{1}{20}(1 - 2x)^{10} + C.$$

10. Let $u = 1 + \cos t$. Then $du = -\sin t\,dt$ and $\sin t\,dt = -du$, so

$$\int \sin t\,\sqrt{1 + \cos t}\,dt = \int \sqrt{u}\,(-du) = -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}(1 + \cos t)^{3/2} + C.$$

11. Let $u = \frac{\pi}{2}t$. Then $du = \frac{\pi}{2}\,dt$ and $dt = \frac{2}{\pi}\,du$, so $\int \cos\left(\frac{\pi}{2}t\right)dt = \int \cos u\left(\frac{2}{\pi}\,du\right) = \frac{2}{\pi}\sin u + C = \frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right) + C.$

12. Let $u = 2\theta$. Then $du = 2\,d\theta$ and $d\theta = \frac{1}{2}\,du$, so $\int \sec^2 2\theta\,d\theta = \int \sec^2 u\left(\frac{1}{2}\,du\right) = \frac{1}{2}\tan u + C = \frac{1}{2}\tan 2\theta + C.$

13. Let $u = 5 - 3x$. Then $du = -3\,dx$ and $dx = -\frac{1}{3}\,du$, so

$$\int \frac{dx}{5 - 3x} = \int \frac{1}{u}\left(-\frac{1}{3}\,du\right) = -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln|5 - 3x| + C.$$

14. Let $u = 4 - y^3$. Then $du = -3y^2\,dy$ and $y^2\,dy = -\frac{1}{3}\,du$, so

$$\int y^2(4 - y^3)^{2/3}\,dy = \int u^{2/3}\left(-\frac{1}{3}\,du\right) = -\frac{1}{3}\cdot\frac{3}{5}u^{5/3} + C = -\frac{1}{5}(4 - y^3)^{5/3} + C.$$

15. Let $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$ and $\sin\theta\,d\theta = -du$, so

$$\int \cos^3\theta\,\sin\theta\,d\theta = \int u^3(-du) = -\frac{1}{4}u^4 + C = -\frac{1}{4}\cos^4\theta + C.$$

16. Let $u = -5r$. Then $du = -5\,dr$ and $dr = -\frac{1}{5}\,du$, so $\int e^{-5r}\,dr = \int e^u\left(-\frac{1}{5}\,du\right) = -\frac{1}{5}e^u + C = -\frac{1}{5}e^{-5r} + C.$

17. Let $x = 1 - e^u$. Then $dx = -e^u\,du$ and $e^u\,du = -dx$, so

$$\int \frac{e^u}{(1 - e^u)^2}\,du = \int \frac{1}{x^2}(-dx) = -\int x^{-2}\,dx = -(-x^{-1}) + C = \frac{1}{x} + C = \frac{1}{1 - e^u} + C.$$

18. Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$ and $2\,du = \frac{1}{\sqrt{x}}\,dx$, so

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \sin u\,(2\,du) = -2\cos u + C = -2\cos\sqrt{x} + C.$$

19. Let $u = 3ax + bx^3$. Then $du = (3a + 3bx^2)\,dx = 3(a + bx^2)\,dx$, so

$$\int \frac{a + bx^2}{\sqrt{3ax + bx^3}}\,dx = \int \frac{\frac{1}{3}\,du}{u^{1/2}} = \frac{1}{3}\int u^{-1/2}\,du = \frac{1}{3}\cdot 2u^{1/2} + C = \frac{2}{3}\sqrt{3ax + bx^3} + C.$$

20. Let $u = z^3 + 1$. Then $du = 3z^2\,dz$ and $\frac{1}{3}\,du = z^2\,dz + C$, so

$$\int \frac{z^2}{z^3 + 1}\,dz = \int \frac{1}{u}\left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\ln|u| + C = \tfrac{1}{3}\ln|z^3 + 1| + C.$$

21. Let $u = \ln x$. Then $du = \dfrac{dx}{x}$, so $\displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}(\ln x)^3 + C.$

22. Let $u = \cos x$. Then $du = -\sin x\,dx$ and $-du = \sin x\,dx$, so

$\int \sin x \, \sin(\cos x)\,dx = \int \sin u\,(-du) = (-\cos u)(-1) + C = \cos(\cos x) + C.$

23. Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$, so $\int \sec^2\theta \, \tan^3\theta\,d\theta = \int u^3\,du = \tfrac{1}{4}u^4 + C = \tfrac{1}{4}\tan^4\theta + C.$

24. Let $u = x + 2$. Then $du = dx$ and $x = u - 2$, so

$\int x\sqrt{x+2}\,dx = \int (u-2)\sqrt{u}\,du = \int (u^{3/2} - 2u^{1/2})\,du = \tfrac{2}{5}u^{5/2} - 2\cdot\tfrac{2}{3}u^{3/2} + C = \tfrac{2}{5}(x+2)^{5/2} - \tfrac{4}{3}(x+2)^{3/2} + C.$

25. Let $u = 1 + e^x$. Then $du = e^x\,dx$, so $\int e^x\sqrt{1 + e^x}\,dx = \int \sqrt{u}\,du = \tfrac{2}{3}u^{3/2} + C = \tfrac{2}{3}(1 + e^x)^{3/2} + C.$

Or: Let $u = \sqrt{1 + e^x}$. Then $u^2 = 1 + e^x$ and $2u\,du = e^x\,dx$, so

$\int e^x\sqrt{1 + e^x}\,dx = \int u\cdot 2u\,du = \tfrac{2}{3}u^3 + C = \tfrac{2}{3}(1 + e^x)^{3/2} + C.$

26. Let $u = ax + b$. Then $du = a\,dx$ and $dx = (1/a)\,du$, so

$$\int \frac{dx}{ax + b} = \int \frac{(1/a)\,du}{u} = \frac{1}{a}\int \frac{1}{u}\,du = \frac{1}{a}\ln|u| + C = \frac{1}{a}\ln|ax + b| + C.$$

27. Let $u = x^3 + 3x$. Then $du = (3x^2 + 3)\,dx$ and $\frac{1}{3}\,du = (x^2 + 1)\,dx$, so

$\int (x^2 + 1)(x^3 + 3x)^4\,dx = \int u^4\left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\cdot\tfrac{1}{5}u^5 + C = \tfrac{1}{15}(x^3 + 3x)^5 + C.$

28. Let $u = \cos t$. Then $du = -\sin t\,dt$ and $\sin t\,dt = -du$, so $\int e^{\cos t}\sin t\,dt = \int e^u\,(-du) = -e^u + C = -e^{\cos t} + C.$

29. Let $u = 5^t$. Then $du = 5^t\ln 5\,dt$ and $5^t\,dt = \dfrac{1}{\ln 5}\,du$, so

$$\int 5^t\sin(5^t)\,dt = \int \sin u\left(\frac{1}{\ln 5}\,du\right) = -\frac{1}{\ln 5}\cos u + C = -\frac{1}{\ln 5}\cos(5^t) + C.$$

30. Let $u = \tan x$. Then $du = \sec^2 x\,dx$, so

$$\int \frac{\sec^2 x}{\tan^2 x}\,dx = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = -1u^{-1} + C = -\frac{1}{\tan x} + C = -\cot x + C.$$

Or: $\displaystyle\int \frac{\sec^2 x}{\tan^2 x}\,dx = \int \left(\frac{1}{\cos^2 x}\cdot\frac{\cos^2 x}{\sin^2 x}\right)dx = \int \csc^2 x\,dx = -\cot x + C$

31. Let $u = \arctan x$. Then $du = \dfrac{1}{x^2 + 1}\,dx$, so $\displaystyle\int \frac{(\arctan x)^2}{x^2 + 1}\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}(\arctan x)^3 + C.$

32. Let $u = x^2 + 4$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so

$$\int \frac{x}{x^2 + 4}\,dx = \int \frac{1}{u}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2 + 4| + C = \frac{1}{2}\ln(x^2 + 4) + C \quad [\text{since } x^2 + 4 > 0].$$

33. Let $u = 1 + 5t$. Then $du = 5\,dt$ and $dt = \frac{1}{5}\,du$, so

$\int \cos(1 + 5t)\,dt = \int \cos u \left(\frac{1}{5}\,du \right) = \frac{1}{5}\sin u + C = \frac{1}{5}\sin(1 + 5t) + C.$

34. Let $u = \dfrac{\pi}{x}$. Then $du = -\dfrac{\pi}{x^2}\,dx$ and $\dfrac{1}{x^2}\,dx = -\dfrac{1}{\pi}\,du$, so

$$\int \frac{\cos(\pi/x)}{x^2}\,dx = \int \cos u \left(-\frac{1}{\pi}\,du \right) = -\frac{1}{\pi}\sin u + C = -\frac{1}{\pi}\sin\frac{\pi}{x} + C$$

35. Let $u = \cot x$. Then $du = -\csc^2 x\,dx$ and $\csc^2 x\,dx = -du$, so

$$\int \sqrt{\cot x}\,\csc^2 x\,dx = \int \sqrt{u}\,(-du) = -\frac{u^{3/2}}{3/2} + C = -\frac{2}{3}(\cot x)^{3/2} + C.$$

36. Let $u = 2^t + 3$. Then $du = 2^t \ln 2\,dt$ and $2^t\,dt = \dfrac{1}{\ln 2}\,du$, so

$$\int \frac{2^t}{2^t + 3}\,dt = \int \frac{1}{u}\left(\frac{1}{\ln 2}\,du \right) = \frac{1}{\ln 2}\ln|u| + C = \frac{1}{\ln 2}\ln(2^t + 3) + C.$$

37. Let $u = \sinh x$. Then $du = \cosh x\,dx$, so $\int \sinh^2 x \cosh x\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sinh^3 x + C.$

38. Let $u = 1 + \tan t$. Then $du = \sec^2 t\,dt$, so

$$\int \frac{dt}{\cos^2 t\,\sqrt{1 + \tan t}} = \int \frac{\sec^2 t\,dt}{\sqrt{1 + \tan t}} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{1 + \tan t} + C.$$

39. $\displaystyle\int \frac{\sin 2x}{1 + \cos^2 x}\,dx = 2\int \frac{\sin x \cos x}{1 + \cos^2 x}\,dx = 2I.$ Let $u = \cos x$. Then $du = -\sin x\,dx$, so

$2I = -2\displaystyle\int \frac{u\,du}{1 + u^2} = -2 \cdot \frac{1}{2}\ln(1 + u^2) + C = -\ln(1 + u^2) + C = -\ln(1 + \cos^2 x) + C.$

Or: Let $u = 1 + \cos^2 x$.

40. Let $u = \cos x$. Then $du = -\sin x\,dx$ and $\sin x\,dx = -du$, so

$$\int \frac{\sin x}{1 + \cos^2 x}\,dx = \int \frac{-du}{1 + u^2} = -\tan^{-1} u + C = -\tan^{-1}(\cos x) + C.$$

41. $\displaystyle\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx.$ Let $u = \sin x$. Then $du = \cos x\,dx$, so $\displaystyle\int \cot x\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sin x| + C.$

42. Let $u = \ln t$. Then $du = \dfrac{1}{t}\,dt$, so $\displaystyle\int \frac{\cos(\ln t)}{t}\,dt = \int \cos u\,du = \sin u + C = \sin(\ln t) + C.$

43. Let $u = \sin^{-1} x$. Then $du = \dfrac{1}{\sqrt{1 - x^2}}\,dx$, so $\displaystyle\int \frac{dx}{\sqrt{1 - x^2}\,\sin^{-1} x} = \int \frac{1}{u}\,du = \ln|u| + C = \ln\left|\sin^{-1} x\right| + C.$

44. Let $u = x^2$. Then $du = 2x\,dx$, so $\displaystyle\int \frac{x}{1 + x^4}\,dx = \int \frac{\frac{1}{2}\,du}{1 + u^2} = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1}(x^2) + C.$

45. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so

$$\int \frac{1 + x}{1 + x^2}\,dx = \int \frac{1}{1 + x^2}\,dx + \int \frac{x}{1 + x^2}\,dx = \tan^{-1} x + \int \frac{\frac{1}{2}\,du}{u} = \tan^{-1} x + \frac{1}{2}\ln|u| + C$$

$$= \tan^{-1} x + \frac{1}{2}\ln\left|1 + x^2\right| + C = \tan^{-1} x + \frac{1}{2}\ln\left(1 + x^2\right) + C \quad [\text{since } 1 + x^2 > 0].$$

46. Let $u = 2 + x$. Then $du = dx$, $x = u - 2$, and $x^2 = (u - 2)^2$, so

$$\int x^2 \sqrt{2 + x}\, dx = \int (u - 2)^2 \sqrt{u}\, du = \int (u^2 - 4u + 4)u^{1/2}\, du = \int (u^{5/2} - 4u^{3/2} + 4u^{1/2})\, du$$

$$= \tfrac{2}{7}u^{7/2} - \tfrac{8}{5}u^{5/2} + \tfrac{8}{3}u^{3/2} + C = \tfrac{2}{7}(2 + x)^{7/2} - \tfrac{8}{5}(2 + x)^{5/2} + \tfrac{8}{3}(2 + x)^{3/2} + C$$

47. Let $u = 2x + 5$. Then $du = 2\, dx$ and $x = \tfrac{1}{2}(u - 5)$, so

$$\int x(2x + 5)^8\, dx = \int \tfrac{1}{2}(u - 5)u^8 \left(\tfrac{1}{2}\, du\right) = \tfrac{1}{4}\int (u^9 - 5u^8)\, du$$

$$= \tfrac{1}{4}\left(\tfrac{1}{10}u^{10} - \tfrac{5}{9}u^9\right) + C = \tfrac{1}{40}(2x + 5)^{10} - \tfrac{5}{36}(2x + 5)^9 + C$$

48. Let $u = x^2 + 1$ [so $x^2 = u - 1$]. Then $du = 2x\, dx$ and $x\, dx = \tfrac{1}{2}\, du$, so

$$\int x^3 \sqrt{x^2 + 1}\, dx = \int x^2 \sqrt{x^2 + 1}\, x\, dx = \int (u - 1)\sqrt{u}\, \left(\tfrac{1}{2}\, du\right) = \tfrac{1}{2}\int (u^{3/2} - u^{1/2})\, du$$

$$= \tfrac{1}{2}\left(\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right) + C = \tfrac{1}{5}(x^2 + 1)^{5/2} - \tfrac{1}{3}(x^2 + 1)^{3/2} + C.$$

Or: Let $u = \sqrt{x^2 + 1}$. Then $u^2 = x^2 + 1 \Rightarrow 2u\, du = 2x\, dx \Rightarrow u\, du = x\, dx$, so
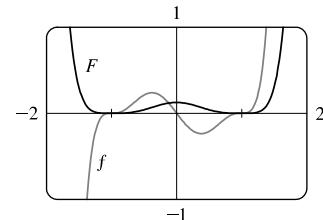
$$\int x^3 \sqrt{x^2 + 1}\, dx = \int x^2 \sqrt{x^2 + 1}\, x\, dx = \int (u^2 - 1)\, u \cdot u\, du = \int (u^4 - u^2)\, du$$

$$= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}(x^2 + 1)^{5/2} - \tfrac{1}{3}(x^2 + 1)^{3/2} + C.$$

Note: This answer can be written as $\tfrac{1}{15}\sqrt{x^2 + 1}(3x^4 + x^2 - 2) + C$.

49. $f(x) = x(x^2 - 1)^3$. $\quad u = x^2 - 1 \Rightarrow du = 2x\, dx$, so

$$\int x(x^2 - 1)^3\, dx = \int u^3 \left(\tfrac{1}{2}\, du\right) = \tfrac{1}{8}u^4 + C = \tfrac{1}{8}(x^2 - 1)^4 + C$$
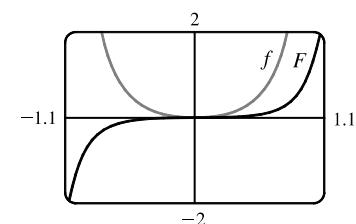
Where f is positive (negative), F is increasing (decreasing). Where f

changes from negative to positive (positive to negative), F has a local

minimum (maximum).

50. $f(\theta) = \tan^2 \theta \sec^2 \theta$. $u = \tan \theta \Rightarrow du = \sec^2 \theta\, d\theta$, so

$$\int \tan^2 \theta \sec^2 \theta\, d\theta = \int u^2\, du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\tan^3 \theta + C$$
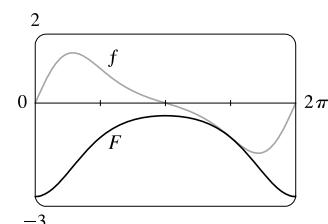
Note that f is positive and F is increasing. At $x = 0$, $f = 0$ and F has a

horizontal tangent.

51. $f(x) = e^{\cos x} \sin x$. $u = \cos x \Rightarrow du = -\sin x\, dx$, so

$$\int e^{\cos x} \sin x\, dx = \int e^u (-du) = -e^u + C = -e^{\cos x} + C$$

Note that at $x = \pi$, f changes from positive to negative and F has a local

maximum. Also, both f and F are periodic with period 2π, so at $x = 0$ and

at $x = 2\pi$, f changes from negative to positive and F has a local minimum.
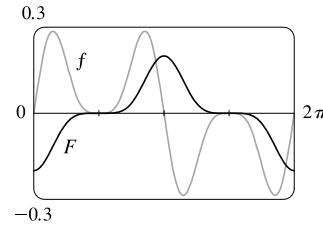
52. $f(x) = \sin x \cos^4 x$. $u = \cos x \quad \Rightarrow \quad du = -\sin x\, dx$, so

$$\int \sin x \cos^4 x\, dx = \int u^4 (-du) = -\tfrac{1}{5} u^5 + C = -\tfrac{1}{5}\cos^5 x + C$$

Note that at $x = \pi$, f changes from positive to negative and F has a local

maximum. Also, both f and F are periodic with period 2π, so at $x = 0$ and

at $x = 2\pi$, f changes from negative to positive and F has a local minimum.

53. Let $u = \tfrac{\pi}{2} t$, so $du = \tfrac{\pi}{2}\, dt$. When $t = 0$, $u = 0$; when $t = 1$, $u = \tfrac{\pi}{2}$. Thus,

$$\int_0^1 \cos(\pi t/2)\, dt = \int_0^{\pi/2} \cos u \left(\tfrac{2}{\pi}\, du\right) = \tfrac{2}{\pi}\left[\sin u\right]_0^{\pi/2} = \tfrac{2}{\pi}\left(\sin\tfrac{\pi}{2} - \sin 0\right) = \tfrac{2}{\pi}(1 - 0) = \tfrac{2}{\pi}$$

54. Let $u = 3t - 1$, so $du = 3\, dt$. When $t = 0$, $u = -1$; when $t = 1$, $u = 2$. Thus,

$$\int_0^1 (3t - 1)^{50}\, dt = \int_{-1}^2 u^{50}\left(\tfrac{1}{3}\, du\right) = \tfrac{1}{3}\left[\tfrac{1}{51}u^{51}\right]_{-1}^2 = \tfrac{1}{153}\left[2^{51} - (-1)^{51}\right] = \tfrac{1}{153}(2^{51} + 1)$$

55. Let $u = 1 + 7x$, so $du = 7\, dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 8$. Thus,

$$\int_0^1 \sqrt[3]{1 + 7x}\, dx = \int_1^8 u^{1/3}(\tfrac{1}{7}\, du) = \tfrac{1}{7}\left[\tfrac{3}{4}u^{4/3}\right]_1^8 = \tfrac{3}{28}(8^{4/3} - 1^{4/3}) = \tfrac{3}{28}(16 - 1) = \tfrac{45}{28}$$

56. Let $u = 5x + 1$, so $du = 5\, dx$. When $x = 0$, $u = 1$; when $x = 3$, $u = 16$. Thus,

$$\int_0^3 \frac{dx}{5x + 1} = \int_1^{16} \frac{1}{u}\left(\frac{1}{5}\, du\right) = \frac{1}{5}\left[\ln|u|\right]_1^{16} = \frac{1}{5}(\ln 16 - \ln 1) = \frac{1}{5}\ln 16.$$

57. Let $u = \cos t$, so $du = -\sin t\, dt$. When $t = 0$, $u = 1$; when $t = \tfrac{\pi}{6}$, $u = \sqrt{3}/2$. Thus,

$$\int_0^{\pi/6} \frac{\sin t}{\cos^2 t}\, dt = \int_1^{\sqrt{3}/2} \frac{1}{u^2}(-du) = \left[\frac{1}{u}\right]_1^{\sqrt{3}/2} = \frac{2}{\sqrt{3}} - 1.$$

58. Let $u = \tfrac{1}{2}t$, so $du = \tfrac{1}{2}\, dt$. When $t = \tfrac{\pi}{3}$, $u = \tfrac{\pi}{6}$; when $t = \tfrac{2\pi}{3}$, $u = \tfrac{\pi}{3}$. Thus,

$$\int_{\pi/3}^{2\pi/3} \csc^2\left(\tfrac{1}{2}t\right) dt = \int_{\pi/6}^{\pi/3} \csc^2 u\, (2\, du) = 2\left[-\cot u\right]_{\pi/6}^{\pi/3} = -2\left(\cot\frac{\pi}{3} - \cot\frac{\pi}{6}\right)$$

$$= -2\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right) = -2(\tfrac{1}{3}\sqrt{3} - \sqrt{3}) = \tfrac{4}{3}\sqrt{3}$$

59. Let $u = 1/x$, so $du = -1/x^2\, dx$. When $x = 1$, $u = 1$; when $x = 2$, $u = \tfrac{1}{2}$. Thus,

$$\int_1^2 \frac{e^{1/x}}{x^2}\, dx = \int_1^{1/2} e^u (-du) = -\left[e^u\right]_1^{1/2} = -(e^{1/2} - e) = e - \sqrt{e}.$$

60. Let $u = -x^2$, so $du = -2x\, dx$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Thus,

$$\int_0^1 xe^{-x^2}\, dx = \int_0^{-1} e^u \left(-\tfrac{1}{2}\, du\right) = -\tfrac{1}{2}\left[e^u\right]_0^{-1} = -\tfrac{1}{2}\left(e^{-1} - e^0\right) = \tfrac{1}{2}(1 - 1/e).$$

61. $\int_{-\pi/4}^{\pi/4} (x^3 + x^4 \tan x)\, dx = 0$ by Theorem 7(b), since $f(x) = x^3 + x^4 \tan x$ is an odd function.

62. Let $u = \sin x$, so $du = \cos x\, dx$. When $x = 0$, $u = 0$; when $x = \tfrac{\pi}{2}$, $u = 1$. Thus,

$$\int_0^{\pi/2} \cos x \sin(\sin x)\, dx = \int_0^1 \sin u\, du = \left[-\cos u\right]_0^1 = -(\cos 1 - 1) = 1 - \cos 1.$$

63. Let $u = 1 + 2x$, so $du = 2\,dx$. When $x = 0$, $u = 1$; when $x = 13$, $u = 27$. Thus,

$$\int_0^{13} \frac{dx}{\sqrt[3]{(1 + 2x)^2}} = \int_1^{27} u^{-2/3}\left(\tfrac{1}{2}\,du\right) = \left[\tfrac{1}{2} \cdot 3u^{1/3}\right]_1^{27} = \tfrac{3}{2}(3 - 1) = 3.$$

64. Assume $a > 0$. Let $u = a^2 - x^2$, so $du = -2x\,dx$. When $x = 0$, $u = a^2$; when $x = a$, $u = 0$. Thus,

$$\int_0^a x\sqrt{a^2 - x^2}\,dx = \int_{a^2}^0 u^{1/2}\left(-\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int_0^{a^2} u^{1/2}\,du = \tfrac{1}{2} \cdot \left[\tfrac{2}{3}u^{3/2}\right]_0^{a^2} = \tfrac{1}{3}a^3.$$

65. Let $u = x^2 + a^2$, so $du = 2x\,dx$ and $x\,dx = \tfrac{1}{2}\,du$. When $x = 0$, $u = a^2$; when $x = a$, $u = 2a^2$. Thus,

$$\int_0^a x\sqrt{x^2 + a^2}\,dx = \int_{a^2}^{2a^2} u^{1/2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_{a^2}^{2a^2} = \left[\tfrac{1}{3}u^{3/2}\right]_{a^2}^{2a^2} = \tfrac{1}{3}\left[(2a^2)^{3/2} - (a^2)^{3/2}\right] = \tfrac{1}{3}\left(2\sqrt{2} - 1\right)a^3$$

66. $\int_{-\pi/3}^{\pi/3} x^4 \sin x\,dx = 0$ by Theorem 7(b), since $f(x) = x^4 \sin x$ is an odd function.

67. Let $u = x - 1$, so $u + 1 = x$ and $du = dx$. When $x = 1$, $u = 0$; when $x = 2$, $u = 1$. Thus,

$$\int_1^2 x\sqrt{x - 1}\,dx = \int_0^1 (u + 1)\sqrt{u}\,du = \int_0^1 (u^{3/2} + u^{1/2})\,du = \left[\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right]_0^1 = \tfrac{2}{5} + \tfrac{2}{3} = \tfrac{16}{15}.$$

68. Let $u = 1 + 2x$, so $x = \tfrac{1}{2}(u - 1)$ and $du = 2\,dx$. When $x = 0$, $u = 1$; when $x = 4$, $u = 9$. Thus,

$$\int_0^4 \frac{x\,dx}{\sqrt{1 + 2x}} = \int_1^9 \frac{\tfrac{1}{2}(u - 1)}{\sqrt{u}}\frac{du}{2} = \tfrac{1}{4}\int_1^9 (u^{1/2} - u^{-1/2})\,du = \tfrac{1}{4}\left[\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^9 = \tfrac{1}{4} \cdot \tfrac{2}{3}\left[u^{3/2} - 3u^{1/2}\right]_1^9$$

$$= \tfrac{1}{6}[(27 - 9) - (1 - 3)] = \tfrac{20}{6} = \tfrac{10}{3}$$

69. Let $u = \ln x$, so $du = \dfrac{dx}{x}$. When $x = e$, $u = 1$; when $x = e^4$; $u = 4$. Thus,

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 u^{-1/2}\,du = 2\left[u^{1/2}\right]_1^4 = 2(2 - 1) = 2.$$

70. Let $u = (x - 1)^2$, so $du = 2(x - 1)\,dx$. When $x = 0$, $u = 1$; when $x = 2$, $u = 1$. Thus,

$$\int_0^2 (x - 1)e^{(x-1)^2}\,dx = \int_1^1 e^u\left(\tfrac{1}{2}\,du\right) = 0 \text{ since the limits are equal.}$$

71. Let $u = e^z + z$, so $du = (e^z + 1)\,dz$. When $z = 0$, $u = 1$; when $z = 1$, $u = e + 1$. Thus,

$$\int_0^1 \frac{e^z + 1}{e^z + z}\,dz = \int_1^{e+1} \frac{1}{u}\,du = \left[\ln|u|\right]_1^{e+1} = \ln|e + 1| - \ln|1| = \ln(e + 1).$$

72. Let $u = \dfrac{2\pi t}{T} - \alpha$, so $du = \dfrac{2\pi}{T}\,dt$. When $t = 0$, $u = -\alpha$; when $t = \dfrac{T}{2}$, $u = \pi - \alpha$. Thus,

$$\int_0^{T/2} \sin\left(\frac{2\pi t}{T} - \alpha\right)dt = \int_{-\alpha}^{\pi-\alpha} \sin u\left(\frac{T}{2\pi}\,du\right) = \frac{T}{2\pi}\left[-\cos u\right]_{-\alpha}^{\pi-\alpha} = -\frac{T}{2\pi}[\cos(\pi - \alpha) - \cos(-\alpha)]$$

$$= -\frac{T}{2\pi}(-\cos\alpha - \cos\alpha) = -\frac{T}{2\pi}(-2\cos\alpha) = \frac{T}{\pi}\cos\alpha$$

73. Let $u = 1 + \sqrt{x}$, so $du = \dfrac{1}{2\sqrt{x}}\,dx \quad \Rightarrow \quad 2\sqrt{x}\,du = dx \quad \Rightarrow \quad 2(u-1)\,du = dx$. When $x = 0$, $u = 1$; when $x = 1$,

$u = 2$. Thus,

$$\int_0^1 \frac{dx}{(1+\sqrt{x})^4} = \int_1^2 \frac{1}{u^4} \cdot [2(u-1)\,du] = 2\int_1^2 \left(\frac{1}{u^3} - \frac{1}{u^4}\right) du = 2\left[-\frac{1}{2u^2} + \frac{1}{3u^3}\right]_1^2$$

$$= 2\left[\left(-\tfrac{1}{8} + \tfrac{1}{24}\right) - \left(-\tfrac{1}{2} + \tfrac{1}{3}\right)\right] = 2\left(\tfrac{1}{12}\right) = \tfrac{1}{6}$$

74. If $f(x) = \sin\sqrt[3]{x}$, then $f(-x) = \sin\sqrt[3]{-x} = \sin(-\sqrt[3]{x}) = -\sin\sqrt[3]{x} = -f(x)$, so f is an odd function. Now

$I = \int_{-2}^3 \sin\sqrt[3]{x}\,dx = \int_{-2}^2 \sin\sqrt[3]{x}\,dx + \int_2^3 \sin\sqrt[3]{x}\,dx = I_1 + I_2$. $\quad I_1 = 0$ by Theorem 7(b). To estimate I_2, note that

$2 \le x \le 3 \quad \Rightarrow \quad \sqrt[3]{2} \le \sqrt[3]{x} \le \sqrt[3]{3}\ [\approx 1.44] \quad \Rightarrow \quad 0 \le \sqrt[3]{x} \le \frac{\pi}{2}\ [\approx 1.57] \quad \Rightarrow \quad \sin 0 \le \sin\sqrt[3]{x} \le \sin\frac{\pi}{2}$ [since sine is

increasing on this interval] $\quad \Rightarrow \quad 0 \le \sin\sqrt[3]{x} \le 1$. By comparison property 8, $0(3-2) \le I_2 \le 1(3-2) \quad \Rightarrow$

$0 \le I_2 \le 1 \quad \Rightarrow \quad 0 \le I \le 1$.
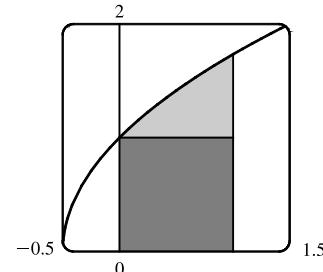
75. From the graph, it appears that the area under the curve is about

$1 + \left(\text{a little more than } \frac{1}{2} \cdot 1 \cdot 0.7\right)$, or about 1.4. The exact area is given by

$A = \int_0^1 \sqrt{2x+1}\,dx$. Let $u = 2x+1$, so $du = 2\,dx$. The limits change to

$2 \cdot 0 + 1 = 1$ and $2 \cdot 1 + 1 = 3$, and
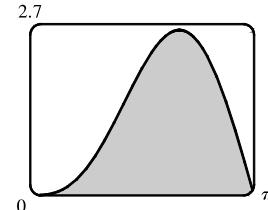
$A = \int_1^3 \sqrt{u}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^3 = \frac{1}{3}\left(3\sqrt{3} - 1\right) = \sqrt{3} - \frac{1}{3} \approx 1.399$.

76. From the graph, it appears that the area under the curve is almost $\frac{1}{2} \cdot \pi \cdot 2.6$,

or about 4. The exact area is given by

$A = \int_0^\pi (2\sin x - \sin 2x)\,dx = -2\left[\cos x\right]_0^\pi - \int_0^\pi \sin 2x\,dx$

$= -2(-1-1) - 0 = 4$

Note: $\int_0^\pi \sin 2x\,dx = 0$ since it is clear from the graph of $y = \sin 2x$ that $\int_{\pi/2}^\pi \sin 2x\,dx = -\int_0^{\pi/2} \sin 2x\,dx$.

77. First write the integral as a sum of two integrals:

$I = \int_{-2}^2 (x+3)\sqrt{4-x^2}\,dx = I_1 + I_2 = \int_{-2}^2 x\sqrt{4-x^2}\,dx + \int_{-2}^2 3\sqrt{4-x^2}\,dx$. $I_1 = 0$ by Theorem 7(b), since

$f(x) = x\sqrt{4-x^2}$ is an odd function and we are integrating from $x = -2$ to $x = 2$. We interpret I_2 as three times the area of

a semicircle with radius 2, so $I = 0 + 3 \cdot \frac{1}{2}\left(\pi \cdot 2^2\right) = 6\pi$.

78. Let $u = x^2$. Then $du = 2x\,dx$ and the limits are unchanged $(0^2 = 0$ and $1^2 = 1)$, so

$I = \int_0^1 x\sqrt{1-x^4}\,dx = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du$. But this integral can be interpreted as the area of a quarter-circle with radius 1.

So $I = \frac{1}{2} \cdot \frac{1}{4}\left(\pi \cdot 1^2\right) = \frac{1}{8}\pi$.

79. First Figure   Let $u = \sqrt{x}$, so $x = u^2$ and $dx = 2u\,du$. When $x = 0$, $u = 0$; when $x = 1$, $u = 1$. Thus,

$$A_1 = \int_0^1 e^{\sqrt{x}}\,dx = \int_0^1 e^u (2u\,du) = 2\int_0^1 u e^u\,du.$$

Second Figure $A_2 = \int_0^1 2xe^x\,dx = 2\int_0^1 u e^u\,du.$

Third Figure   Let $u = \sin x$, so $du = \cos x\,dx$. When $x = 0$, $u = 0$; when $x = \frac{\pi}{2}$, $u = 1$. Thus,

$$A_3 = \int_0^{\pi/2} e^{\sin x} \sin 2x\,dx = \int_0^{\pi/2} e^{\sin x}(2\sin x\,\cos x)\,dx = \int_0^1 e^u (2u\,du) = 2\int_0^1 u e^u\,du.$$

Since $A_1 = A_2 = A_3$, all three areas are equal.

80. Let $u = \dfrac{\pi t}{12}$. Then $du = \dfrac{\pi}{12}\,dt$ and

$$\int_0^{24} R(t)\,dt = \int_0^{24}\left[85 - 0.18\cos\left(\frac{\pi t}{12}\right)\right] dt = \int_0^{2\pi}(85 - 0.18\cos u)\left(\frac{12}{\pi}\,du\right) = \frac{12}{\pi}\left[85u - 0.18\sin u\right]_0^{2\pi}$$

$$= \frac{12}{\pi}\left[(85\cdot 2\pi - 0) - (0 - 0)\right] = 2040 \text{ kcal}$$

81. The rate is measured in liters per minute. Integrating from $t = 0$ minutes to $t = 60$ minutes will give us the total amount of oil

that leaks out (in liters) during the first hour.

$$\int_0^{60} r(t)\,dt = \int_0^{60} 100 e^{-0.01t}\,dt \qquad [u = -0.01t, du = -0.01dt]$$

$$= 100\int_0^{-0.6} e^u(-100\,du) = -10,000\left[e^u\right]_0^{-0.6} = -10,000(e^{-0.6} - 1) \approx 4511.9 \approx 4512 \text{ liters}$$

82. Let $r(t) = ae^{bt}$ with $a = 450.268$ and $b = 1.12567$, and $n(t) = $ population after t hours. Since $r(t) = n'(t)$,

$\int_0^3 r(t)\,dt = n(3) - n(0)$ is the total change in the population after three hours. Since we start with 400 bacteria, the

population will be

$$n(3) = 400 + \int_0^3 r(t)\,dt = 400 + \int_0^3 ae^{bt}\,dt = 400 + \frac{a}{b}\left[e^{bt}\right]_0^3 = 400 + \frac{a}{b}\left(e^{3b} - 1\right)$$

$$\approx 400 + 11,313 = 11,713 \text{ bacteria}$$

83. The volume of inhaled air in the lungs at time t is

$$V(t) = \int_0^t f(u)\,du = \int_0^t \frac{1}{2}\sin\left(\frac{2\pi}{5}u\right) du = \int_0^{2\pi t/5} \frac{1}{2}\sin v\left(\frac{5}{2\pi}\,dv\right) \qquad \left[\text{substitute } v = \tfrac{2\pi}{5}u, dv = \tfrac{2\pi}{5}\,du\right]$$

$$= \frac{5}{4\pi}\left[-\cos v\right]_0^{2\pi t/5} = \frac{5}{4\pi}\left[-\cos\left(\frac{2\pi}{5}t\right) + 1\right] = \frac{5}{4\pi}\left[1 - \cos\left(\frac{2\pi}{5}t\right)\right] \text{ liters}$$

84. The rate G is measured in kilograms per year. Integrating from $t = 0$ years (2000) to $t = 20$ years (2020) will give us the net

change in biomass from 2000 to 2020.

$$\int_0^{20} \frac{60,000e^{-0.6t}}{(1 + 5e^{-0.6t})^2}\,dt = \int_6^{1+5e^{-12}} \frac{60,000}{u^2}\left(-\frac{1}{3}\,du\right) \qquad \begin{bmatrix} u = 1 + 5e^{-0.6t}, \\ du = -3e^{-0.6t}dt \end{bmatrix}$$

$$= \left[\frac{20,000}{u}\right]_6^{1+5e^{-12}} = \frac{20,000}{1 + 5e^{-12}} - \frac{20,000}{6} \approx 16,666$$

Thus, the predicted biomass for the year 2020 is approximately $25,000 + 16,666 = 41,666$ kg.

85. $\int_0^{30} u(t)\, dt = \int_0^{30} \frac{r}{V} C_0 e^{-rt/V}\, dt = C_0 \int_1^{e^{-30r/V}} (-dx)$ $\qquad \begin{bmatrix} x = e^{-rt/V}, \\ dx = -\frac{r}{V} e^{-rt/V}\, dt \end{bmatrix}$

$\qquad\qquad = C_0 \Big[-x \Big]_1^{e^{-30r/V}} = C_0(-e^{-30r/V} + 1)$

The integral $\int_0^{30} u(t)\, dt$ represents the total amount of urea removed from the blood in the first 30 minutes of dialysis.

86. Number of calculators $= x(4) - x(2) = \int_2^4 5000 \left[1 - 100(t + 10)^{-2} \right] dt$

$\qquad\qquad = 5000 \left[t + 100(t + 10)^{-1} \right]_2^4 = 5000 \left[\left(4 + \frac{100}{14}\right) - \left(2 + \frac{100}{12}\right) \right] \approx 4048$
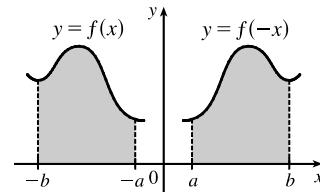
87. Let $u = 2x$. Then $du = 2\, dx$, so $\int_0^2 f(2x)\, dx = \int_0^4 f(u)\left(\frac{1}{2}\, du\right) = \frac{1}{2} \int_0^4 f(u)\, du = \frac{1}{2}(10) = 5$.

88. Let $u = x^2$. Then $du = 2x\, dx$, so $\int_0^3 x f(x^2)\, dx = \int_0^9 f(u)\left(\frac{1}{2}\, du\right) = \frac{1}{2} \int_0^9 f(u)\, du = \frac{1}{2}(4) = 2$.

89. Let $u = -x$. Then $du = -dx$, so

$$\int_a^b f(-x)\, dx = \int_{-a}^{-b} f(u)(-du) = \int_{-b}^{-a} f(u)\, du = \int_{-b}^{-a} f(x)\, dx$$
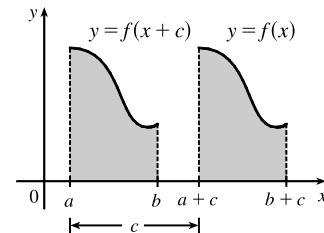
From the diagram, we see that the equality follows from the fact that we are

reflecting the graph of f, and the limits of integration, about the y-axis.

90. Let $u = x + c$. Then $du = dx$, so

$$\int_a^b f(x + c)\, dx = \int_{a+c}^{b+c} f(u)\, du = \int_{a+c}^{b+c} f(x)\, dx$$

From the diagram, we see that the equality follows from the fact that we are

translating the graph of f, and the limits of integration, by a distance c.

91. Let $u = 1 - x$. Then $x = 1 - u$ and $dx = -du$, so

$\int_0^1 x^a (1 - x)^b\, dx = \int_1^0 (1 - u)^a u^b (-du) = \int_0^1 u^b (1 - u)^a\, du = \int_0^1 x^b (1 - x)^a\, dx$.

92. Let $u = \pi - x$. Then $du = -dx$. When $x = \pi$, $u = 0$ and when $x = 0$, $u = \pi$. So

$\qquad\qquad \int_0^\pi x f(\sin x)\, dx = -\int_\pi^0 (\pi - u) f(\sin(\pi - u))\, du = \int_0^\pi (\pi - u) f(\sin u)\, du$

$\qquad\qquad\qquad = \pi \int_0^\pi f(\sin u)\, du - \int_0^\pi u f(\sin u)\, du = \pi \int_0^\pi f(\sin x)\, dx - \int_0^\pi x f(\sin x)\, dx \quad \Rightarrow$

$2 \int_0^\pi x f(\sin x)\, dx = \pi \int_0^\pi f(\sin x)\, dx \quad \Rightarrow \quad \int_0^\pi x f(\sin x)\, dx = \frac{\pi}{2} \int_0^\pi f(\sin x)\, dx$.

93. $\dfrac{x \sin x}{1 + \cos^2 x} = x \cdot \dfrac{\sin x}{2 - \sin^2 x} = x f(\sin x)$, where $f(t) = \dfrac{t}{2 - t^2}$. By Exercise 92,

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x}\, dx = \int_0^\pi x f(\sin x)\, dx = \frac{\pi}{2} \int_0^\pi f(\sin x)\, dx = \frac{\pi}{2} \int_0^\pi \frac{\sin x}{1 + \cos^2 x}\, dx$$

Let $u = \cos x$. Then $du = -\sin x\, dx$. When $x = \pi$, $u = -1$ and when $x = 0$, $u = 1$. So

$$\frac{\pi}{2} \int_0^\pi \frac{\sin x}{1 + \cos^2 x}\, dx = -\frac{\pi}{2} \int_1^{-1} \frac{du}{1 + u^2} = \frac{\pi}{2} \int_{-1}^1 \frac{du}{1 + u^2} = \frac{\pi}{2} \Big[\tan^{-1} u \Big]_{-1}^1$$

$$= \frac{\pi}{2} [\tan^{-1} 1 - \tan^{-1}(-1)] = \frac{\pi}{2} \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] = \frac{\pi^2}{4}$$

94. (a) $\int_0^{\pi/2} f(\cos x)\, dx = \int_0^{\pi/2} f\left[\sin\left(\frac{\pi}{2} - x\right)\right] dx$ $\qquad \left[u = \frac{\pi}{2} - x,\, du = -dx\right]$

$\qquad\qquad = \int_{\pi/2}^0 f(\sin u)(-du) = \int_0^{\pi/2} f(\sin u)\, du = \int_0^{\pi/2} f(\sin x)\, dx$

Continuity of f is needed in order to apply the substitution rule for definite integrals.

(b) In part (a), take $f(x) = x^2$, so $\int_0^{\pi/2} \cos^2 x\, dx = \int_0^{\pi/2} \sin^2 x\, dx$. Now

$\int_0^{\pi/2} \cos^2 x\, dx + \int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} (\cos^2 x + \sin^2 x)\, dx = \int_0^{\pi/2} 1\, dx = \left[x\right]_0^{\pi/2} = \frac{\pi}{2}$,

so $2\int_0^{\pi/2} \cos^2 x\, dx = \frac{\pi}{2}$ $\;\Rightarrow\;$ $\int_0^{\pi/2} \cos^2 x\, dx = \frac{\pi}{4}$ $\;\left[= \int_0^{\pi/2} \sin^2 x\, dx\right]$.

5   Review

TRUE-FALSE QUIZ

1. True by Property 2 of the Integral in Section 5.2.

2. False. Try $a = 0$, $b = 2$, $f(x) = g(x) = 1$ as a counterexample.

3. True by Property 3 of the Integral in Section 5.2.

4. False. You can't take a variable outside the integral sign. For example, using $f(x) = 1$ on $[0, 1]$,

$\int_0^1 x\, f(x)\, dx = \int_0^1 x\, dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$ (a constant) while $x\int_0^1 1\, dx = x\,[x]_0^1 = x \cdot 1 = x$ (a variable).

5. False. For example, let $f(x) = x^2$. Then $\int_0^1 \sqrt{x^2}\, dx = \int_0^1 x\, dx = \frac{1}{2}$, but $\sqrt{\int_0^1 x^2\, dx} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$.

6. True by the Net Change Theorem.

7. True by Comparison Property 7 of the Integral in Section 5.2.

8. False. For example, let $a = 0$, $b = 1$, $f(x) = 3$, $g(x) = x$. $f(x) > g(x)$ for each x in $(0, 1)$, but $f'(x) = 0 < 1 = g'(x)$
for $x \in (0, 1)$.

9. True. The integrand is an odd function that is continuous on $[-1, 1]$.

10. True. $\int_{-5}^5 \left(ax^2 + bx + c\right) dx = \int_{-5}^5 \left(ax^2 + c\right) dx + \int_{-5}^5 bx\, dx$

$\qquad\qquad\qquad\qquad = 2\int_0^5 \left(ax^2 + c\right) dx + 0$ [because $ax^2 + c$ is even and bx is odd]

11. False. For example, the function $y = |x|$ is continuous on $\mathbb{R}$, but has no derivative at $x = 0$.

12. True by FTC1.

13. True by Property 5 of Integrals.

14. False. For example, $\int_0^1 \left(x - \frac{1}{2}\right) dx = \left[\frac{1}{2}x^2 - \frac{1}{2}x\right]_0^1 = \left(\frac{1}{2} - \frac{1}{2}\right) - (0 - 0) = 0$, but $f(x) = x - \frac{1}{2} \neq 0$.

15. False. $\int_a^b f(x)\, dx$ is a constant, so $\dfrac{d}{dx}\left(\int_a^b f(x)\, dx\right) = 0$, not $f(x)$ [unless $f(x) = 0$]. Compare the given statement

carefully with FTC1, in which the upper limit in the integral is x.
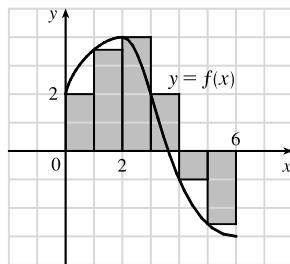
16. False.   See the paragraph before Note 4 and Figure 4 in Section 5.2, and notice that $y = x - x^3 < 0$ for $1 < x \leq 2$.

17. False.   The function $f(x) = 1/x^4$ is not bounded on the interval $[-2, 1]$. It has an infinite discontinuity at $x = 0$, so it is

not integrable on the interval. (If the integral were to exist, a positive value would be expected, by Comparison

Property 6 of Integrals.)

18. False.   For example, if $f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } -1 \leq x < 0 \end{cases}$ then f has a jump discontinuity at 0, but $\int_{-1}^1 f(x)\, dx$ exists and is

equal to 1.

EXERCISES

1. (a)
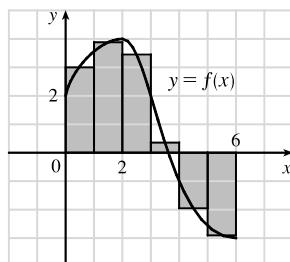
$L_6 = \sum\limits_{i=1}^{6} f(x_{i-1})\, \Delta x \quad [\Delta x = \frac{6-0}{6} = 1]$

$= f(x_0) \cdot 1 + f(x_1) \cdot 1 + f(x_2) \cdot 1 + f(x_3) \cdot 1 + f(x_4) \cdot 1 + f(x_5) \cdot 1$

$\approx 2 + 3.5 + 4 + 2 + (-1) + (-2.5) = 8$

The Riemann sum represents the sum of the areas of the four rectangles

above the x-axis minus the sum of the areas of the two rectangles below the

x-axis.

(b)
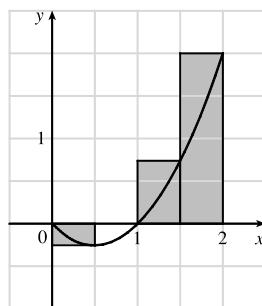
$M_6 = \sum\limits_{i=1}^{6} f(\overline{x}_i)\, \Delta x \quad [\Delta x = \frac{6-0}{6} = 1]$

$= f(\overline{x}_1) \cdot 1 + f(\overline{x}_2) \cdot 1 + f(\overline{x}_3) \cdot 1 + f(\overline{x}_4) \cdot 1 + f(\overline{x}_5) \cdot 1 + f(\overline{x}_6) \cdot 1$

$= f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)$

$\approx 3 + 3.9 + 3.4 + 0.3 + (-2) + (-2.9) = 5.7$

The Riemann sum represents the sum of the areas of the four rectangles

above the x-axis minus the sum of the areas of the two rectangles below the

x-axis.

2. (a)

$f(x) = x^2 - x$ and $\Delta x = \frac{2-0}{4} = 0.5 \quad \Rightarrow$

$R_4 = 0.5 f(0.5) + 0.5 f(1) + 0.5 f(1.5) + 0.5 f(2)$

$= 0.5(-0.25 + 0 + 0.75 + 2) = 1.25$

The Riemann sum represents the sum of the areas of the two rectangles

above the x-axis minus the area of the rectangle below the x-axis. (The

second rectangle vanishes.)
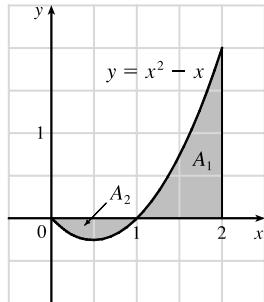
(b) $\int_0^2 (x^2 - x)\, dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\, \Delta x \qquad [\Delta x = 2/n \text{ and } x_i = 2i/n]$

$$= \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \left(\frac{4i^2}{n^2} - \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \lim\limits_{n\to\infty} \frac{2}{n}\left[\frac{4}{n^2} \sum\limits_{i=1}^{n} i^2 - \frac{2}{n} \sum\limits_{i=1}^{n} i\right]$$

$$= \lim\limits_{n\to\infty} \left[\frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{4}{n^2} \cdot \frac{n(n+1)}{2}\right] = \lim\limits_{n\to\infty} \left[\frac{4}{3} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n} - 2 \cdot \frac{n+1}{n}\right]$$

$$= \lim\limits_{n\to\infty} \left[\frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - 2\left(1 + \frac{1}{n}\right)\right] = \frac{4}{3} \cdot 1 \cdot 2 - 2 \cdot 1 = \frac{2}{3}$$

(c) $\int_0^2 (x^2 - x)\, dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_0^2 = \left(\frac{8}{3} - 2\right) = \frac{2}{3}$
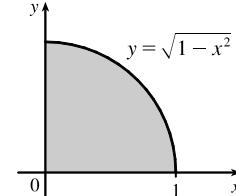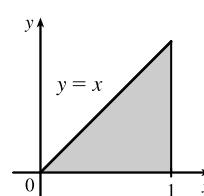
(d)

$\int_0^2 (x^2 - x)\, dx = A_1 - A_2$, where A_1 and A_2 are the areas shown in the diagram.

3. $\int_0^1 \left(x + \sqrt{1 - x^2}\right) dx = \int_0^1 x\, dx + \int_0^1 \sqrt{1 - x^2}\, dx = I_1 + I_2$.

I_1 can be interpreted as the area of the triangle shown in the figure

and I_2 can be interpreted as the area of the quarter-circle.

Area $= \frac{1}{2}(1)(1) + \frac{1}{4}(\pi)(1)^2 = \frac{1}{2} + \frac{\pi}{4}$.

4. On $[0, \pi]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \sin x_i\, \Delta x = \int_0^\pi \sin x\, dx = [-\cos x]_0^\pi = -(-1) - (-1) = 2$.

5. $\int_0^6 f(x)\, dx = \int_0^4 f(x)\, dx + \int_4^6 f(x)\, dx \;\Rightarrow\; 10 = 7 + \int_4^6 f(x)\, dx \;\Rightarrow\; \int_4^6 f(x)\, dx = 10 - 7 = 3$

6. (a) $\int_1^5 (x + 2x^5)\, dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\, \Delta x \qquad \left[\Delta x = \dfrac{5 - 1}{n} = \dfrac{4}{n},\, x_i = 1 + \dfrac{4i}{n}\right]$

$$= \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \left[\left(1 + \frac{4i}{n}\right) + 2\left(1 + \frac{4i}{n}\right)^5\right] \cdot \frac{4}{n} \stackrel{\text{CAS}}{=} \lim\limits_{n\to\infty} \frac{1305n^4 + 3126n^3 + 2080n^2 - 256}{n^3} \cdot \frac{4}{n}$$

$$= 5220$$

(b) $\int_1^5 (x + 2x^5)\, dx = \left[\frac{1}{2}x^2 + \frac{2}{6}x^6\right]_1^5 = \left(\frac{25}{2} + \frac{15{,}625}{3}\right) - \left(\frac{1}{2} + \frac{1}{3}\right) = 12 + 5208 = 5220$

7. First note that either a or b must be the graph of $\int_0^x f(t)\, dt$, since $\int_0^0 f(t)\, dt = 0$, and $c(0) \neq 0$. Now notice that $b > 0$ when c

is increasing, and that $c > 0$ when a is increasing. It follows that c is the graph of $f(x)$, b is the graph of $f'(x)$, and a is the

graph of $\int_0^x f(t)\, dt$.

8. (a) By the Net Change Theorem (FTC2), $\int_0^1 \frac{d}{dx}\left(e^{\arctan x}\right) dx = \left[e^{\arctan x}\right]_0^1 = e^{\pi/4} - 1$

(b) $\frac{d}{dx} \int_0^1 e^{\arctan x} \, dx = 0$ since this is the derivative of a constant.

(c) By FTC1, $\frac{d}{dx} \int_0^x e^{\arctan t} \, dt = e^{\arctan x}$.

9. $g(4) = \int_0^4 f(t) \, dt = \int_0^1 f(t) \, dt + \int_1^2 f(t) \, dt + \int_2^3 f(t) \, dt + \int_3^4 f(t) \, dt$

$= -\frac{1}{2} \cdot 1 \cdot 2 \quad \begin{bmatrix} \text{area of triangle,} \\ \text{below } t\text{-axis} \end{bmatrix} \quad + \frac{1}{2} \cdot 1 \cdot 2 + 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 = 3$

By FTC1, $g'(x) = f(x)$, so $g'(4) = f(4) = 0$.

10. $g(x) = \int_0^x f(t) \, dt \quad \Rightarrow \quad g'(x) = f(x)$ [by FTC1] $\Rightarrow \quad g''(x) = f'(x)$, so $g''(4) = f'(4) = -2$, which is the slope of

the line segment at $x = 4$.

11. $\int_1^2 \left(8x^3 + 3x^2\right) dx = \left[8 \cdot \frac{1}{4}x^4 + 3 \cdot \frac{1}{3}x^3\right]_1^2 = \left[2x^4 + x^3\right]_1^2 = (2 \cdot 2^4 + 2^3) - (2 + 1) = 40 - 3 = 37$

12. $\int_0^T \left(x^4 - 8x + 7\right) dx = \left[\frac{1}{5}x^5 - 4x^2 + 7x\right]_0^T = \left(\frac{1}{5}T^5 - 4T^2 + 7T\right) - 0 = \frac{1}{5}T^5 - 4T^2 + 7T$

13. $\int_0^1 \left(1 - x^9\right) dx = \left[x - \frac{1}{10}x^{10}\right]_0^1 = \left(1 - \frac{1}{10}\right) - 0 = \frac{9}{10}$

14. Let $u = 1 - x$, so $du = -dx$ and $dx = -du$. When $x = 0$, $u = 1$; when $x = 1$, $u = 0$. Thus,

$\int_0^1 (1 - x)^9 \, dx = \int_1^0 u^9 (-du) = \int_0^1 u^9 \, du = \frac{1}{10} \left[u^{10}\right]_0^1 = \frac{1}{10}(1 - 0) = \frac{1}{10}$.

15. $\int_1^9 \frac{\sqrt{u} - 2u^2}{u} \, du = \int_1^9 \left(u^{-1/2} - 2u\right) du = \left[2u^{1/2} - u^2\right]_1^9 = (6 - 81) - (2 - 1) = -76$

16. $\int_0^1 \left(\sqrt[4]{u} + 1\right)^2 du = \int_0^1 \left(u^{1/2} + 2u^{1/4} + 1\right) du = \left[\frac{2}{3}u^{3/2} + \frac{8}{5}u^{5/4} + u\right]_0^1 = \left(\frac{2}{3} + \frac{8}{5} + 1\right) - 0 = \frac{49}{15}$

17. Let $u = y^2 + 1$, so $du = 2y \, dy$ and $y \, dy = \frac{1}{2} du$. When $y = 0$, $u = 1$; when $y = 1$, $u = 2$. Thus,

$\int_0^1 y(y^2 + 1)^5 \, dy = \int_1^2 u^5 \left(\frac{1}{2} du\right) = \frac{1}{2} \left[\frac{1}{6}u^6\right]_1^2 = \frac{1}{12}(64 - 1) = \frac{63}{12} = \frac{21}{4}$.

18. Let $u = 1 + y^3$, so $du = 3y^2 \, dy$ and $y^2 \, dy = \frac{1}{3} du$. When $y = 0$, $u = 1$; when $y = 2$, $u = 9$. Thus,

$\int_0^2 y^2 \sqrt{1 + y^3} \, dy = \int_1^9 u^{1/2} \left(\frac{1}{3} du\right) = \frac{1}{3} \left[\frac{2}{3}u^{3/2}\right]_1^9 = \frac{2}{9}(27 - 1) = \frac{52}{9}$.

19. $\int_1^5 \frac{dt}{(t-4)^2}$ does not exist because the function $f(t) = \frac{1}{(t-4)^2}$ has an infinite discontinuity at $t = 4$;

that is, f is discontinuous on the interval $[1, 5]$.

20. Let $u = 3\pi t$, so $du = 3\pi \, dt$. When $t = 0$, $u = 1$; when $t = 1$, $u = 3\pi$. Thus,

$\int_0^1 \sin(3\pi t) \, dt = \int_0^{3\pi} \sin u \left(\frac{1}{3\pi} du\right) = \frac{1}{3\pi} \left[-\cos u\right]_0^{3\pi} = -\frac{1}{3\pi}(-1 - 1) = \frac{2}{3\pi}$.

21. Let $u = v^3$, so $du = 3v^2\,dv$. When $v = 0$, $u = 0$; when $v = 1$, $u = 1$. Thus,

$$\int_0^1 v^2 \cos(v^3)\,dv = \int_0^1 \cos u \left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\left[\sin u\right]_0^1 = \tfrac{1}{3}(\sin 1 - 0) = \tfrac{1}{3}\sin 1.$$

22. $\displaystyle\int_{-1}^1 \frac{\sin x}{1 + x^2}\,dx = 0$ by Theorem 5.5.7(b), since $f(x) = \dfrac{\sin x}{1 + x^2}$ is an odd function.

23. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{t^4 \tan t}{2 + \cos t}\,dt = 0$ by Theorem 5.5.7(b), since $f(t) = \dfrac{t^4 \tan t}{2 + \cos t}$ is an odd function.

24. Let $u = e^x$, so $du = e^x\,dx$. When $x = 0, u = 1$; when $x = 1, u = e$. Thus,

$$\int_0^1 \frac{e^x}{1 + e^{2x}}\,dx = \int_1^e \frac{1}{1 + u^2}\,du = \left[\arctan u\right]_1^e = \arctan e - \arctan 1 = \arctan e - \tfrac{\pi}{4}.$$

25. $\displaystyle\int \left(\frac{1 - x}{x}\right)^2 dx = \int \left(\frac{1}{x} - 1\right)^2 dx = \int \left(\frac{1}{x^2} - \frac{2}{x} + 1\right) dx = -\frac{1}{x} - 2\ln|x| + x + C$

26. $\displaystyle\int_1^{10} \frac{x}{x^2 - 4}\,dx$ does not exist because the function $f(x) = \dfrac{x}{x^2 - 4}$ has an infinite discontinuity at $x = 2$;

that is, f is discontinuous on the interval $[1, 10]$.

27. Let $u = x^2 + 4x$. Then $du = (2x + 4)\,dx = 2(x + 2)\,dx$, so

$$\int \frac{x + 2}{\sqrt{x^2 + 4x}}\,dx = \int u^{-1/2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2} \cdot 2u^{1/2} + C = \sqrt{u} + C = \sqrt{x^2 + 4x} + C.$$

28. Let $u = 1 + \cot x$. Then $du = -\csc^2 x\,dx$, so $\displaystyle\int \frac{\csc^2 x}{1 + \cot x}\,dx = \int \frac{1}{u}\,(-du) = -\ln|u| + C = -\ln|1 + \cot x| + C.$

29. Let $u = \sin \pi t$. Then $du = \pi \cos \pi t\,dt$, so $\int \sin \pi t \cos \pi t\,dt = \int u\left(\tfrac{1}{\pi}\,du\right) = \tfrac{1}{\pi} \cdot \tfrac{1}{2}u^2 + C = \tfrac{1}{2\pi}(\sin \pi t)^2 + C.$

30. Let $u = \cos x$. Then $du = -\sin x\,dx$, so $\int \sin x \cos(\cos x)\,dx = -\int \cos u\,du = -\sin u + C = -\sin(\cos x) + C.$

31. Let $u = \sqrt{x}$. Then $du = \dfrac{dx}{2\sqrt{x}}$, so $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C.$

32. Let $u = \ln x$. Then $du = \dfrac{1}{x}\,dx$, so $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx = \int \sin u\,du = -\cos u + C = -\cos(\ln x) + C.$

33. Let $u = \ln(\cos x)$. Then $du = \dfrac{-\sin x}{\cos x}\,dx = -\tan x\,dx$, so

$$\int \tan x \ln(\cos x)\,dx = -\int u\,du = -\tfrac{1}{2}u^2 + C = -\tfrac{1}{2}[\ln(\cos x)]^2 + C.$$

34. Let $u = x^2$. Then $du = 2x\,dx$, so $\displaystyle\int \frac{x}{\sqrt{1 - x^4}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{1 - u^2}} = \tfrac{1}{2}\sin^{-1}u + C = \tfrac{1}{2}\sin^{-1}(x^2) + C.$

35. Let $u = 1 + x^4$. Then $du = 4x^3\,dx$, so $\displaystyle\int \frac{x^3}{1 + x^4}\,dx = \frac{1}{4}\int \frac{1}{u}\,du = \tfrac{1}{4}\ln|u| + C = \tfrac{1}{4}\ln(1 + x^4) + C.$

36. Let $u = 1 + 4x$. Then $du = 4\,dx$, so $\int \sinh(1 + 4x)\,dx = \tfrac{1}{4}\int \sinh u\,du = \tfrac{1}{4}\cosh u + C = \tfrac{1}{4}\cosh(1 + 4x) + C.$

37. Let $u = 1 + \sec\theta$. Then $du = \sec\theta\,\tan\theta\,d\theta$, so

$$\int \frac{\sec\theta\,\tan\theta}{1+\sec\theta}\,d\theta = \int \frac{1}{1+\sec\theta}\,(\sec\theta\,\tan\theta\,d\theta) = \int \frac{1}{u}\,du = \ln|u| + C = \ln|1+\sec\theta| + C.$$

38. Let $u = 1 + \tan t$, so $du = \sec^2 t\,dt$. When $t = 0$, $u = 1$; when $t = \frac{\pi}{4}$, $u = 2$. Thus,

$$\int_0^{\pi/4}(1+\tan t)^3\sec^2 t\,dt = \int_1^2 u^3\,du = \left[\tfrac{1}{4}u^4\right]_1^2 = \tfrac{1}{4}\left(2^4 - 1^4\right) = \tfrac{1}{4}(16-1) = \tfrac{15}{4}.$$

39. Since $x^2 - 4 < 0$ for $0 \le x < 2$ and $x^2 - 4 > 0$ for $2 < x \le 3$, we have $\left|x^2 - 4\right| = -(x^2 - 4) = 4 - x^2$ for $0 \le x < 2$ and

$\left|x^2 - 4\right| = x^2 - 4$ for $2 < x \le 3$. Thus,

$$\int_0^3 \left|x^2 - 4\right|\,dx = \int_0^2 (4 - x^2)\,dx + \int_2^3 (x^2 - 4)\,dx = \left[4x - \frac{x^3}{3}\right]_0^2 + \left[\frac{x^3}{3} - 4x\right]_2^3$$
$$= \left(8 - \tfrac{8}{3}\right) - 0 + (9 - 12) - \left(\tfrac{8}{3} - 8\right) = \tfrac{16}{3} - 3 + \tfrac{16}{3} = \tfrac{32}{3} - \tfrac{9}{3} = \tfrac{23}{3}$$
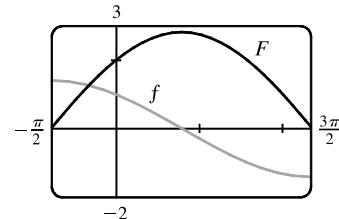
40. Since $\sqrt{x} - 1 < 0$ for $0 \le x < 1$ and $\sqrt{x} - 1 > 0$ for $1 < x \le 4$, we have $\left|\sqrt{x} - 1\right| = -\left(\sqrt{x} - 1\right) = 1 - \sqrt{x}$

for $0 \le x < 1$ and $\left|\sqrt{x} - 1\right| = \sqrt{x} - 1$ for $1 < x \le 4$. Thus,

$$\int_0^4 \left|\sqrt{x} - 1\right|\,dx = \int_0^1 \left(1 - \sqrt{x}\right)\,dx + \int_1^4 \left(\sqrt{x} - 1\right)\,dx = \left[x - \tfrac{2}{3}x^{3/2}\right]_0^1 + \left[\tfrac{2}{3}x^{3/2} - x\right]_1^4$$
$$= \left(1 - \tfrac{2}{3}\right) - 0 + \left(\tfrac{16}{3} - 4\right) - \left(\tfrac{2}{3} - 1\right) = \tfrac{1}{3} + \tfrac{16}{3} - 4 + \tfrac{1}{3} = 6 - 4 = 2$$
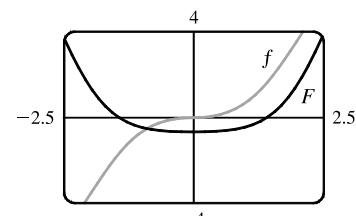
41. Let $u = 1 + \sin x$. Then $du = \cos x\,dx$, so

$$\int \frac{\cos x\,dx}{\sqrt{1+\sin x}} = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{1+\sin x} + C.$$
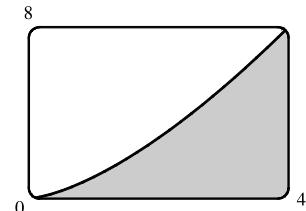
42. Let $u = x^2 + 1$. Then $x^2 = u - 1$ and $x\,dx = \tfrac{1}{2}\,du$, so

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \int \frac{(u-1)}{\sqrt{u}}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\int (u^{1/2} - u^{-1/2})\,du$$
$$= \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right) + C = \tfrac{1}{3}(x^2+1)^{3/2} - (x^2+1)^{1/2} + C$$
$$= \tfrac{1}{3}(x^2+1)^{1/2}\left[(x^2+1) - 3\right] + C = \tfrac{1}{3}\sqrt{x^2+1}\,(x^2 - 2) + C$$
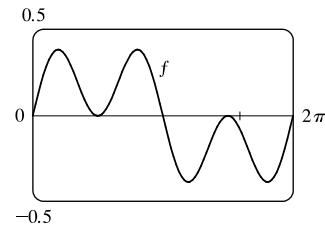

43. From the graph, it appears that the area under the curve $y = x\sqrt{x}$ between $x = 0$
and $x = 4$ is somewhat less than half the area of an 8×4 rectangle, so perhaps
about 13 or 14. To find the exact value, we evaluate

$$\int_0^4 x\sqrt{x}\,dx = \int_0^4 x^{3/2}\,dx = \left[\tfrac{2}{5}x^{5/2}\right]_0^4 = \tfrac{2}{5}(4)^{5/2} = \tfrac{64}{5} = 12.8.$$

44. From the graph, it seems as though $\int_0^{2\pi} \cos^2 x \sin x\, dx$ is equal to 0. To evaluate

the integral, let $u = \cos x \quad \Rightarrow \quad du = -\sin x\, dx$. Thus, $I = \int_1^1 u^2\,(-du) = 0$.

45. $F(x) = \displaystyle\int_0^x \frac{t^2}{1+t^3}\, dt \quad \Rightarrow \quad F'(x) = \frac{d}{dx}\int_0^x \frac{t^2}{1+t^3}\, dt = \frac{x^2}{1+x^3}$

46. $F(x) = \displaystyle\int_x^1 \sqrt{t + \sin t}\, dt = -\int_1^x \sqrt{t + \sin t}\, dt \quad \Rightarrow \quad F'(x) = -\frac{d}{dx}\int_1^x \sqrt{t + \sin t}\, dt = -\sqrt{x + \sin x}$

47. Let $u = x^4$. Then $\dfrac{du}{dx} = 4x^3$. Also, $\dfrac{dg}{dx} = \dfrac{dg}{du}\dfrac{du}{dx}$, so

$$g'(x) = \frac{d}{dx}\int_0^{x^4} \cos(t^2)\, dt = \frac{d}{du}\int_0^u \cos(t^2)\, dt \cdot \frac{du}{dx} = \cos(u^2)\,\frac{du}{dx} = 4x^3 \cos(x^8).$$

48. Let $u = \sin x$. Then $\dfrac{du}{dx} = \cos x$. Also, $\dfrac{dg}{dx} = \dfrac{dg}{du}\dfrac{du}{dx}$, so

$$g'(x) = \frac{d}{dx}\int_1^{\sin x} \frac{1-t^2}{1+t^4}\, dt = \frac{d}{du}\int_1^u \frac{1-t^2}{1+t^4}\, dt \cdot \frac{du}{dx} = \frac{1-u^2}{1+u^4}\cdot\frac{du}{dx} = \frac{1-\sin^2 x}{1+\sin^4 x}\cdot \cos x = \frac{\cos^3 x}{1+\sin^4 x}$$

49. $y = \displaystyle\int_{\sqrt{x}}^x \frac{e^t}{t}\, dt = \int_{\sqrt{x}}^1 \frac{e^t}{t}\, dt + \int_1^x \frac{e^t}{t}\, dt = -\int_1^{\sqrt{x}} \frac{e^t}{t}\, dt + \int_1^x \frac{e^t}{t}\, dt \quad \Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{d}{dx}\left(\displaystyle\int_1^{\sqrt{x}} \frac{e^t}{t}\, dt\right) + \dfrac{d}{dx}\left(\displaystyle\int_1^x \frac{e^t}{t}\, dt\right)$. Let $u = \sqrt{x}$. Then

$$\frac{d}{dx}\int_1^{\sqrt{x}} \frac{e^t}{t}\, dt = \frac{d}{dx}\int_1^u \frac{e^t}{t}\, dt = \frac{d}{du}\left(\int_1^u \frac{e^t}{t}\, dt\right)\frac{du}{dx} = \frac{e^u}{u}\cdot\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2x},$$

so $\dfrac{dy}{dx} = -\dfrac{e^{\sqrt{x}}}{2x} + \dfrac{e^x}{x} = \dfrac{2e^x - e^{\sqrt{x}}}{2x}.$

50. $y = \int_{2x}^{3x+1} \sin(t^4)\,dt = \int_{2x}^0 \sin(t^4)\, dt + \int_0^{3x+1} \sin(t^4)\, dt = \int_0^{3x+1} \sin(t^4)\, dt - \int_0^{2x} \sin(t^4)\, dt \quad \Rightarrow$

$y' = \sin\big[(3x+1)^4\big]\cdot\dfrac{d}{dx}(3x+1) - \sin\big[(2x)^4\big]\cdot\dfrac{d}{dx}(2x) = 3\sin\big[(3x+1)^4\big] - 2\sin\big[(2x)^4\big]$

51. If $1 \le x \le 3$, then $\sqrt{1^2+3} \le \sqrt{x^2+3} \le \sqrt{3^2+3} \quad \Rightarrow \quad 2 \le \sqrt{x^2+3} \le 2\sqrt{3}$, so

$2(3-1) \le \int_1^3 \sqrt{x^2+3}\, dx \le 2\sqrt{3}(3-1)$; that is, $4 \le \int_1^3 \sqrt{x^2+3}\, dx \le 4\sqrt{3}.$

52. If $3 \le x \le 5$, then $4 \le x+1 \le 6$ and $\dfrac{1}{6} \le \dfrac{1}{x+1} \le \dfrac{1}{4}$, so $\dfrac{1}{6}(5-3) \le \displaystyle\int_3^5 \frac{1}{x+1}\, dx \le \dfrac{1}{4}(5-3)$;

that is, $\dfrac{1}{3} \le \displaystyle\int_3^5 \frac{1}{x+1}\, dx \le \dfrac{1}{2}.$

53. $0 \le x \le 1 \quad \Rightarrow \quad 0 \le \cos x \le 1 \quad \Rightarrow \quad x^2 \cos x \le x^2 \quad \Rightarrow \quad \int_0^1 x^2 \cos x\, dx \le \int_0^1 x^2\, dx = \frac{1}{3}\big[x^3\big]_0^1 = \frac{1}{3}$ [Property 7].

54. On the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, x is increasing and $\sin x$ is decreasing, so $\frac{\sin x}{x}$ is decreasing. Therefore, the largest value of $\frac{\sin x}{x}$ on

$\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ is $\frac{\sin(\pi/4)}{\pi/4} = \frac{\sqrt{2}/2}{\pi/4} = \frac{2\sqrt{2}}{\pi}$. By Property 8 with $M = \frac{2\sqrt{2}}{\pi}$ we get $\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\, dx \le \frac{2\sqrt{2}}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

55. $\cos x \le 1 \;\Rightarrow\; e^x \cos x \le e^x \;\Rightarrow\; \int_0^1 e^x \cos x\, dx \le \int_0^1 e^x\, dx = [e^x]_0^1 = e - 1$

56. For $0 \le x \le 1$, $0 \le \sin^{-1} x \le \frac{\pi}{2}$, so $\int_0^1 x\sin^{-1} x\, dx \le \int_0^1 x\left(\frac{\pi}{2}\right)dx = \left[\frac{\pi}{4}x^2\right]_0^1 = \frac{\pi}{4}$.

57. $\Delta x = (3-0)/6 = \frac{1}{2}$, so the endpoints are 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, and 3, and the midpoints are $\frac{1}{4}$, $\frac{3}{4}$, $\frac{5}{4}$, $\frac{7}{4}$, $\frac{9}{4}$, and $\frac{11}{4}$.

The Midpoint Rule gives

$\int_0^3 \sin(x^3)\, dx \approx \sum_{i=1}^{6} f(\overline{x}_i)\,\Delta x = \frac{1}{2}\left[\sin\left(\frac{1}{4}\right)^3 + \sin\left(\frac{3}{4}\right)^3 + \sin\left(\frac{5}{4}\right)^3 + \sin\left(\frac{7}{4}\right)^3 + \sin\left(\frac{9}{4}\right)^3 + \sin\left(\frac{11}{4}\right)^3\right] \approx 0.280981$.

58. (a) Displacement $= \int_0^5 \left(t^2 - t\right)dt = \left[\frac{1}{3}t^3 - \frac{1}{2}t^2\right]_0^5 = \frac{125}{3} - \frac{25}{2} = \frac{175}{6} = 29.1\overline{6}$ meters

(b) Distance traveled $= \int_0^5 \left|t^2 - t\right|dt = \int_0^5 |t(t-1)|\, dt = \int_0^1 \left(t - t^2\right)dt + \int_1^5 \left(t^2 - t\right)dt$

$= \left[\frac{1}{2}t^2 - \frac{1}{3}t^3\right]_0^1 + \left[\frac{1}{3}t^3 - \frac{1}{2}t^2\right]_1^5 = \frac{1}{2} - \frac{1}{3} - 0 + \left(\frac{125}{3} - \frac{25}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{177}{6} = 29.5$ meters

59. Note that $r(t) = b'(t)$, where $b(t) = $ the number of barrels of oil consumed up to time t. So, by the Net Change Theorem,

$\int_0^8 r(t)\, dt = b(8) - b(0)$ represents the number of barrels of oil consumed from Jan. 1, 2000, through Jan. 1, 2008.

60. Distance covered $= \int_0^{5.0} v(t)\, dt \approx M_5 = \frac{5.0 - 0}{5}[v(0.5) + v(1.5) + v(2.5) + v(3.5) + v(4.5)]$

$= 1(4.67 + 8.86 + 10.22 + 10.67 + 10.81) = 45.23$ m

61. We use the Midpoint Rule with $n = 6$ and $\Delta t = \frac{24 - 0}{6} = 4$. The increase in the bee population was
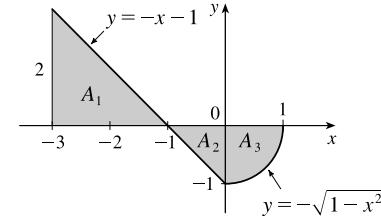
$\int_0^{24} r(t)\, dt \approx M_6 = 4[r(2) + r(6) + r(10) + r(14) + r(18) + r(22)]$

$\approx 4[50 + 1000 + 7000 + 8550 + 1350 + 150] = 4(18{,}100) = 72{,}400$

62. $A_1 = \frac{1}{2}bh = \frac{1}{2}(2)(2) = 2$, $A_2 = \frac{1}{2}bh = \frac{1}{2}(1)(1) = \frac{1}{2}$, and since

$y = -\sqrt{1 - x^2}$ for $0 \le x \le 1$ represents a quarter-circle with radius 1,

$A_3 = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$. So

$\int_{-3}^1 f(x)\, dx = A_1 - A_2 - A_3 = 2 - \frac{1}{2} - \frac{\pi}{4} = \frac{1}{4}(6 - \pi)$

63. Let $u = 2\sin\theta$. Then $du = 2\cos\theta\, d\theta$ and when $\theta = 0$, $u = 0$; when $\theta = \frac{\pi}{2}$, $u = 2$. Thus,

$\int_0^{\pi/2} f(2\sin\theta)\cos\theta\, d\theta = \int_0^2 f(u)\left(\frac{1}{2}\, du\right) = \frac{1}{2}\int_0^2 f(u)\, du = \frac{1}{2}\int_0^2 f(x)\, dx = \frac{1}{2}(6) = 3$.

64. (a) C is increasing on those intervals where C' is positive. By the Fundamental Theorem of Calculus,
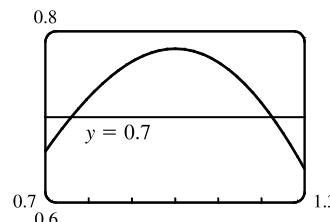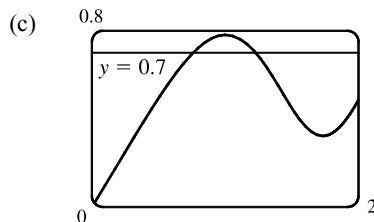
$C'(x) = \frac{d}{dx}\left[\int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt\right] = \cos\left(\frac{\pi}{2}x^2\right)$. This is positive when $\frac{\pi}{2}x^2$ is in the interval $\left(\left(2n - \frac{1}{2}\right)\pi, \left(2n + \frac{1}{2}\right)\pi\right)$,

n any integer. This implies that $\left(2n - \frac{1}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n + \frac{1}{2}\right)\pi \;\Leftrightarrow\; 0 \le |x| < 1$ or $\sqrt{4n - 1} < |x| < \sqrt{4n + 1}$,

n any positive integer. So C is increasing on the intervals $(-1, 1)$, $(\sqrt{3}, \sqrt{5})$, $(-\sqrt{5}, -\sqrt{3})$, $(\sqrt{7}, 3)$, $(-3, -\sqrt{7})$, ...
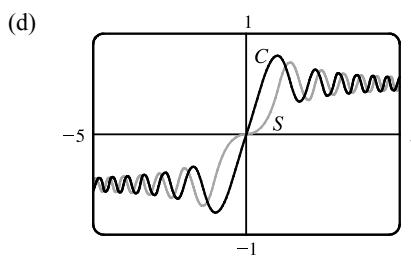
(b) C is concave upward on those intervals where $C'' > 0$. We differentiate C' to find C'': $C'(x) = \cos\left(\frac{\pi}{2}x^2\right)$ $\Rightarrow$

$C''(x) = -\sin\left(\frac{\pi}{2}x^2\right)\left(\frac{\pi}{2}\cdot 2x\right) = -\pi x \sin\left(\frac{\pi}{2}x^2\right)$. For $x > 0$, this is positive where $(2n-1)\pi < \frac{\pi}{2}x^2 < 2n\pi$, n any

positive integer $\Leftrightarrow$ $\sqrt{2(2n-1)} < x < 2\sqrt{n}$, n any positive integer. Since there is a factor of $-x$ in C'', the intervals

of upward concavity for $x < 0$ are $\left(-\sqrt{2(2n+1)}, -2\sqrt{n}\right)$, n any nonnegative integer. That is, C is concave upward on

$\left(-\sqrt{2}, 0\right), \left(\sqrt{2}, 2\right), \left(-\sqrt{6}, -2\right), \left(\sqrt{6}, 2\sqrt{2}\right), \ldots$.

(c)

From the graphs, we can determine

that $\int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt = 0.7$ at

$x \approx 0.76$ and $x \approx 1.22$.

(d)

The graphs of $S(x)$ and $C(x)$ have similar shapes, except that S's flattens out

near the origin, while C's does not. Note that for $x > 0$, C is increasing

where S is concave up, and C is decreasing where S is concave down.

Similarly, S is increasing where C is concave down, and S is decreasing

where C is concave up. For $x < 0$, these relationships are reversed; that is, C

is increasing where S is concave down, and S is increasing where C is

concave up. See Example 5.3.3 and Exercise 5.3.65 for a discussion of $S(x)$.
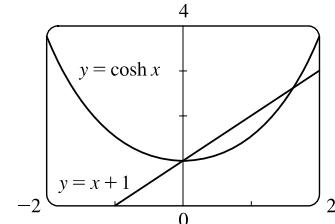
65. Area under the curve $y = \sinh cx$ between $x = 0$ and $x = 1$ is equal to 1 $\Rightarrow$

$\int_0^1 \sinh cx\, dx = 1$ $\Rightarrow$ $\frac{1}{c}\left[\cosh cx\right]_0^1 = 1$ $\Rightarrow$ $\frac{1}{c}(\cosh c - 1) = 1$ $\Rightarrow$

$\cosh c - 1 = c$ $\Rightarrow$ $\cosh c = c + 1$. From the graph, we get $c = 0$ and

$c \approx 1.6161$, but $c = 0$ isn't a solution for this problem since the curve

$y = \sinh cx$ becomes $y = 0$ and the area under it is 0. Thus, $c \approx 1.6161$.

66. Both numerator and denominator approach 0 as $a \to 0$, so we use l'Hospital's Rule. (Note that we are differentiating *with*

respect to a, since that is the quantity which is changing.) We also use FTC1:

$$\lim_{a \to 0} T(x,t) = \lim_{a \to 0} \frac{C\int_0^a e^{-(x-u)^2/(4kt)}\,du}{a\sqrt{4\pi kt}} \overset{\text{H}}{=} \lim_{a \to 0} \frac{Ce^{-(x-a)^2/(4kt)}}{\sqrt{4\pi kt}} = \frac{Ce^{-x^2/(4kt)}}{\sqrt{4\pi kt}}$$

67. Using FTC1, we differentiate both sides of the given equation, $\int_1^x f(t)\,dt = (x-1)e^{2x} + \int_1^x e^{-t}f(t)\,dt$, and get

$f(x) = e^{2x} + 2(x-1)e^{2x} + e^{-x}f(x)$ $\Rightarrow$ $f(x)(1 - e^{-x}) = e^{2x} + 2(x-1)e^{2x}$ $\Rightarrow$ $f(x) = \dfrac{e^{2x}(2x-1)}{1 - e^{-x}}$.

68. The second derivative is the derivative of the first derivative, so we'll apply the Net Change Theorem with $F = h'$.

$\int_1^2 h''(u)\,du = \int_1^2 (h')'(u)\,du = h'(2) - h'(1) = 5 - 2 = 3$. The other information is unnecessary.

69. Let $u = f(x)$ and $du = f'(x)\,dx$. So $2\int_a^b f(x)f'(x)\,dx = 2\int_{f(a)}^{f(b)} u\,du = \left[u^2\right]_{f(a)}^{f(b)} = [f(b)]^2 - [f(a)]^2$.

70. Let $F(x) = \displaystyle\int_2^x \sqrt{1 + t^3}\,dt$. Then $F'(2) = \displaystyle\lim_{h\to 0} \frac{F(2+h) - F(2)}{h} = \lim_{h\to 0} \frac{1}{h}\int_2^{2+h} \sqrt{1+t^3}\,dt$, and $F'(x) = \sqrt{1+x^3}$, so

$$\lim_{h\to 0} \frac{1}{h}\int_2^{2+h} \sqrt{1+t^3}\,dt = F'(2) = \sqrt{1 + 2^3} = \sqrt{9} = 3.$$

71. Let $u = 1 - x$. Then $du = -dx$, so $\int_0^1 f(1-x)\,dx = \int_1^0 f(u)(-du) = \int_0^1 f(u)\,du = \int_0^1 f(x)\,dx$.
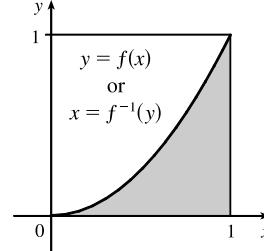
72. $\displaystyle\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9\right] = \lim_{n\to\infty} \frac{1-0}{n}\sum_{i=1}^n \left(\frac{i}{n}\right)^9 = \int_0^1 x^9\,dx = \left[\frac{x^{10}}{10}\right]_0^1 = \frac{1}{10}$

The limit is based on Riemann sums using right endpoints and subintervals of equal length.

73. The shaded region has area $\int_0^1 f(x)\,dx = \frac{1}{3}$. The integral $\int_0^1 f^{-1}(y)\,dy$

gives the area of the unshaded region, which we know to be $1 - \frac{1}{3} = \frac{2}{3}$.

So $\int_0^1 f^{-1}(y)\,dy = \frac{2}{3}$.

PROBLEMS PLUS

1. Differentiating both sides of the equation $x \sin \pi x = \int_0^{x^2} f(t)\, dt$ (using FTC1 and the Chain Rule for the right side) gives

$\sin \pi x + \pi x \cos \pi x = 2x f(x^2)$. Letting $x = 2$ so that $f(x^2) = f(4)$, we obtain $\sin 2\pi + 2\pi \cos 2\pi = 4f(4)$, so

$f(4) = \frac{1}{4}(0 + 2\pi \cdot 1) = \frac{\pi}{2}$.

2. The area A under the curve $y = x + 1/x$ from $x = a$ to $x = a + 1.5$ is given by $A(a) = \int_a^{a+1.5} \left(x + \frac{1}{x} \right) dx$.

To find the minimum value of A, we'll differentiate A using FTC1 and set the derivative equal to 0.

$$
\begin{aligned}
A'(a) &= \frac{d}{da} \int_a^{a+1.5} \left(x + \frac{1}{x} \right) dx \\
&= \frac{d}{da} \int_a^{1} \left(x + \frac{1}{x} \right) dx + \frac{d}{da} \int_1^{a+1.5} \left(x + \frac{1}{x} \right) dx \\
&= -\frac{d}{da} \int_1^{a} \left(x + \frac{1}{x} \right) dx + \frac{d}{da} \int_1^{a+1.5} \left(x + \frac{1}{x} \right) dx \\
&= -\left(a + \frac{1}{a} \right) + \left(a + 1.5 + \frac{1}{a+1.5} \right) = 1.5 + \frac{1}{a+1.5} - \frac{1}{a}
\end{aligned}
$$
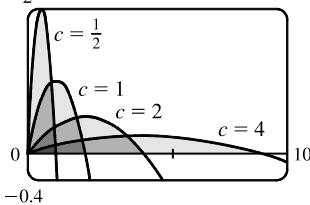
$A'(a) = 0 \quad \Leftrightarrow \quad 1.5 + \frac{1}{a+1.5} - \frac{1}{a} = 0 \quad \Leftrightarrow \quad 1.5a(a+1.5) + a - (a+1.5) = 0 \quad \Leftrightarrow$

$1.5a^2 + 2.25a - 1.5 = 0 \quad \left[\text{multiply by } \frac{4}{3}\right] \quad \Leftrightarrow \quad 2a^2 + 3a - 2 = 0 \quad \Leftrightarrow \quad (2a-1)(a+2) = 0 \quad \Leftrightarrow \quad a = \frac{1}{2}$ or

$a = -2$. Since $a > 0, a = \frac{1}{2}$. $\quad A''(a) = -\frac{1}{(a+1.5)^2} + \frac{1}{a^2} > 0$, so

$A\left(\frac{1}{2}\right) = \int_{1/2}^{2} \left(x + \frac{1}{x} \right) dx = \left[\frac{1}{2}x^2 + \ln|x| \right]_{1/2}^{2} = (2 + \ln 2) - \left(\frac{1}{8} - \ln 2 \right) = \frac{15}{8} + 2\ln 2$ is the minimum value of A.

3. For $I = \int_0^4 x e^{(x-2)^4}\, dx$, let $u = x - 2$ so that $x = u + 2$ and $dx = du$. Then

$I = \int_{-2}^{2} (u+2)e^{u^4}\, du = \int_{-2}^{2} u e^{u^4}\, du + \int_{-2}^{2} 2e^{u^4}\, du = 0$ [by 5.5.7(b)] $+ 2\int_0^4 e^{(x-2)^4}\, dx = 2k$.

4. (a)

From the graph of $f(x) = \dfrac{2cx - x^2}{c^3}$, it appears that the areas

are equal; that is, the area enclosed is independent of c.
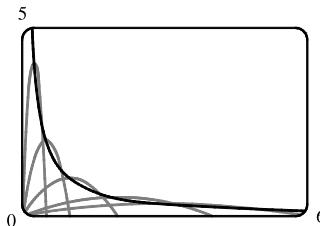
(b) We first find the x-intercepts of the curve, to determine the limits of integration: $y = 0 \quad \Leftrightarrow \quad 2cx - x^2 = 0 \quad \Leftrightarrow \quad x = 0$

or $x = 2c$. Now we integrate the function between these limits to find the enclosed area:

$A = \int_0^{2c} \frac{2cx - x^2}{c^3}\, dx = \frac{1}{c^3} \left[cx^2 - \frac{1}{3}x^3 \right]_0^{2c} = \frac{1}{c^3} \left[c(2c)^2 - \frac{1}{3}(2c)^3 \right] = \frac{1}{c^3} \left[4c^3 - \frac{8}{3}c^3 \right] = \frac{4}{3}$, a constant.

(c)

The vertices of the family of parabolas seem to determine a branch of a hyperbola.

(d) For a particular c, the vertex is the point where the maximum occurs. We have seen that the x-intercepts are 0 and $2c$, so by symmetry, the maximum occurs at $x = c$, and its value is $\dfrac{2c(c) - c^2}{c^3} = \dfrac{1}{c}$. So we are interested in the curve consisting of all points of the form $\left(c, \dfrac{1}{c}\right)$, $c > 0$. This is the part of the hyperbola $y = 1/x$ lying in the first quadrant.

5. $f(x) = \displaystyle\int_0^{g(x)} \dfrac{1}{\sqrt{1 + t^3}}\, dt$, where $g(x) = \displaystyle\int_0^{\cos x} [1 + \sin(t^2)]\, dt$. Using FTC1 and the Chain Rule (twice) we have

$f'(x) = \dfrac{1}{\sqrt{1 + [g(x)]^3}}\, g'(x) = \dfrac{1}{\sqrt{1 + [g(x)]^3}}\, [1 + \sin(\cos^2 x)](-\sin x)$. Now $g\left(\frac{\pi}{2}\right) = \displaystyle\int_0^0 [1 + \sin(t^2)]\, dt = 0$, so

$f'\left(\frac{\pi}{2}\right) = \dfrac{1}{\sqrt{1 + 0}}\, (1 + \sin 0)(-1) = 1 \cdot 1 \cdot (-1) = -1$.

6. If $f(x) = \int_0^x x^2 \sin(t^2)\, dt = x^2 \int_0^x \sin(t^2)\, dt$, then $f'(x) = x^2 \sin(x^2) + 2x \int_0^x \sin(t^2)\, dt$, by the Product Rule and FTC1.

7. By l'Hospital's Rule and the Fundamental Theorem, using the notation $\exp(y) = e^y$,

$$\lim_{x \to 0} \frac{\int_0^x (1 - \tan 2t)^{1/t}\, dt}{x} \overset{\mathrm{H}}{=} \lim_{x \to 0} \frac{(1 - \tan 2x)^{1/x}}{1} = \exp\left(\lim_{x \to 0} \frac{\ln(1 - \tan 2x)}{x}\right)$$

$$\overset{\mathrm{H}}{=} \exp\left(\lim_{x \to 0} \frac{-2\sec^2 2x}{1 - \tan 2x}\right) = \exp\left(\frac{-2 \cdot 1^2}{1 - 0}\right) = e^{-2}$$

8. The area $A(t) = \int_0^t \sin(x^2)\, dx$, and the area $B(t) = \frac{1}{2} t \sin(t^2)$. Since $\displaystyle\lim_{t \to 0^+} A(t) = 0 = \lim_{t \to 0^+} B(t)$, we can use l'Hospital's Rule:

$$\lim_{t \to 0^+} \frac{A(t)}{B(t)} \overset{\mathrm{H}}{=} \lim_{t \to 0^+} \frac{\sin(t^2)}{\frac{1}{2}\sin(t^2) + \frac{1}{2}t[2t\cos(t^2)]} \qquad \text{[by FTC1 and the Product Rule]}$$

$$\overset{\mathrm{H}}{=} \lim_{t \to 0^+} \frac{2t\cos(t^2)}{t\cos(t^2) - 2t^3\sin(t^2) + 2t\cos(t^2)} = \lim_{t \to 0^+} \frac{2\cos(t^2)}{3\cos(t^2) - 2t^2\sin(t^2)} = \frac{2}{3 - 0} = \frac{2}{3}$$

9. $f(x) = 2 + x - x^2 = (-x + 2)(x + 1) = 0 \;\Leftrightarrow\; x = 2$ or $x = -1$. $f(x) \geq 0$ for $x \in [-1, 2]$ and $f(x) < 0$ everywhere else. The integral $\int_a^b (2 + x - x^2)\, dx$ has a maximum on the interval where the integrand is positive, which is $[-1, 2]$. So $a = -1$, $b = 2$. (Any larger interval gives a smaller integral since $f(x) < 0$ outside $[-1, 2]$. Any smaller interval also gives a smaller integral since $f(x) \geq 0$ in $[-1, 2]$.)

10. This sum can be interpreted as a Riemann sum, with the right endpoints of the subintervals as sample points and with $a = 0$, $b = 10{,}000$, and $f(x) = \sqrt{x}$. So we approximate

$$\sum_{i=1}^{10{,}000} \sqrt{i} \approx \lim_{n \to \infty} \frac{10{,}000}{n} \sum_{i=1}^{n} \sqrt{\frac{10{,}000i}{n}} = \int_0^{10{,}000} \sqrt{x}\, dx = \left[\frac{2}{3} x^{3/2}\right]_0^{10{,}000} = \frac{2}{3}(1{,}000{,}000) \approx 666{,}667.$$

Alternate method: We can use graphical methods as follows:
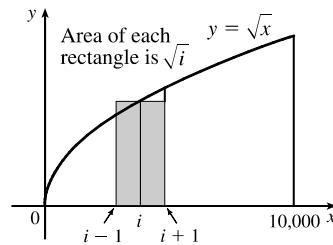
From the figure we see that $\int_{i-1}^{i} \sqrt{x}\, dx < \sqrt{i} < \int_{i}^{i+1} \sqrt{x}\, dx$, so

Area of each rectangle is $\sqrt{i}$ $y = \sqrt{x}$

$\int_{0}^{10,000} \sqrt{x}\, dx < \sum_{i=1}^{10,000} \sqrt{i} < \int_{1}^{10,001} \sqrt{x}\, dx$. Since

$\int \sqrt{x}\, dx = \frac{2}{3} x^{3/2} + C$, we get $\int_{0}^{10,000} \sqrt{x}\, dx = 666{,}666.\overline{6}$ and

$\int_{1}^{10,001} \sqrt{x}\, dx = \frac{2}{3}\left[(10{,}001)^{3/2} - 1\right] \approx 666{,}766$.

Hence, $666{,}666.\overline{6} < \sum_{i=1}^{10,000} \sqrt{i} < 666{,}766$. We can estimate the sum by averaging these bounds:

$\sum_{i=1}^{10,000} \approx \dfrac{666{,}666.\overline{6} + 666{,}766}{2} \approx 666{,}716$. The actual value is about $666{,}716.46$.

11. (a) We can split the integral $\int_{0}^{n} [\![x]\!]\, dx$ into the sum $\sum_{i=1}^{n} \left[\int_{i-1}^{i} [\![x]\!]\, dx \right]$. But on each of the intervals $[i-1, i)$ of integration,

$[\![x]\!]$ is a constant function, namely $i - 1$. So the ith integral in the sum is equal to $(i-1)[i - (i-1)] = (i-1)$. So the

original integral is equal to $\sum_{i=1}^{n} (i-1) = \sum_{i=1}^{n-1} i = \dfrac{(n-1)n}{2}$.

(b) We can write $\int_{a}^{b} [\![x]\!]\, dx = \int_{0}^{b} [\![x]\!]\, dx - \int_{0}^{a} [\![x]\!]\, dx$.

Now $\int_{0}^{b} [\![x]\!]\, dx = \int_{0}^{[\![b]\!]} [\![x]\!]\, dx + \int_{[\![b]\!]}^{b} [\![x]\!]\, dx$. The first of these integrals is equal to $\frac{1}{2}([\![b]\!] - 1)\,[\![b]\!]$,

by part (a), and since $[\![x]\!] = [\![b]\!]$ on $[[\![b]\!], b]$, the second integral is just $[\![b]\!]\,(b - [\![b]\!])$. So

$\int_{0}^{b} [\![x]\!]\, dx = \frac{1}{2}([\![b]\!] - 1)\,[\![b]\!] + [\![b]\!]\,(b - [\![b]\!]) = \frac{1}{2}[\![b]\!]\,(2b - [\![b]\!] - 1)$ and similarly $\int_{0}^{a} [\![x]\!]\, dx = \frac{1}{2}[\![a]\!]\,(2a - [\![a]\!] - 1)$.

Therefore, $\int_{a}^{b} [\![x]\!]\, dx = \frac{1}{2}[\![b]\!]\,(2b - [\![b]\!] - 1) - \frac{1}{2}[\![a]\!]\,(2a - [\![a]\!] - 1)$.

12. By FTC1, $\dfrac{d}{dx} \int_{0}^{x}\left(\int_{1}^{\sin t} \sqrt{1 + u^4}\, du\right) dt = \int_{1}^{\sin x} \sqrt{1 + u^4}\, du$. Again using FTC1,

$\dfrac{d^2}{dx^2} \int_{0}^{x}\left(\int_{1}^{\sin t} \sqrt{1 + u^4}\, du\right) dt = \dfrac{d}{dx} \int_{1}^{\sin x} \sqrt{1 + u^4}\, du = \sqrt{1 + \sin^4 x}\,\cos x$.

13. Let $Q(x) = \displaystyle\int_{0}^{x} P(t)\, dt = \left[at + \dfrac{b}{2}t^2 + \dfrac{c}{3}t^3 + \dfrac{d}{4}t^4\right]_{0}^{x} = ax + \dfrac{b}{2}x^2 + \dfrac{c}{3}x^3 + \dfrac{d}{4}x^4$. Then $Q(0) = 0$, and $Q(1) = 0$ by the

given condition, $a + \dfrac{b}{2} + \dfrac{c}{3} + \dfrac{d}{4} = 0$. Also, $Q'(x) = P(x) = a + bx + cx^2 + dx^3$ by FTC1. By Rolle's Theorem, applied to

Q on $[0, 1]$, there is a number r in $(0, 1)$ such that $Q'(r) = 0$, that is, such that $P(r) = 0$. Thus, the equation $P(x) = 0$ has a

root between 0 and 1.

More generally, if $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ and if $a_0 + \dfrac{a_1}{2} + \dfrac{a_2}{3} + \cdots + \dfrac{a_n}{n+1} = 0$, then the equation

$P(x) = 0$ has a root between 0 and 1. The proof is the same as before:
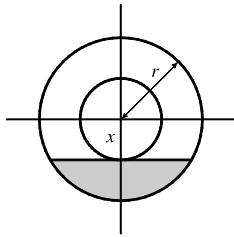
Let $Q(x) = \displaystyle\int_{0}^{x} P(t)\, dt = a_0 x + \dfrac{a_1}{2} x^2 + \dfrac{a_2}{3} x^3 + \cdots + \dfrac{a_n}{n+1} x^n$. Then $Q(0) = Q(1) = 0$ and $Q'(x) = P(x)$. By

Rolle's Theorem applied to Q on $[0, 1]$, there is a number r in $(0, 1)$ such that $Q'(r) = 0$, that is, such that $P(r) = 0$.

14.

Let x be the distance between the center of the disk and the surface of the liquid.
The wetted circular region has area $\pi r^2 - \pi x^2$ while the unexposed wetted region
(shaded in the diagram) has area $2 \int_x^r \sqrt{r^2 - t^2}\, dt$, so the exposed wetted region
has area $A(x) = \pi r^2 - \pi x^2 - 2 \int_x^r \sqrt{r^2 - t^2}\, dt$, $0 \le x \le r$. By FTC1, we have
$A'(x) = -2\pi x + 2\sqrt{r^2 - x^2}$.

Now $A'(x) > 0 \;\Rightarrow\; -2\pi x + 2\sqrt{r^2 - x^2} > 0 \;\Rightarrow\; \sqrt{r^2 - x^2} > \pi x \;\Rightarrow\; r^2 - x^2 > \pi^2 x^2 \;\Rightarrow$

$r^2 > \pi^2 x^2 + x^2 \;\Rightarrow\; r^2 > x^2(\pi^2 + 1) \;\Rightarrow\; x^2 < \dfrac{r^2}{\pi^2 + 1} \;\Rightarrow\; x < \dfrac{r}{\sqrt{\pi^2 + 1}}$, and we'll call this value x^*.

Since $A'(x) > 0$ for $0 < x < x^*$ and $A'(x) < 0$ for $x^* < x < r$, we have an absolute maximum when $x = x^*$.

15. Note that $\dfrac{d}{dx}\left(\int_0^x \left[\int_0^u f(t)\, dt \right] du \right) = \int_0^x f(t)\, dt$ by FTC1, while

$$\frac{d}{dx}\left[\int_0^x f(u)(x - u)\, du \right] = \frac{d}{dx}\left[x \int_0^x f(u)\, du \right] - \frac{d}{dx}\left[\int_0^x f(u)u\, du \right]$$

$$= \int_0^x f(u)\, du + x f(x) - f(x)x = \int_0^x f(u)\, du$$
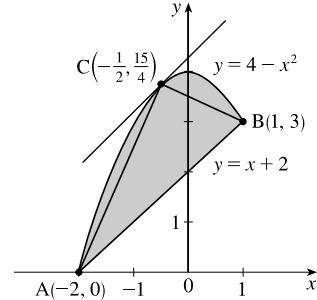
Hence, $\int_0^x f(u)(x - u)\, du = \int_0^x \left[\int_0^u f(t)\, dt \right] du + C$. Setting $x = 0$ gives $C = 0$.

16. The parabola $y = 4 - x^2$ and the line $y = x + 2$ intersect when

$4 - x^2 = x + 2 \;\Leftrightarrow\; x^2 + x - 2 = 0 \;\Leftrightarrow\; (x + 2)(x - 1) = 0 \;\Leftrightarrow$

$x = -2$ or 1. So the point A is $(-2, 0)$ and B is $(1, 3)$. The slope of the line

$y = x + 2$ is 1 and the slope of the parabola $y = 4 - x^2$ at x-coordinate x is

$-2x$. These slopes are equal when $x = -\frac{1}{2}$, so the point C is $\left(-\frac{1}{2}, \frac{15}{4}\right)$.

The area A_1 of the parabolic segment is the area under the parabola from

$x = -2$ to $x = 1$, minus the area under the line $y = x + 2$ from -2 to 1. Thus,

$$A_1 = \int_{-2}^1 (4 - x^2)\, dx - \int_{-2}^1 (x + 2)\, dx = \left[4x - \tfrac{1}{3}x^3 \right]_{-2}^1 - \left[\tfrac{1}{2}x^2 + 2x \right]_{-2}^1$$

$$= \left[(4 - \tfrac{1}{3}) - (-8 + \tfrac{8}{3}) \right] - \left[(\tfrac{1}{2} + 2) - (2 - 4) \right] = 9 - \tfrac{9}{2} = \tfrac{9}{2}.$$

The area A_2 of the inscribed triangle is the area under the line segment AC plus the area under the line segment CB minus

the area under the line segment AB. The line through A and C has slope $\dfrac{15/4 - 0}{-1/2 + 2} = \dfrac{5}{2}$ and equation $y - 0 = \frac{5}{2}(x + 2)$, or
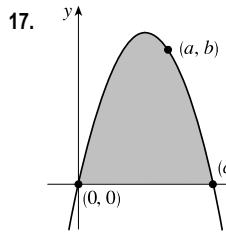
$y = \frac{5}{2}x + 5$. The line through C and B has slope $\dfrac{3 - 15/4}{1 + 1/2} = -\dfrac{1}{2}$ and equation $y - 3 = -\frac{1}{2}(x - 1)$, or $y = -\frac{1}{2}x + \frac{7}{2}$.

Thus,

$$A_2 = \int_{-2}^{-1/2} \left(\tfrac{5}{2}x + 5 \right) dx + \int_{-1/2}^1 \left(-\tfrac{1}{2}x + \tfrac{7}{2} \right) dx - \int_{-2}^1 (x + 2)\, dx = \left[\tfrac{5}{4}x^2 + 5x \right]_{-2}^{-1/2} + \left[-\tfrac{1}{4}x^2 + \tfrac{7}{2}x \right]_{-1/2}^1 - \tfrac{9}{2}$$

$$= \left[\left(\tfrac{5}{16} - \tfrac{5}{2} \right) - (5 - 10) \right] + \left[\left(-\tfrac{1}{4} + \tfrac{7}{2} \right) - \left(-\tfrac{1}{16} - \tfrac{7}{4} \right) \right] - \tfrac{9}{2} = \tfrac{45}{16} + \tfrac{81}{16} - \tfrac{72}{16} = \tfrac{54}{16} = \tfrac{27}{8}$$

Archimedes' result states that $A_1 = \frac{4}{3}A_2$, which is verified in this case since $\frac{4}{3} \cdot \frac{27}{8} = \frac{9}{2}$.

17.

Let c be the nonzero x-intercept so that the parabola has equation $f(x) = kx(x - c)$, or $y = kx^2 - ckx$, where $k < 0$. The area A under the parabola is

$$A = \int_0^c kx(x - c)\, dx = k \int_0^c (x^2 - cx)\, dx = k\left[\tfrac{1}{3}x^3 - \tfrac{1}{2}cx^2\right]_0^c$$
$$= k\left(\tfrac{1}{3}c^3 - \tfrac{1}{2}c^3\right) = -\tfrac{1}{6}kc^3$$
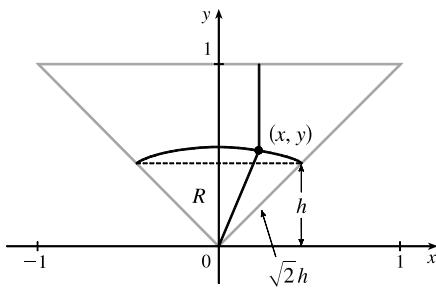
The point (a, b) is on the parabola, so $f(a) = b \Rightarrow b = ka(a - c) \Rightarrow$

$k = \dfrac{b}{a(a - c)}$. Substituting for k in A gives $A(c) = -\dfrac{b}{6a} \cdot \dfrac{c^3}{a - c} \Rightarrow$

$$A'(c) = -\frac{b}{6a} \cdot \frac{(a - c)(3c^2) - c^3(-1)}{(a - c)^2} = -\frac{b}{6a} \cdot \frac{c^2[3(a - c) + c]}{(a - c)^2} = -\frac{bc^2(3a - 2c)}{6a(a - c)^2}$$

Now $A' = 0 \Rightarrow c = \tfrac{3}{2}a$. Since $A'(c) < 0$ for $a < c < \tfrac{3}{2}a$ and $A'(c) > 0$ for $c > \tfrac{3}{2}a$, so A has an absolute

minimum when $c = \tfrac{3}{2}a$. Substituting for c in k gives us $k = \dfrac{b}{a\left(a - \tfrac{3}{2}a\right)} = -\dfrac{2b}{a^2}$, so $f(x) = -\dfrac{2b}{a^2}\, x\left(x - \tfrac{3}{2}a\right)$, or

$f(x) = -\dfrac{2b}{a^2}\, x^2 + \dfrac{3b}{a}\, x$. Note that the vertex of the parabola is $\left(\tfrac{3}{4}a, \tfrac{9}{8}b\right)$ and the minimal area under the parabola

is $A\left(\tfrac{3}{2}a\right) = \tfrac{9}{8}ab$.

18.

We restrict our attention to the triangle shown. A point in this triangle is closer to the side shown than to any other side, so if we find the area of the region R consisting of all points in the triangle that are closer to the center than to that side, we can multiply this area by 4 to find the total area. We find the equation of the set of points which are equidistant from the center and the side: the distance of the point (x, y) from the side is $1 - y$, and its distance from the center is $\sqrt{x^2 + y^2}$.

So the distances are equal if $\sqrt{x^2 + y^2} = 1 - y \Leftrightarrow x^2 + y^2 = 1 - 2y + y^2 \Leftrightarrow y = \tfrac{1}{2}(1 - x^2)$. Note that the area

we are interested in is equal to the area of a triangle plus a crescent-shaped area. To find these areas, we have to find the

y-coordinate h of the horizontal line separating them. From the diagram, $1 - h = \sqrt{2}\, h \Leftrightarrow h = \dfrac{1}{1 + \sqrt{2}} = \sqrt{2} - 1$.

We calculate the areas in terms of h, and substitute afterward.

The area of the triangle is $\tfrac{1}{2}(2h)(h) = h^2$, and the area of the crescent-shaped section is

$$\int_{-h}^h \left[\tfrac{1}{2}(1 - x^2) - h\right] dx = 2\int_0^h \left(\tfrac{1}{2} - h - \tfrac{1}{2}x^2\right) dx = 2\left[\left(\tfrac{1}{2} - h\right)x - \tfrac{1}{6}x^3\right]_0^h = h - 2h^2 - \tfrac{1}{3}h^3.$$

So the area of the whole region is

$$4\left[\left(h - 2h^2 - \tfrac{1}{3}h^3\right) + h^2\right] = 4h\left(1 - h - \tfrac{1}{3}h^2\right) = 4\left(\sqrt{2} - 1\right)\left[1 - \left(\sqrt{2} - 1\right) - \tfrac{1}{3}\left(\sqrt{2} - 1\right)^2\right]$$
$$= 4\left(\sqrt{2} - 1\right)\left(1 - \tfrac{1}{3}\sqrt{2}\right) = \tfrac{4}{3}\left(4\sqrt{2} - 5\right)$$

19. $\displaystyle\lim_{n \to \infty} \left(\frac{1}{\sqrt{n}\,\sqrt{n+1}} + \frac{1}{\sqrt{n}\,\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\,\sqrt{n+n}} \right)$

$$= \lim_{n \to \infty} \frac{1}{n} \left(\sqrt{\frac{n}{n+1}} + \sqrt{\frac{n}{n+2}} + \cdots + \sqrt{\frac{n}{n+n}} \right)$$

$$= \lim_{n \to \infty} \frac{1}{n} \left(\frac{1}{\sqrt{1+1/n}} + \frac{1}{\sqrt{1+2/n}} + \cdots + \frac{1}{\sqrt{1+1}} \right)$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \qquad \left[\text{where } f(x) = \frac{1}{\sqrt{1+x}} \right]$$

$$= \int_0^1 \frac{1}{\sqrt{1+x}}\, dx = \left[2\sqrt{1+x} \right]_0^1 = 2\left(\sqrt{2}-1\right)$$
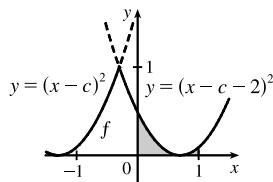
20. Note that the graphs of $(x-c)^2$ and $[(x-c)-2]^2$ intersect when $|x-c| = |x-c-2|$ $\Leftrightarrow$

$c - x = x - c - 2$ $\Leftrightarrow$ $x = c+1$. The integration will proceed differently depending on the value of c.

Case 1: $-2 \le c < -1$

In this case, $f_c(x) = (x-c-2)^2$ for $x \in [0,1]$, so

$$g(c) = \int_0^1 (x-c-2)^2\, dx = \tfrac{1}{3}\left[(x-c-2)^3\right]_0^1 = \tfrac{1}{3}\left[(-c-1)^3 - (-c-2)^3\right]$$

$$= \tfrac{1}{3}(3c^2 + 9c + 7) = c^2 + 3c + \tfrac{7}{3} = \left(c+\tfrac{3}{2}\right)^2 + \tfrac{1}{12}$$
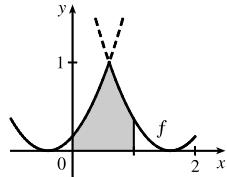

This is a parabola; its maximum value for $-2 \le c < -1$ is $g(-2) = \tfrac{1}{3}$, and its minimum value is $g\left(-\tfrac{3}{2}\right) = \tfrac{1}{12}$.

Case 2: $-1 \le c < 0$

In this case, $f_c(x) = \begin{cases} (x-c)^2 & \text{if } 0 \le x \le c+1 \\ (x-c-2)^2 & \text{if } c+1 < x \le 1 \end{cases}$

Therefore,

$$g(c) = \int_0^1 f_c(x)\, dx = \int_0^{c+1} (x-c)^2\, dx + \int_{c+1}^1 (x-c-2)^2\, dx$$

$$= \tfrac{1}{3}\left[(x-c)^3\right]_0^{c+1} + \tfrac{1}{3}\left[(x-c-2)^3\right]_{c+1}^1 = \tfrac{1}{3}\left[1 + c^3 + (-c-1)^3 - (-1)\right]$$

$$= -c^2 - c + \tfrac{1}{3} = -\left(c+\tfrac{1}{2}\right)^2 + \tfrac{7}{12}$$

Again, this is a parabola, whose maximum value for $-1 \le c < 0$ is $g\left(-\tfrac{1}{2}\right) = \tfrac{7}{12}$, and whose minimum value on this c-interval is $g(-1) = \tfrac{1}{3}$.
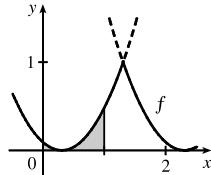
Case 3: $0 \le c \le 2$

In this case, $f_c(x) = (x - c)^2$ for $x \in [0, 1]$, so

$$g(c) = \int_0^1 (x - c)^2 \, dx = \tfrac{1}{3} \big[(x - c)^3\big]_0^1 = \tfrac{1}{3} \big[(1 - c)^3 - (-c)^3\big]$$

$$= c^2 - c + \tfrac{1}{3} = \big(c - \tfrac{1}{2}\big)^2 + \tfrac{1}{12}$$

This parabola has a maximum value of $g(2) = \tfrac{7}{3}$ and a minimum value of $g\big(\tfrac{1}{2}\big) = \tfrac{1}{12}$.

We conclude that $g(c)$ has an absolute maximum value of $g(2) = \tfrac{7}{3}$, and absolute minimum values of $g\big(-\tfrac{3}{2}\big) = g\big(\tfrac{1}{2}\big) = \tfrac{1}{12}$.