Homework 4
Key Answers

1. Use the definition of the derivative to calculate the derivatives of the following functions at the given points:

(a) \( f(x) = x^2 \cos(x) \) at \( x = 0 \).
(b) \( f(x) = \frac{3x+4}{2x-1} \) at \( x = 1 \).

Proof:

(a) Let us compute:

\[
\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \cos(h)}{h} = \lim_{h \to 0} h \cos(h).
\]

Since \( \cos(x) \) is a continuous function at \( x = 0 \), we know that \( \lim_{h \to 0} \cos(h) = \cos(0) = 1 \). So the limit will be: \( \lim_{h \to 0} h \cos(h) = (\lim_{h \to 0} h)(\lim_{h \to 0} \cos(h)) = 0 \cdot 1 = 0 \). We have \( f'(0) = 0 \).

(b) Let us compute:

\[
\lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{3(h+1)+4}{2(h+1)-1} - \frac{7}{2} = \lim_{h \to 0} \frac{-11(h + 1) + 11}{(2(h + 1) - 1)h} = \lim_{h \to 0} \frac{-11h}{(2h + 1)h} = \lim_{h \to 0} \frac{-11}{2h + 1} = -11.
\]

We then have \( f'(1) = -11 \).

2. Let \( f(x) = x^2 \sin\left(\frac{1}{x}\right) \) for \( x \neq 0 \) and \( f(x) = 0 \) at \( x = 0 \).

(a) What is the derivative of \( f(x) \) for any point \( x \neq 0 \)? You can use the fact that \( (\sin(x))' = \cos(x) \).

Hint: You can use the product and chain rules.

(b) Use the definition of the derivative to prove that \( f'(0) = 0 \).

Proof:

(a) At \( x \neq 0 \), we have \( f'(x) = (x^2 \sin\left(\frac{1}{x}\right))' = 2x \sin\left(\frac{1}{x}\right) + x^2\left(-\frac{1}{x^2} \cos\left(\frac{1}{x}\right)\right) \) that is \( f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \).

(b) Let us compute:

\[
\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right).
\]

Since the function sine is bounded between \(-1\) and \(1\), we have \(-h \leq h \sin\left(\frac{1}{h}\right) \leq h \). The limit when \( h \to 0 \) is then 0. We have \( f'(0) = 0 \).
3. (3 points each) Let \( f(x) = x \sin(\frac{1}{x}) \) for \( x \neq 0 \) and \( f(x) = 0 \) at \( x = 0 \).

(a) Prove that \( f(x) \) is continuous at \( x = 0 \).

(b) Prove that \( f(x) \) is not differentiable at \( x = 0 \).

Proof:

(a) Let us consider:

\[
|f(x) - f(0)| = |f(x)| = |x \sin(\frac{1}{x})| \leq |x|.
\]

According to this result, we see that \( |f(x) - f(0)| < \epsilon \) for any \( \epsilon > 0 \), whenever \( |x - 0| < \delta \) (when \( \delta = \epsilon \)). The function \( f(x) \) is then continuous at \( x = 0 \).

(b) Let us compute:

\[
\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin(\frac{1}{h})}{h} = \lim_{h \to 0} \sin(\frac{1}{h}).
\]

Since the function \( \sin(1/h) \) does not have a limit as \( h \to 0 \), the function is not differentiable at \( x = 0 \).

4. (4 points each) Let \( f(x) = x^2 \) for \( x \in \mathbb{Q} \) (\( x \) is rational) and \( f(x) = 0 \) for \( x \in \mathbb{R} \setminus \mathbb{Q} \) (\( x \) is irrational).

(a) Prove that \( f(x) \) is continuous at \( x = 0 \).

(b) Prove that \( f(x) \) is not continuous at \( x \neq 0 \).

(c) Prove that \( f(x) \) is not differentiable for any \( x \neq 0 \) but is differentiable at \( x = 0 \).

Proof:

(a) At \( x = 0 \), we have \( f(0) = 0 \). Let us prove now that \( \lim_{x \to 0} f(x) = f(0) \). Let \( \delta > 0 \) and \( \epsilon > 0 \). For any \( x \in \mathbb{R} \) such that \( |x - 0| = |x| < \delta \), we have either \( x \in \mathbb{Q} \) or \( x \in \mathbb{R} \setminus \mathbb{Q} \) (by denseness of \( \mathbb{Q} \), we can always find a rational, or an irrational number in any interval of \( \mathbb{R} \)). Assume that \( x \) is a rational number. In this case, the following statement:

\[
|x - 0| = |x| < \delta \rightarrow |f(x) - f(0)| = 0 < \epsilon,
\]

is obviously always true for any \( \epsilon \). Assume now that \( x \) is irrational. Then the following statement:

\[
|x - 0| = |x| < \delta \rightarrow |f(x) - f(0)| = x^2 < \epsilon,
\]

is true, if we simply set \( \delta = \sqrt{\epsilon} \). So for any \( \epsilon > 0 \), there exists \( \delta > 0 \) (= \( \sqrt{\epsilon} \)) such that:

\[
|x - 0| < \delta \rightarrow |f(x) - f(0)| < \epsilon,
\]

therefore, \( f(x) \) is continuous at \( 0 \).
(b) Let \( x_0 \neq 0 \), and assume that \( f(x) \) is continuous at \( x_0 \). Then, for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[
|x - x_0| < \delta \quad \rightarrow \quad |f(x) - f(x_0)| < \epsilon.
\]

Let \( x_0 \) be an irrational number. Then \( f(x_0) = 0 \). By denseness of \( \mathbb{Q} \) in \( \mathbb{R} \), we know that, for any \( \delta > 0 \), there exist at least one rational number \( x \) such that \( |x - x_0| < \delta \). Then \( f(x) = x^2 \). We then get

\[
|x - x_0| < \delta \quad \rightarrow \quad |f(x) - f(x_0)| = x^2 < \epsilon.
\]

By denseness of \( \mathbb{Q} \) in \( \mathbb{R} \), we can pick \( x \) rational, as close as we want to \( x_0 \). Since \( x_0 \neq 0 \), we can have \( |x| > 0 \). Then, there must exist some \( \epsilon > 0 \) such that \( |x| > \epsilon \). Then the definition is not true for any \( \epsilon > 0 \) and \( f(x) \) can not be continuous (if \( x_0 \) is a rational number, the proof is exactly the same, except that \( x \) will be picked as an irrational number).

(c) Since \( f(x) \) is not continuous at \( x \neq 0 \), it can not be differentiable. At \( x = 0 \), we have

\[
\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}.
\]

Either \( h \) is a non-rational number. In this case \( \frac{f(h)}{h} = 0 \). Or \( h \) is a rational number. In this case \( \frac{f(h)}{h} = h \), which converges to 0 as \( h \to 0 \). In both cases, the limit is 0. We then have \( f'(0) = 0 \).

**Extra Credits:**

1. **(5 points)** Suppose that \( f(x) \) is differentiable at \( x = a \). Prove that

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a - h)}{2h}.
\]

**Proof:** Let us write:

\[
\frac{f(a + h) - f(a - h)}{2h} = \frac{f(a + h) - f(a)}{2h} + \frac{f(a) - f(a - h)}{2h}.
\]

If we take now the limit \( h \to 0 \), we get:

\[
\lim_{h \to 0} \frac{f(a + h) - f(a - h)}{2h} = \lim_{h \to 0} \left( \frac{f(a + h) - f(a)}{2h} + \frac{f(a) - f(a - h)}{2h} \right).
\]

We know that, by definition:

\[
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a).
\]

Which gives: \( \lim_{h \to 0} \frac{f(a + h) - f(a)}{2h} = \frac{f'(a)}{2} \). Let us consider now the other term:

\[
\lim_{h \to 0} \frac{f(a) - f(a - h)}{h} = \lim_{h \to 0} \frac{f(a) - f(a)}{(-h)}.
\]
We need to remember the meaning of the limit \( \lim_{h \to 0} \frac{f(a-h) - f(a)}{(-h)} \). It means we are considering the limit \( \lim_{n \to \infty} \frac{f(a-h_n) - f(a)}{(-h_n)} \) of the possible sequences \( h_n \) in the set \( S = \{ h_n, \ n \in \mathbb{N}, \lim_{n \to \infty} h_n = 0 \} \). This implies that, for each sequence \( (h_n) \in S \), there is also a sequence \( (-h_n) \in S \), since \( \lim(-h_n) = 0 \) (in other words, for every sequences converging to 0, the set \( S \) includes also the same sequence with the opposite sign). There is then no difference between \( \lim_{h \to 0} \) and \( \lim_{-h \to 0} \) since the set of sequences we are considering are exactly the same. Setting \( \delta = -h \), we get:

\[
\lim_{h \to 0} \frac{f(a-h) - f(a)}{(-h)} = \lim_{-h \to 0} \frac{f(a-h) - f(a)}{(-h)} = \lim_{\delta \to 0} \frac{f(a+\delta) - f(a)}{\delta} = f'(a).
\]

Which gives: \( \lim_{h \to 0} \frac{f(a) - f(a-h)}{2h} = \frac{f'(a)}{2} \). We finally get:

\[
\lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h} = \lim_{h \to 0} \left( \frac{f(a+h) - f(a)}{2h} + \frac{f(a) - f(a-h)}{2h} \right) = \frac{f'(a)}{2} + \frac{f'(a)}{2} = f'(a).
\]

2. **(5 points)** Suppose that \( f(x) \) is differentiable on \( \mathbb{R} \) and that \( 1 \leq f'(x) \leq 2 \) for any \( x \in \mathbb{R} \) and \( f(0) = 0 \). Prove that \( x \leq f(x) \leq 2x \) for all \( x \geq 0 \). (You can not integrate the function!!).

**Proof:** Let us pick a number \( x \in \mathbb{R} \). According to the Mean Value Theorem, there exists a number \( y \in (0, x) \) such that:

\[
\frac{f(x) - f(0)}{x - 0} = f'(y).
\]

Since \( f(0) = 0 \), we get:

\[
\frac{f(x)}{x} = f'(y).
\]

We know that \( 1 \leq f'(y) \leq 2 \) for any \( y \in \mathbb{R} \). This leads to:

\[
1 \leq \frac{f(x)}{x} \leq 2.
\]

The number \( x \) being strictly positive, we get then: \( x \leq f(x) \leq 2x \).