1. (5 points) Find the Taylor series for \( f(x) = \cos(x) \) about \( x = 0 \) and show that \( \cos(x) \) equals its Taylor series for any \( x \in \mathbb{R} \).

2. (5 points each) Find the Taylor series for \( f(x) = \cosh(x) = \frac{1}{2}(e^x + e^{-x}) \) and \( g(x) = \frac{1}{2}(e^x - e^{-x}) \) about \( x = 0 \).

3. (5 points) Find the Taylor series for \( f(x) = \frac{x^2}{1+x} \) about \( x = 0 \). On what interval does \( f(x) \) equal its Taylor series?
   Hint: Compute \( f'(x) \), \( f''(x) \) and \( f^{(3)}(x) \) and then find \( f^{(n)}(x) \) for \( n > 3 \) by induction.

4. In the following exercise, we will define a compactly supported function \( f(x) \) as being a function that is non zero (that is \( |f(x)| > 0 \)) on some interval \( x \in (a, b) \) and zero outside this interval. Let \( C^\infty_0 \) be the set of infinitely differentiable on \( \mathbb{R} \), compactly supported functions.
   The set \( C^\infty_0 \) is of great importance in mathematics. We want to prove that there are infinitely many functions that belong to \( C^\infty_0 \). We will consider first the function:
   \[
   f(x) = \begin{cases} 
   0, & \text{if } x \leq 0 \\
   e^{-\frac{1}{x}}, & \text{if } x > 0 
   \end{cases}
   \]
   (a) (1 points) Sketch a graph of \( f(x) \).
   (b) (4 points) Show that \( f(x) \) is infinitely differentiable on \( \mathbb{R} \).
      Hint: you can use \( \lim_{x \to 0} x^{-k}e^{-\frac{1}{x}} = 0 \) for any \( k \in \mathbb{Z} \).
   (c) (1 points) Let \( g(x) \) be an infinitely differentiable function on \( \mathbb{R} \). Show that \( f(x)g(x) \) is also infinitely differentiable \( \mathbb{R} \).
   (d) (4 points) Using the above results, construct a function that belongs to \( C^\infty_0 \) and show that there exists infinitely many of them.
      Hint: use functions of the form \( f(x - a) \) and \( f(-x + b) \).