1. (a) **Proof:** Let us write the upper and lower Darboux sums for $f(x) = 1$ on $[0, 1]$:

$$U(f, P_n) = \sum_{k=0}^{n-1} M(f, [t_k, t_{k+1}]) (t_{k+1} - t_k),$$

$$L(f, P_n) = \sum_{k=0}^{n-1} m(f, [t_k, t_{k+1}]) (t_{k+1} - t_k).$$

Since $f(x) = 1$ for any $x \in [0, 1]$, we have $M(f, [t_k, t_{k+1}]) = m(f, [t_k, t_{k+1}]) = 1$ for any $k$. Moreover, $\sum_{k=0}^{n-1} (t_{k+1} - t_k) = t_n - t_0 = 1$. We get then:

$$U(f, P_n) = 1,$$

$$L(f, P_n) = 1.$$

The 2 sums are constant. Therefore $\lim U(f, P_n) = \lim L(f, P_n) = 1$. This shows that $f$ is integrable on $[0, 1]$ and $\int_0^1 f(x) \, dx = 1$.

(b) **Proof:** Let us write the upper and lower Darboux sums for $f(x) = x^2 - 1$ on $[0, 1]$:

$$U(f, P_n) = \sum_{k=0}^{n-1} M(f, [t_k, t_{k+1}]) (t_{k+1} - t_k),$$

$$L(f, P_n) = \sum_{k=0}^{n-1} m(f, [t_k, t_{k+1}]) (t_{k+1} - t_k).$$

Since $f(x)$ is increasing on $[0, 1]$, we get $M(f, [t_k, t_{k+1}]) = f(t_{k+1}) = t_{k+1}^2 - 1 = \left(\frac{k+1}{n}\right)^2 + 1$ and $m(f, [t_k, t_{k+1}]) = f(t_k) = t_k^2 - 1 = \left(\frac{k}{n}\right)^2 + 1$. Now, using the fact that $t_{k+1} - t_k = \frac{1}{n}$ and the relation $\sum_{k=0}^{n-1} k^2 = \frac{n(n+1)(2n+1)}{6}$, we get:

$$U(f, P_n) = \frac{n-1}{n} M(f, [t_k, t_{k+1}]) (t_{k+1} - t_k) = \frac{n(n+1)(2n+1)}{6n^2} - 1,$$

$$L(f, P_n) = \frac{n-1}{n} m(f, [t_k, t_{k+1}]) (t_{k+1} - t_k) = \frac{n(n-1)(2n-1)}{6n^2} - 1.$$ 

We could compute compute now $\inf(U(f, P_n))$ and $\sup(L(f, P_n))$ (keep in mind that it should the inf and sup over all the possible partition of $[0, 1]$). However, since we restricted ourselves to a particular set of partitions depending only on $n$, the infimum and supremum would be computed with respect to $n$). It is however simpler to just take the limit $n \to \infty$. This gives:

$$\lim_{n \to \infty} U(f, P_n) = \frac{1}{3} - 1 = -\frac{2}{3},$$

$$\lim_{n \to \infty} L(f, P_n) = \frac{1}{3} - 1 = -\frac{2}{3}.$$
This gives \( \lim_{n \to \infty} U(f, P_n) = \lim_{n \to \infty} L(f, P_n) \). The function is then integrable and \( \int_0^1 f(x) dx = -\frac{2}{3} \).

(c) **Proof:** Let us write the upper and lower Darboux sums for \( f(x) = e^x \) on \([0, 1]\):

\[
U(f, P_n) = \sum_{k=0}^{n-1} M(f, [t_k, t_{k+1}]) (t_{k+1} - t_k),
\]

\[
L(f, P_n) = \sum_{k=0}^{n-1} m(f, [t_k, t_{k+1}]) (t_{k+1} - t_k).
\]

Since \( f(x) \) is increasing on \([0, 1]\), we get \( M(f, [t_k, t_{k+1}]) = f(t_{k+1}) = e^{t_{k+1}} = e^{\frac{k+1}{n}} \) and \( m(f, [t_k, t_{k+1}]) = f(t_k) = e^{t_k} = e^{\frac{k}{n}} \). Now, using the fact that \( t_{k+1} - t_k = \frac{1}{n} \) and the relation \( \sum_{k=0}^{n-1} e^{\frac{k}{n}} = \frac{1-e^{1/n}}{1-e^{1}} \) (the sum is a geometric sum), we get:

\[
U(f, P_n) = \sum_{k=0}^{n-1} M(f, [t_k, t_{k+1}]) (t_{k+1} - t_k) = \frac{e^{1/n}}{n} \left( \frac{1-e^{1/n}}{1-e^{1}} \right),
\]

\[
L(f, P_n) = \sum_{k=0}^{n-1} m(f, [t_k, t_{k+1}]) (t_{k+1} - t_k) = \frac{1}{n} \left( \frac{1-e^{1/n}}{1-e^{1}} \right).
\]

Using L'Hospital rule, we can prove that \( \lim n(1-e^{1/n}) = -1 \). We then get \( \lim U(f, P_n) = \lim L(f, P_n) = e^1 - 1 \). Therefore, \( f \) is integrable on \([0, 1]\) and \( \int_0^1 e^x dx = e^1 - 1 \).

2. Let us compute first the lower Darboux sum:

\[
L(f, P_n) = \sum_{k=0}^{n-1} m(f, [t_k, t_{k+1}]) (t_{k+1} - t_k) = 0.
\]

By denseness of the irrational numbers in \( \mathbb{R} \), we can always find an irrational number in any subinterval of \([0, 1]\), and since \( x \geq 0 \) on \([0, 1]\), we see that \( m(f, [t_k, t_{k+1}]) = 0 \) for any partition. And therefore \( L(f) = \sup L(f, P) = 0 \).

The upper Darboux sum is:

\[
U(f, P_n) = \sum_{k=0}^{n-1} M(f, [t_k, t_{k+1}]) (t_{k+1} - t_k).
\]

Regarding the expression of \( f(x) \), we can see that the maximum of \( f \) on a subinterval \([t_k, t_{k+1}]\) will be given by \( f(x_k) = x_k \), \( x_k \) being the closest rational number to \( t_{k+1} \). Either \( t_{k+1} \) is a rational and \( x_k = t_{k+1} \) or \( t_{k+1} \) is irrational and \( x_k \) can actually be as close as we want to \( t_{k+1} \) (by denseness of \( \mathbb{Q} \) in \( \mathbb{R} \)). This implies that \( f(x_k) \geq \frac{t_{k+1}}{2} \) (the densess actually implies that \( f(x_k) \geq (1 - \epsilon) t_{k+1}, \) for any \( \epsilon > 0 \). We decide to set \( \epsilon = 1/2 \). This gives:

\[
U(f, P_n) = \sum_{k=0}^{n-1} M(f, [t_k, t_{k+1}]) (t_{k+1} - t_k) \geq \frac{1}{2} \sum_{k=0}^{n-1} t_{k+1}(t_{k+1} - t_k).
\]

The Darboux sum on the right-hand side is actually \( U(x, P) \). We know that \( x \) is integrable on \([0, 1]\) and this gives \( \inf U(x, P) = \int_0^1 x dx = \frac{1}{2} \). Then \( U(f) = \inf U(f, P) \geq \frac{1}{2} \inf U(x, P) = \frac{1}{4} \).

Therefore we have \( U(f) \neq L(f) \) and the function is not integrable.
3. **Proof:** Let \((P_n)\) be a sequence of partitions of \([a, b]\) such that \(\lim U(f, P_n) = \lim L(f, P_n) = \int_a^b f(x)dx\) (such a partition exists since \(f\) is integrable on \([a, b]\)).

Using the following relations, for any interval \(I \subset \mathbb{R}\)

\[
M(\alpha f, I) = \begin{cases} 
\alpha M(f, I), & \text{if } \alpha > 0 \\
\alpha m(f, I), & \text{if } \alpha < 0
\end{cases}
\]

and

\[
m(\alpha f, I) = \begin{cases} 
\alpha m(f, I), & \text{if } \alpha > 0 \\
\alpha M(f, I), & \text{if } \alpha < 0
\end{cases}
\]

This implies that

\[
\lim U(\alpha f, P_n) = \begin{cases} 
\alpha \lim U(f, P_n), & \text{if } \alpha > 0 \\
\alpha \lim L(f, P_n), & \text{if } \alpha < 0
\end{cases}
\]

since \(\lim U(f, P_n) = \lim L(f, P_n) = \int_a^b f(x)dx\), both cases give \(\lim U(\alpha f, P_n) = \alpha \int_a^b f(x)dx\).

The same computation for \(L(\alpha f, P_n)\) would give the same result: \(\lim L(\alpha f, P_n) = \alpha \int_a^b f(x)dx\).

This implies that \(\lim U(\alpha f, P_n) = \lim L(\alpha f, P_n)\). Therefore \(\alpha f\) is integrable on \([a, b]\) and \(\int_a^b \alpha f(x)dx = \lim U(\alpha f, P_n) = \alpha \int_a^b f(x)dx\).

**Extra Credits:**

1. **Proof:** Let us start with the assumption that \(f(x) = g(x)\) except at one point in \([a, b]\). Therefore the difference of the sums \(U(f, P) - L(f, P)\) and \(U(g, P) - L(g, P)\) might differ only at one interval. Let \(\text{mesh}(P) < \delta\) and assume that \(|f(x) - g(x)| = B\) at that point of difference. Therefore

\[
U(g, P) - L(g, P) < U(f, P) - L(f, P) + B\delta.
\]

Since \(f\) is integrable: for any \(\epsilon > 0\), there exists \(\delta' > 0\) such that

\[
\text{mesh}(P) < \delta' \rightarrow U(f, P) - L(f, P) < \frac{\epsilon}{2}.
\]

We can now select a \(\delta\) small enough such that \(\delta < \delta'\) and \(B\delta < \epsilon/2\). Therefore, we would get

\[
U(g, P) - L(g, P) < \epsilon.
\]

This implies that \(g\) is integrable on \([a, b]\). And

\[
\int_a^b g(x)dx = U(g) \leq U(g, P) \leq U(f, P) + \epsilon/2 < L(f, P) + \epsilon \leq L(f) + \epsilon = \int_a^b f(x)dx + \epsilon.
\]

In the same way

\[
\int_a^b g(x)dx = L(g) \geq L(g, P) \geq L(f, P) - \epsilon/2 > U(f, P) - \epsilon \leq U(f) - \epsilon = \int_a^b f(x)dx - \epsilon.
\]

This is true for any \(\epsilon > 0\). Therefore \(\int_a^b g(x)dx = \int_a^b f(x)dx\).

To prove for any function \(g\) that differs from \(f\) by finitely many points, we can then use an induction argument.
2. Let $P$ be a partition of $[a, b]$ such that $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ and assume there exists $0 < q < m < n$ such that $t_q = c$ and $t_m = d$. Let us call $Q$ the partition $Q = \{c = t_q < t_{q+1} < \ldots < t_m = d\}$.
Let us assume that $f$ is integrable on $[a, b]$ but not on $[c, d]$. Then

$$U(f, P) - L(f, P) \geq U(f, Q) - L(f, Q).$$

Since $f$ is not integrable on $[c, d]$, there exists some $\epsilon > 0$, such that $U(f, Q) - L(f, Q) > \epsilon$ for any partition $Q$ of $[c, d]$. But this would imply that $U(f, P) - L(f, P) \geq \epsilon$ and therefore is not integrable. Then $f$ must be integrable on $[c, d]$.

**Remark:** Actually, the proof is not completely rigorous since we imposed a constraint on the partition $P$ ($P$ contains two point $t_q = c$ and $t_m = d$). It actually does not matter, since we can set $\text{mesh}(P)$ to be be small enough such that we can get two points as close as we want to $c$ and $d$. And therefore the argument still holds.