140B - First Midterm
Solutions

1 Exercise 1 (30 points)
Find the radius of convergence and interval of convergence of the power series:

\[ \sum \frac{\sqrt{n}}{3^n} x^n. \]

Solution:
Let us compute:

\[ \alpha = \lim \sup \left| \frac{\sqrt{n}}{3^n} \right|^{1/n} = \lim \frac{n^{1/(2n)}}{3} = \frac{1}{3}. \]

The radius of convergence is then \( R = \frac{1}{\alpha} = 3 \). At the boundaries \( x = \pm 3 \), we can see that the series can not be convergent since we would have either \( \sum \sqrt{n} \) or \( \sum (-1)^n \sqrt{n} \) and \( \lim \sqrt{n} = \infty \) (also \( \lim (-1)^n \sqrt{n} \) does not exist). Since the limit of the terms of the series is not zero, the series can not be convergent. The interval of convergence is then \((-3, 3)\).
2 Exercise 2 (30 points)

Find the pointwise limit of the sequence of functions:

\[ f_n(x) = \frac{\cos(nx)}{n^2}, \quad \text{on } x \in \mathbb{R} \]

Give a rigorous proof.

**Solution:** Since \( \cos(nx) \) is bounded by 1 on \( \mathbb{R} \), we should expect the sequence to go to 0 for any \( x \) as \( n \) goes to \( \infty \). Let us prove this rigorously:

\[ |f_n(x) - 0| = \left| \frac{\cos(nx)}{n^2} \right| \leq \frac{1}{n^2}. \]

We can see that \( \frac{1}{n^2} < \epsilon \) for any \( \epsilon > 0 \) if \( n > \frac{1}{\sqrt{\epsilon}} = N \). Therefore the series is pointwise convergent to 0 on \( \mathbb{R} \).
3 Exercise 3 (20 points each)

Determine whether the following series are convergent:

1) \( \sum \frac{n^2}{n!} \)

2) \( \sum \frac{1}{2^n + n} \).

Hint: use the comparison test for the second.

Solutions:

1. Using the ratio test with the term \( a_n = \frac{n^2}{n!} \), we see that

\[
\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left( \frac{n+1}{n} \right)^2 \frac{1}{n} = 0.
\]

Then \( \lim \left| \frac{a_{n+1}}{a_n} \right| < 1 \). The series is convergent.

2. Since \( n \geq 1 \) we have the upper bound \( \frac{1}{2^n + n} \leq \frac{1}{2^n} \). We know that the series \( \sum \frac{1}{2^n} \) is convergent (it actually gives \( \sum \frac{1}{2^n} = 2 \)). By comparison test, the series \( \sum \frac{1}{2^n + n} \) is also convergent.
4 EXTRA CREDITS: Exercise 5 (10 points)

Let \((a_n)\) be a sequence of non-zero real numbers such that the sequence \(\left(\frac{a_{n+1}}{a_n}\right)\) is a constant sequence: \(\frac{a_{n+1}}{a_n} = q\), for any \(n \in \mathbb{N}\). Show that \(\sum a_n\) is a geometric series: \(\sum a_k = \sum q^k\).

Solution:
The relation \(\frac{a_{n+1}}{a_n} = q\) can be written as \(a_{n+1} = qa_n\). This gives the following relations:

\[
\begin{align*}
a_1 &= qa_0, \\
a_2 &= qa_1 = q(qa_0) = q^2a_0, \\
a_3 &= qa_2 = q(q^2a_0) = q^3a_0...
\end{align*}
\]

We can see that this implies \(a_n = q^na_0\) (we can show this by induction very easily). Therefore the sum becomes:

\[
\sum a_n = \sum q^na_0 = a_0 \sum q^n,
\]

which is a geometric sum (up to the term \(a_0\) which is just a constant and can be set to be 1).