Homework 1
Three-dimensional coordinate systems.

September 28, 2004

1 Exercises

1. Sketch a graph of the plane defined by the equation \( y = 2 \).

2. Sketch a graph of the plane defined by the equation \( 2x + y = 2 \).

3. Find the equation of the sphere that has the center \( C(3, 8, 1) \) and passes through the point \( P(4, 3, -1) \).

4. What region in the three-dimensional sphere is defined by the equation \( 1 \leq x^2 + y^2 + z^2 \leq 4 \).

5. Find the equation of the sphere that has the center \( C(1, -4, 3) \) and the radius 5. Describe the intersection of the sphere with the z-axis, with the y-axis and with the xy-plane.

2 Solutions

1. The plane defined by the equation \( y = 2 \) corresponds to all the points \( P(x, y, z) \) such that \( y = 2 \). The 2 other coordinates \( x \) and \( z \) can take every possible value. This corresponds to a plane that is parallel to the \( xz \)-plane and passes through the point \( y = 2 \) on the y-axis.
2. Let us look first at the equation $2x + y = 2$ in the $xy$-plane. It corresponds to the line $y = -2x + 2$:

The equation does not impose a value for $z$, so, in the 3-dimensional space, we can translate every point of the line $y = -2x + 2$ along the $z$-axis. We then get the plane:

3. The equation of sphere is of the form:

$$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = r^2,$$

with $(x_c, y_c, z_c)$ the coordinates of the center and $r$ the radius. The coordinates of the center $C$ are given: $(3, 8, 1)$. We can rewrite then the above equation:

$$(x - 3)^2 + (y - 8)^2 + (z - 1)^2 = r^2.$$

We need now to find the radius. We know the coordinates of a point $P(4, 3, 1)$ belonging to the sphere. The coordinates of this point are, by definition, a solution of the equation of the sphere. We can then replace $x, y$ and $z$ by $(4, 3, -1)$ in the equation of the sphere and we get

$$(4 - 3)^2 + (3 - 8)^2 + (-1 - 1)^2 = r^2.$$
This relation gives us the radius \( r^2 = 1 + 25 + 4 = 30 \), that is \( r = \sqrt{30} \). We now have the equation for the sphere that has the center \( C(3, 8, 1) \) and passes through \( P(4, 3, -1) \):

\[
(x - 3)^2 + (y - 8)^2 + (z - 1)^2 = 30.
\]

4. The equation \( x^2 + y^2 + z^2 = r^2 \) corresponds to a sphere centered at the origin \( O(0, 0, 0) \), with a radius \( r \). The double inequality \( 1 \leq x^2 + y^2 + z^2 \leq 4 \) means that we are considering all the points belonging to the spheres centered at the origin and whose radius can take any value between 1 and 2. In other words, the double inequality corresponds to all the points between the sphere of radius 1 centered at the origin and the sphere of radius 2 centered at the origin:

![Diagram](image.png)

5. The equation of a sphere with center \( C(1, -4, 3) \) and radius 5 is by definition

\[
(x - 1)^2 + (y + 4)^2 + (z - 3)^2 = 25.
\]

The sphere intersects the z-axis when \( x = 0 \) and \( y = 0 \) (the z-axis corresponds to all the points such that \( x \) and \( y \) are zero). If we replace then \( x \) and \( y \) by zero in the equation of the sphere, we get

\[
1 + 16 + (z - 3)^2 = 25 \Rightarrow (z - 3)^2 = 8.
\]

The solutions of the equation \((z - 3)^2 = 8\) are \( z = 3 + \sqrt{8} \) and \( z = 3 - \sqrt{8} \) (it means the sphere intersects the z-axis at this 2 points).

The sphere intersects the y-axis when \( x = 0 \) and \( z = 0 \) (the y-axis corresponds to all the points such that \( x \) and \( z \) are zero). If we replace then \( x \) and \( z \) by zero in the equation of the sphere, we get

\[
1 + (y + 4)^2 + 9 = 25 \Rightarrow (y + 4)^2 = 15.
\]

The solutions of the equation \((y + 4)^2 = 15\) are \( y = -4 + \sqrt{15} \) and \( y = -4 - \sqrt{15} \) (it means the sphere intersects the y-axis at this 2 points).
The xy-plane corresponds to all the points in the 3-dimensional space such that \( z = 0 \). If we replace \( z \) by zero in the equation of the sphere, we get

\[
(x - 1)^2 + (y + 4)^2 + 9 = 25 \Rightarrow (x - 1)^2 + (y + 4)^2 = 16.
\]

The intersection of the sphere with the xy-plane corresponds to all the points solution of the equation \((x - 1)^2 + (y + 4)^2 = 16\), that is: a circle of center \((1, -4)\) and of radius 4 (the equation of a circle in a plane is \((x - x_c)^2 + (y - y_c)^2 = r^2\) with \((x_c, y_c)\) the coordinates of the center and \(r\) the radius).