Homework 7
Double integrals.

November 15, 2004

1 Exercises

1.1 Double integrals over rectangles

1. Compute the integrals of the following functions over the rectangle \( R = [-1, 2] \times [0, 4] \).

\[
\begin{align*}
    f(x, y) &= xe^y, \\
    g(x, y) &= \frac{x + 2}{y + 1} + \frac{y + 1}{x + 2}
\end{align*}
\]

2. Compute the integral of the following functions over the rectangle \( R = [1, 2] \times [0, 5] \).

\[
\begin{align*}
    f(x, y) &= \frac{1}{(x + y)^2}, \\
    g(x, y) &= \cos(x + 5y).
\end{align*}
\]

3. Find the volume of the solid that lies under the plane of equation \( 3x + y - z = 0 \) and above the rectangle \( D = [1, 4] \times [0, 5] \).

4. Find the volume of the solid that lies under the ellipsoid of equation \( z = x^2 + 3y^2 \) and above the rectangle \( D = [-1, 1] \times [-1, 1] \)

1.2 Double integrals over general regions

1. Compute the integrals of the following functions over the domain \( D = \{(x, y)| 0 \leq y \leq 4, \ 0 \leq x \leq \sqrt{y}\} \):

\[
\begin{align*}
    f(x, y) &= e^{\sqrt{y}}, \\
    g(x, y) &= x\sqrt{y}.
\end{align*}
\]

2. Compute the integral of the following function over the domain \( D = \{(x, y)| 1 \leq x \leq 2, \ 0 \leq y \leq 2x\} \):

\[
    f(x, y) = \frac{4y}{x^3 + 2}
\]
3. Let $D$ be a trapezoid:

and let $f(x, y)$ be a function such that $f(x, y) = 5$ on the domain $D$. What is the value of the integral? :

$$\int \int_D f(x, y) \, dx \, dy.$$

4. Let $f(x, y)$ be the function:

$$f(x, y) = (y - \frac{1}{3})(x^2 - \frac{1}{4}).$$

and let $D$ be the domain $D = \{(x, y) \mid 0 \leq x \leq 1, \ 0 \leq y \leq x\}$. What is the value of the integral of $f(x, y)$ over this domain?

In the previous homework, we saw that the absolute minimum of $f(x, y)$ on $D$ is $f(1, 0) = -\frac{1}{4}$ and the absolute maximum is $f(1, 1) = \frac{1}{2}$. Can you use this 2 values to bound the integral of $f(x, y)$ over $D$?

2 Solutions

2.1 Double integrals over rectangles

1. The integral of $f(x, y)$ over $R$ gives:

$$\int \int_R f(x, y) \, dx \, dy = \int_{-1}^{2} \int_{0}^{4} x e^y \, dy \, dx.$$

The function $f(x, y)$ is a product of a function of $x$ and a function of $y$. We can then rewrite the double integral as the product of integrals:

$$\int \int_R f(x, y) \, dx \, dy = \int_{-1}^{2} x \, dx \int_{0}^{4} e^y \, dy = \left( \frac{x^2}{2} \right) \bigg|_{-1}^{2} \left( e^y \right) \bigg|_{0}^{4},$$

which is equal to:

$$\int \int_R f(x, y) \, dx \, dy = \frac{3}{2} (e^4 - 1) \simeq 80.4.$$

The integral of $g(x, y)$ over $R$ gives:

$$\int \int_R g(x, y) \, dx \, dy = \int_{-1}^{2} \int_{0}^{4} \left( \frac{x + 2}{y + 1} + \frac{y + 1}{x + 2} \right) \, dy \, dx.$$
The integral of the sum of 2 functions can be written as the sum of the integrals:

\[ \int \int_R g(x, y) \, dxdy = \int_{-1}^{2} \int_{0}^{4} \frac{x + 2}{y + 1} \, dydx + \int_{-1}^{2} \int_{0}^{4} \frac{y + 1}{x + 2} \, dydx. \]

In each integral, we have the product of a function of \( x \) and a function of \( y \). We can then rewrite the double integrals as:

\[ \int \int_R g(x, y) \, dxdy = \int_{-1}^{2} (x + 2) \, dx \int_{0}^{4} \frac{1}{y + 1} \, dy + \int_{0}^{4} (y + 1) \, dy \int_{-1}^{2} \frac{1}{x + 2} \, dy, \]

which gives:

\[ \int \int_R g(x, y) \, dxdy = \left( \frac{x^2}{2} + 2x \right) \left( \ln(y + 1) \right)_{0}^{4} + \left( \frac{y^2}{2} + y \right) \left( \ln(x + 2) \right)_{-1}^{2}, \]

\[ = \frac{15}{2} \ln(5) + 12 \ln(4) \approx 28.71 \]

2. The integral of \( f(x, y) \) over \( R \) gives:

\[ \int \int_R f(x, y) \, dxdy = \int_{-1}^{2} \int_{0}^{5} \frac{1}{(x + y)^2} \, dydx. \]

The anti-derivative of \( \frac{1}{(x+y)^2} \) with respect to \( y \) is \( -\frac{1}{x+y} \). The integral over \( y \) gives then:

\[ \int \int_R f(x, y) \, dxdy = \int_{1}^{2} \left( -\frac{1}{(x + y)^0} \right) \, dx = \int_{1}^{2} \left( -\frac{1}{x + 5} + \frac{1}{x} \right) \, dx. \]

The antiderivative of \( \frac{1}{(x+a)} \) (\( a \) being a constant) is \( \ln(x + a) \). This leads to:

\[ \int \int_R f(x, y) \, dxdy = -\ln(x+5) + \ln(x) \bigg|_{1}^{5} = (-\ln(7) + \ln(2)) - (-\ln(6) + \ln(1)) = \ln \left( \frac{12}{7} \right) \approx .54 \]

The integral of \( f(x, y) \) over \( R \) gives:

\[ \int \int_R f(x, y) \, dxdy = \int_{1}^{2} \int_{0}^{5} \cos(x + 5y) \, dydx. \]

The anti-derivative of \( \cos(x + 5y) \) with respect to \( y \) is \( \frac{1}{5} \sin(x + 5y) \). The integral over \( y \) gives then:

\[ \int \int_R f(x, y) \, dxdy = \int_{1}^{2} \left( \frac{1}{5} \sin(x + 5y) \right)_{0}^{5} \, dx = \int_{1}^{2} \left( \frac{1}{5} \sin(x + 25) - \frac{1}{5} \sin(x) \right) \, dx. \]

The antiderivative of \( \sin(x + a) \) (\( a \) being a constant) is \( -\cos(x + a) \). This leads to:

\[ \int \int_R f(x, y) \, dxdy = -\frac{1}{5} \cos(x+25) + \frac{1}{5} \cos(x) \bigg|_{1}^{5} = \frac{1}{5} \left( \cos(26) + \cos(2) - \cos(27) - \cos(1) \right) \approx -.003 \]

3. The equation of the plane can be written as \( z = 3x + y \). We see that it corresponds actually to the graph of the function \( f(x, y) = 3x + y \). We can also check that the function \( f(x, y) \) is positive on \( D = [1, 4] \times [0, 5] \) (that is, for any \( 1 \leq x \leq 4 \) and \( 0 \leq y \leq 5 \)). Then,
by definition, the volume of the solid that lies between the graph of the function and the rectangle in the $xy$-plane is given by the double integral:

$$
\int \int_D f(x, y) dxdy = \int_1^4 \int_0^5 (3x + y) dydx.
$$

The anti-derivative of $3x + y$ with respect to $y$ is $3xy + \frac{y^2}{2}$. This gives:

$$
\int \int_D f(x, y) dxdy = \int_1^4 \int_0^5 \left(3xy + \frac{y^2}{2}\right) dy = \int_1^4 (15x + \frac{25}{2}) dx,
$$

which leads to:

$$
\int \int_D f(x, y) dxdy = \int_1^4 (15x + \frac{25}{2}) dx = \frac{15x^2}{2} + \frac{25x}{2} \bigg|_1^4 = 150.
$$

4. The equation of the ellipsoid is $z = x^2 + 3y^2$. It corresponds actually to the graph of the function $f(x, y) = x^2 + 3y^2$. We can also check that the function $f(x, y)$ is positive on $D = [-1, 1] \times [-1, 1]$. Then, by definition, the volume of the solid that lies between the graph of the function and the rectangle in the $xy$-plane is given by the double integral:

$$
\int \int_D f(x, y) dxdy = \int_{-1}^1 \int_{-1}^1 (x^2 + 3y^2) dydx.
$$

The anti-derivative of $x^2 + 3y^2$ with respect to $y$ is $x^2y + \frac{3y^3}{3}$. This gives:

$$
\int \int_D f(x, y) dxdy = \int_{-1}^1 \left(x^2y + \frac{3y^3}{3}\right) dy = \int_{-1}^1 (2x^2 + 2) dx,
$$

which leads to:

$$
\int \int_D f(x, y) dxdy = \int_{-1}^1 (2x^2 + 2) dx = \frac{2x^3}{3} + 2x \bigg|_{-1}^1 = \frac{16}{3} \approx 5.33
$$

### 2.2 Double integrals over general regions

1. The integral of $f(x, y)$ over $D$ gives:

$$
\int \int_D f(x, y) dxdy = \int_0^4 \int_0^{\sqrt[4]{y}} e^{\frac{x}{\sqrt[4]{y}}} dydx.
$$

The domain is not a rectangle. It is a domain of type II. We have to compute the integral over $x$ first. The anti-derivative of $e^{\frac{x}{\sqrt[4]{y}}}$ with respect to $x$ is $\sqrt[4]{y}e^{\frac{x}{\sqrt[4]{y}}}$ (keep in mind that $y$ is considered as a constant). This gives then:

$$
\int \int_D f(x, y) dxdy = \int_0^4 \left(\sqrt[4]{y}e^{\frac{x}{\sqrt[4]{y}}}\right) \bigg|_0^{\sqrt[4]{y}} dy = \int_0^4 \sqrt[4]{y}(e^1 - 1) dy.
$$

The integral over $y$ gives us:

$$
\int \int_D f(x, y) dxdy = \int_0^4 \sqrt[4]{y}(e^1 - 1) dy = \frac{2}{3} y^{3/2}(e^1 - 1) \bigg|_0^4 = \frac{2}{3} 4^{3/2}(e^1 - 1) \approx 9.16
$$

The integral of $g(x, y)$ over $D$ gives:

$$
\int \int_D g(x, y) dxdy = \int_0^\sqrt[4]{y} x \sqrt[4]{y} dx dy.
$$
Here again, the domain is a domain of type II. We have to compute the integral over $x$ first. The anti-derivative of $x\sqrt{y}$ with respect to $x$ is $\frac{x^2}{2}\sqrt{y}$. This gives then:

$$\int \int_D g(x, y) \, dx \, dy = \int_0^4 \left( \frac{x^2}{2}\sqrt{y} \right) \, dy = \int_0^4 \frac{y}{2}\sqrt{y} \, dy.$$ 

You can notice that $\frac{y}{2}\sqrt{y} = y^{3/2}$. So, the integral over $y$ gives us:

$$\int \int_D g(x, y) \, dx \, dy = \int_0^4 \frac{y^{3/2}}{2} \, dy = \frac{2}{10}y^{5/2}\Big|_0^4 = \frac{64}{10} = 6.4$$

2. The integral of $f(x, y)$ over $D$ gives:

$$\int \int_D f(x, y) \, dx \, dy = \int_1^2 \int_0^{2x} \frac{4y}{x^3 + 2} \, dy \, dx.$$ 

The domain $D$ is a domain of type I. We have to compute the integral over $y$ first. The anti-derivative of $\frac{4y}{x^3 + 2}$ with respect to $y$ is $\frac{2y^2}{x^3 + 2}$ (keep in mind that $x$ is considered as a constant). This gives then:

$$\int \int_D f(x, y) \, dx \, dy = \int_1^2 \left( \frac{2y^2}{x^3 + 2} \right) \, dy = \int_1^2 \frac{8y^2}{x^3 + 2} \, dy.$$ 

The anti-derivative of $\frac{8y^2}{x^3 + 2}$ is $\frac{8}{3}\ln(x^3 + 2)$. It gives then:

$$\int \int_D f(x, y) \, dx \, dy = \frac{8}{3}\ln(x^3 + 2)\Big|_1^2 = \frac{8}{3}\ln\left(\frac{10}{3}\right) \approx 3.21$$

3. The function is a constant on the domain $D$, that is $f(x, y) = 5$ for any point in $D$. The integral is then:

$$\int \int_D f(x, y) \, dx \, dy = \int \int_D \, 5 \, dx \, dy = 5\int \int_D \, dx \, dy = 5A(D),$$

$A(D)$ being the area of the domain $D$. According to the picture, the area of the trapezoid is $A(D) = 30$ (you can see the trapezoid as 2 right triangles associated with a rectangle). The integral is then:

$$\int \int_D f(x, y) \, dx \, dy = 5A(D) = 150$$

4. The integral of $f(x, y)$ over $D$ gives:

$$\int \int_D f(x, y) \, dx \, dy = \int_0^1 \int_0^{x^2 - \frac{1}{4}} (y - \frac{1}{3})(x^2 - \frac{1}{4}) \, dy \, dx.$$ 

The domain $D$ is a domain of type I. We have to compute the integral over $y$ first. The anti-derivative of $(y - \frac{1}{3})(x^2 - \frac{1}{4})$ with respect to $y$ is $(\frac{y^2}{2} - \frac{y}{3})(x^2 - \frac{1}{4})$. The integral is then:

$$\int \int_D f(x, y) \, dx \, dy = \int_0^1 \left( \left( \frac{y^2}{2} - \frac{y}{3} \right)(x^2 - \frac{1}{4}) \right) \, dx = \int_0^1 \left( \frac{x^2}{2} - \frac{x}{3} \right)(x^2 - \frac{1}{4}) \, dx$$

$$= \int_0^1 \left( \frac{x^4}{2} - \frac{x^2}{8} - \frac{x^3}{3} + \frac{x}{12} \right) \, dx.$$
The integral over $x$ gives:

\[
\int \int_D f(x,y)dx\,dy = \frac{x^5}{10} - \frac{x^3}{24} - \frac{x^4}{12} + \frac{x^2}{24}\bigg|_0^1 = \frac{1}{60}.
\]

We know that the function is bounded by $\frac{1}{4}$ and $\frac{1}{2}$ on $D$, that is $-\frac{1}{4} \leq f(x,y) \leq \frac{1}{2}$. This gives us the bounds for the double integral:

\[-\frac{1}{4} A(D) \leq \int \int_D f(x,y)dx\,dy \leq \frac{1}{2} A(D),\]

$A(D)$ being the area of the domain. In the case of that domain (which is a triangle), the area is $A(D) = \frac{1}{2}$. This gives us:

\[-\frac{1}{8} \leq \int \int_D f(x,y)dx\,dy \leq \frac{1}{4}.\]

Remark: the last inequality is consistent with the computation of the integral since $\int \int_D f(x,y)dx\,dy = \frac{1}{60}$ and:

\[-\frac{1}{8} \leq \frac{1}{60} \leq \frac{1}{4}.
\]