Homework 3
Equation of lines and planes - Cylindrical and spherical coordinates.

1 Exercises
1.1 Lines and planes

1. Let \( A(1, 2, -3) \) and \( B(0, 3, 3) \) be two points of a line. Let \( \vec{u} = (-1, 1, 6) \) and \( \vec{v} = (3, -3, -18) \) be two direction vectors of this line. Find the symmetric equations of the line using first \( A \) and \( \vec{u} \) as a point and direction vector, and then using \( B \) and \( \vec{v} \). Compare these two equations and show they are identical.

2. Find the symmetric equations of the line that passes through the point \((1, 0, 6)\) and that is perpendicular to the plane of equation \(x + 3y + z = 5\).

3. Find the equation of the plane that passes through the points \( A(2, -7, 10), B(1, 1, 1) \) and \( C(5, 0, 1) \).

4. Find the equation of the plane that passes through the origin and is parallel to the plane of equation \(2x - y + 3z = 1\).

5. Find the intersection of the two planes of equations \(x + y + z = -2\) and \(3x - 7z = 12\).

1.2 Cylinder and spherical coordinates

1. Let \( P \) be a point of coordinates \((2, 2, 3)\) in the \((x, y, z)\) coordinate system. What are the coordinates of \( P \) in the cylindrical coordinates system \((r, \theta, z)\) and in the spherical coordinates system \((\rho, \theta, \phi)\) ?

2. The equation of the hyperboloid of one sheet is of the kind:

\[ x^2 + y^2 = 1 + z^2. \]

Rewrite that equation using the cylindrical and spherical coordinate systems.

2 Solutions

2.1 Lines and planes

1. In general: if \( P(x_0, y_0, z_0) \) is a point of the line and \( \vec{w} = (a, b, c) \) is the direction vector, then the symmetric equations that describe the line are:

\[ \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \]
In our case: if $A(1, 2, -3)$ is taken as a point of the line and $\vec{u} = (-1, 1, 6)$ as the direction vector, we have then the following symmetric equations:

$$-x + 1 = y - 2 = \frac{z + 3}{6}.$$ 

Instead of $A$ and $\vec{u}$, we can take the point $B(0, 3, 3)$ as a point of the line and $\vec{v} = (3, -3, -18)$ as a direction vector. We get then the symmetric equations:

$$\frac{x}{3} = \frac{y - 3}{3} = \frac{z - 3}{18}.$$ 

These two equations are actually the same since they describe the same line. If you multiply by $-3$ the second set of symmetric equations, we get:

$$-3 \left( \frac{x}{3} \right) = -3 \left( \frac{y - 3}{3} \right) = -3 \left( \frac{z - 3}{18} \right) \rightarrow -x = y - 3 = \frac{z - 3}{6}.$$ 

We can now add 1 to each of these equations. We recover then the first set of symmetric equations

$$-x + 1 = y - 2 = \frac{z + 3}{6}.$$ 

This proves that the symmetric equations do not depend on a specific point and direction vector of the line.

2. The coefficients of $x$, $y$ and $z$ in the equation of the plane give the components of the normal vector:

$$x + 3y + z = 5 \rightarrow \vec{n} = \langle 1, 3, 1 \rangle.$$ 

The normal vector is by definition perpendicular to the plane. If the line is also perpendicular, then $\vec{n}$ can be a direction vector for the line. Moreover, we know a point of the line: $(1, 0, 6)$. This gives us the symmetric equations:

$$x - 1 = \frac{y}{3} = z - 6.$$ 

3. We have three points of the plane: $A(2, -7, 10)$, $B(1, 1, 1)$ and $C(5, 0, 1)$. We can build (at least) two vectors of the plane. For instance:

$$\vec{AB} = \langle -1, 8, -9 \rangle,$$

$$\vec{BC} = \langle 4, -1, 0 \rangle.$$ 

Using these two vectors and the cross product of them, we can find a vector orthogonal to the plane (since it will be orthogonal to the vectors of the plane):

$$\vec{n} = \begin{vmatrix} -1 & 4 & -9 \\ 8 & -1 & -36 \\ -9 & 0 & -31 \end{vmatrix} = (-1, 1, 6).$$ 

The equation of the plane is then:

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0,$$

with $\vec{r} = \langle x, y, z \rangle$ and $\vec{r}_0$ being a vector whose initial point is the origin and terminal point is a point of the plane. Let us choose $\vec{r}_0 = 0\vec{A} = \langle 2, -7, 10 \rangle$. The equation of the plane is:

$$\vec{n} \cdot (\vec{r} - 0\vec{A}) = 0 \rightarrow 9(x + 1) + 36(y + 7) + 31(z - 10) = 0.$$ 

2
4. The coefficients of $x$, $y$ and $z$ in the equation of the plane give the components of the normal vector of the plane:

$$2x - y + 3z = 1 \rightarrow \vec{n} = \langle 2, -1, 3 \rangle.$$ 

We are looking for a plane parallel to it. Since they are parallel, these two planes can have the same normal vector. The equation of the plane we are looking for is then of the form:

$$2x - y + 3z = c,$$

with $c$ a constant. To find the constant $c$, we have to remind that this equation is true for every point of the plane. The origin $O(0,0,0)$ being a point of it, we can replace $x$, $y$ and $z$ by the coordinates of the origin and we get:

$$c = 0.$$ 

The equation of the plane, parallel to the plane of equation $2x - y + 3z = 1$ and passing through the origin is then

$$2x - y + 3z = 0,$$

5. If a point belongs to the intersection of two planes, it must be a solution of the equation of these two planes:

$$x + y + z = -2,$$

$$3x - 7z = 12.$$ 

The first equation gives us $z = -\frac{12}{7} + \frac{3}{7}x$. We can plug this relation into the first equation and we get:

$$x + y + z = -2$$

$$\rightarrow x + y + \left( -\frac{12}{7} + \frac{3}{7}x \right) = -2$$

$$\rightarrow \frac{10}{7}x = -y - 2 + \frac{12}{7}$$

$$\rightarrow \frac{10}{7}x = -y - \frac{2}{7}.$$ 

If we divide both sides of the equation by 10, we get a first symmetric equation:

$$\frac{x}{7} = -\frac{y + \frac{2}{7}}{10}.$$ 

To get a second symmetric equation, we need to go back to equations of the planes. We see that the second one gives us $x = 4 + \frac{7}{3}z$. If we substitute $x$ in the first plane equation with this expression, we get

$$x + y + z = -2$$

$$\rightarrow \left( 4 + \frac{7}{3}z \right) + y + z = -2$$

$$\rightarrow \frac{10}{3}z + 6 = -y.$$ 

3
To fit with the previous symmetric equation, we can subtract \( \frac{2}{7} \) on both sides and divide both sides by 10. We get a second symmetric equation:

\[
\frac{10}{3} z + 6 - \frac{2}{7} = -y - \frac{2}{7}
\]
\[
\rightarrow \frac{z}{3} + 6 - \frac{2}{3} = \frac{y}{10} + \frac{2}{10},
\]
\[
\rightarrow \frac{z + \frac{12}{3}}{3} = -\frac{y + \frac{2}{10}}{10}.
\]

The intersection of the two planes gives then a line with the symmetric equations:

\[
\frac{x}{7} = -\frac{y + \frac{2}{10}}{10} = \frac{z + \frac{12}{3}}{3}.
\]

### 2.2 Cylinder and spherical coordinates

1. The coordinates of \( P(2, 2, 3) \) in cylindrical coordinates are:

\[
\begin{align*}
\rho &= \sqrt{x^2 + y^2} = \sqrt{2^2 + 2^2} = 2\sqrt{2} \approx 2.83, \\
\theta &= \tan^{-1} \left( \frac{y}{x} \right) = \tan^{-1}(1) = \frac{\pi}{4}, \\
z &= 3.
\end{align*}
\]

In spherical coordinates, we have:

\[
\begin{align*}
\rho &= \sqrt{x^2 + y^2 + z^2} = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{17} \approx 4.12, \\
\theta &= \tan^{-1} \left( \frac{y}{x} \right) = \frac{\pi}{4}, \\
\phi &= \tan^{-1} \left( \frac{\sqrt{x^2 + y^2}}{z} \right) \approx .76
\end{align*}
\]

2. The equation

\[
x^2 + y^2 = 1 + z^2,
\]

can be written in cylindrical coordinates, using the relation \( x^2 + y^2 = r^2 \). We then have:

\[
r^2 = 1 + z^2 \quad \rightarrow \quad r = \sqrt{1 + z^2}.
\]

In spherical coordinates, we know that

\[
x^2 + y^2 = \rho^2 \sin^2(\phi) \cos^2(\theta) + \rho^2 \sin^2(\phi) \sin^2(\theta) = \rho^2 \sin^2(\phi)(\cos^2(\theta) + \sin^2(\theta))
\]
\[
\rightarrow \quad x^2 + y^2 = \rho^2 \sin^2(\phi).
\]

Moreover \( z^2 = \rho^2 \cos^2(\phi) \). If we plug these two relations into the equation of the hyperboloid, we get

\[
x^2 + y^2 = 1 + z^2 \\
\rightarrow \rho^2 \sin^2(\phi) = 1 + \rho^2 \cos^2(\phi) \\
\rightarrow \rho^2(\sin^2(\phi) - \cos^2(\phi)) = 1 \\
\rightarrow \rho = \frac{1}{\sqrt{\sin^2(\phi) - \cos^2(\phi)}}.
\]
We can leave the equation this way or, we can simplify it by noticing that \( \cos^2(\phi) - \sin^2(\phi) = \cos(2\phi) \). This gives us:

\[
\rho = \frac{1}{\sqrt{-\cos(2\phi)}}
\]

This imposes that \( \cos(2\phi) \) has to be negative, that is: \( \frac{\pi}{2} \leq 2\phi \leq \pi \) or \( \frac{3\pi}{2} \leq 2\phi \leq 2\pi \). The angle \( \phi \) is then defined on the intervals \( \left[ \frac{\pi}{4}, \frac{\pi}{2} \right] \) and \( \left[ \frac{3\pi}{4}, \pi \right] \).