First midterm exam - Math2D
Solutions

1 Exercise 1 (10 points)

Let \( \vec{A} = \langle -3, 4, 5 \rangle \), \( \vec{B} = \langle 1, 0, -2 \rangle \) and \( \vec{C} = \langle 4, 2, 1 \rangle \). Compute \((3\vec{A} - \vec{B}) \cdot \vec{C}\).

Solution:
\[ 3\vec{A} - \vec{B} = 3\langle -3, 4, 5 \rangle - \langle 1, 0, -2 \rangle = \langle -9, 12, 15 \rangle - \langle 1, 0, -2 \rangle = \langle -10, 12, 17 \rangle. \]

The dot product gives then
\[
(3\vec{A} - \vec{B}) \cdot \vec{C} = \langle -10, 12, 17 \rangle \cdot \langle 4, 2, 1 \rangle \\
= -40 + 24 + 17 = 1.
\]
2 Exercise 2-(30 points)

Let the function \( f(x, y) \) be defined as:

\[
\begin{cases}
  \frac{xy^2}{x - y}, & \text{if } (x, y) \neq (0, 0) \\
  1, & \text{if } (x, y) = (0, 0).
\end{cases}
\]

a) What are the domain and range of \( f(x, y) \)? (just explain briefly).
b) Write \( f(x, y) \) using polar coordinates.
c) What is \( \lim_{(x, y) \to (0, 0)} f(x, y) \)? Is it continuous?

Solution:
a) We can see that \( f(x, y) \) is defined everywhere except for \( x = y \). The domain is then \( \text{dom}(f) = \{(x, y) \in \mathbb{R}^2, x \neq y\} \) (to be more accurate, we need to add the origin \( x = y = 0 \) since \( f(0, 0) \) is defined even though \( x = y \). This means \( \text{dom}(f) = \{(x, y) \in \mathbb{R}^2, x \neq y\} \cup \{(0, 0)\} \)). We can see that \( f(x, y) \) can take any value, therefore the range of \( f(x, y) \) is \( \mathbb{R} \).

b) Using \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \). This gives:

\[
\frac{xy^2}{x - y} = \frac{r \cos(\theta) r^2 \sin^2(\theta)}{r \cos(\theta) - r \sin(\theta)} = \frac{r^2 \cos(\theta) \sin^2(\theta)}{\cos(\theta) - \sin(\theta)}.
\]

Unless \( (x, y) = (0, 0) \), that is, if \( r = 0 \). In this case, \( f(0, \theta) = 1 \) (since \( f \) is a constant at the origin).

c) We can compute the limit using the polar coordinates:

\[
\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{r \to 0} f(r, \theta) = \lim_{r \to 0} \frac{r^2 \cos(\theta) \sin^2(\theta)}{\cos(\theta) - \sin(\theta)} = 0.
\]

The limit exists. However the limit is different from \( f(0, 0) = 1 \). Therefore the function is not continuous at \( (0, 0) \).
3 Exercise 3-(20 points)

Show that \( \lim_{(x,y) \to (0,0)} \frac{x}{x^2 + y} \) doesn’t exist.

Solution: There are two possible ways to prove this (you can use either of these):

1. Let us compute the limits along different paths:
   \[
   \lim_{x \to 0} \lim_{y \to 0} \frac{x}{x^2 + y} = \lim_{x \to 0} \frac{1}{x} = \infty
   \]
   \[
   \lim_{y \to 0} \lim_{x \to 0} \frac{x}{x^2 + y} = \lim_{y \to 0} 0 = 0.
   \]
   These two limits don’t match. The limit to \((0,0)\) does not exist.

2. Let us use the polar (cylinder) coordinates and compute the limit \( r \to 0 \)
   \[
   \lim_{(x,y) \to (0,0)} \frac{x}{x^2 + y} = \lim_{r \to 0} \frac{r \cos(\theta)}{r^2 \cos(\theta) + r \sin(\theta)}
   \]
   \[
   = \lim_{r \to 0} \frac{\cos(\theta)}{r \cos(\theta) + \sin(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}.
   \]
   The limit still depends on \( \theta \). The limit cannot exist.
Let a curve be described by the following parametric equations and $t \geq 1$:

$$
\begin{align*}
x &= \frac{2}{3}(t - 1)^{3/2}, \\
y &= \frac{1}{\pi} \cos(\pi t), \\
z &= \frac{1}{\pi} \sin(\pi t) + 1.
\end{align*}
$$

a) Find the parametric equation of the line tangent to the curve at $t = 1$.  

b) What is the length of the curve between $t = 1$ and $t = 2$?

**Solution:**

a) Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$. The vector equation of the line tangent to the curve at $t = 1$ is

$$
\langle x, y, z \rangle = \vec{r}(1) + q\vec{r}'(1),
$$

with a parameter $q$. We can calculate $\vec{r}(1) = \langle 0, -\frac{1}{\pi}, 1 \rangle$ and $\vec{r}'(t) = \langle \sqrt{t - 1}, -\sin(\pi t), \cos(\pi t) \rangle$, which gives $\vec{r}'(1) = \langle 0, 0, -1 \rangle$. This gives

$$
\langle x, y, z \rangle = \langle 0, -\frac{1}{\pi}, 1 - q \rangle.
$$

This gives the following parametric equations:

$$
\begin{align*}
x &= 0, \\
y &= -\frac{1}{\pi}, \\
z &= 1 - q.
\end{align*}
$$

b) The length is given by

$$
\int_1^2 \|\vec{r}'(t)\| dt = \int_1^2 \sqrt{(t - 1) + \cos^2(\pi t) + \sin^2(\pi t)} dt = \int_1^2 \sqrt{(t - 1) + 1} dt
$$

$$
= \int_1^2 \sqrt{t} dt = \frac{2}{3} t^{3/2} \bigg|_1^2 = \frac{2}{3} (2^{3/2} - 1).
$$
5 Exercise 5-(20 points)

Find the equation of the plane passing through the points $A = (1, 2, 3)$, $B = (2, 3, -1)$ and $C = (4, -6, 1)$.

Solution: Let us consider the vectors $\vec{AB} = (-1, -1, 4)$, $\vec{AC} = (3, 8, 2)$. We can get a normal vector with $\vec{n} = \vec{AB} \times \vec{AC}$

$$\vec{n} = \begin{vmatrix} -1 & -3 \\ -1 & 8 \\ 4 & 2 \end{vmatrix} = \begin{vmatrix} -34 \\ -10 \\ -11 \end{vmatrix}$$

Using $\vec{n}$ and a point on the plane, for example $A = (1, 2, 3)$, we get:

$$-34(x - 1) - 10(y - 2) - 11(z - 3) = 0.$$