1 Exercise 1 (20 points)

Let us consider the function \( f(x, y) = x^2 y - x^2 - \frac{y^2}{2} \). Find the critical points of the functions. Are these points local maximum, local minimum or neither of those?

Solution: We are looking for the points such that \( \vec{\nabla} f = (f_x(x, y), f_y(x, y)) = \vec{0} \). That is

\[
\begin{align*}
    f_x(x, y) &= 2xy - 2x = 0, \\
    f_y(x, y) &= x^2 - y = 0.
\end{align*}
\]

The first equation gives either \( x = 0 \) or \( y = 1 \). If \( x = 0 \), the second equation becomes \( y = 0 \). If \( y = 1 \), the second equation gives \( x^2 - 1 = 0 \), that is \( x = \pm 1 \). We get then 3 critical points: \((0, 0), (-1, 1)\) and \((1, 1)\).

Let us compute \( J \):

\[
J = \begin{vmatrix}
    f_{xx} & f_{xy} \\
    f_{yx} & f_{yy}
\end{vmatrix} = \begin{vmatrix}
    2y - 2 & 2x \\
    2x & -1
\end{vmatrix} = 2 - 2y - 4x.
\]

For \((0, 0)\), then \( J = 2 > 0 \) and \( f_{xx}(0, 0) = -2 < 0 \). It is a local maximum.

For \((-1, 1)\), then \( J = 4 > 0 \) and \( f_{xx}(-1, 1) = 0 \) and \( f_{yy}(-1, 1) = -1 < 0 \). It is a local maximum.

For \((1, 1)\), then \( J = -4 \). It is a saddle point.
Let us consider the function $f(x, y) = (x^2 + y)\cos\left(\frac{\pi xy}{2}\right)$. What is the equation of the tangent plane of the graph at the point $(x, y) = (1, 3)$. Does the plane pass through the origin?

**Solution:** The equation of the plane is by definition:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

with $x_0 = 1$, $y_0 = 3$ and $z_0 = f(x_0, y_0) = (x_0^2 + y_0)\cos\left(\frac{\pi x_0y_0}{2}\right) = 4\cos\left(\frac{3\pi}{2}\right) = 0$.

$$f_x(x, y) = 2x\cos\left(\frac{\pi xy}{2}\right) - (x^2 + y)\frac{\pi y}{2}\sin\left(\frac{\pi xy}{2}\right) \rightarrow f_x(x_0, y_0) = 6\pi$$

$$f_y(x, y) = \cos\left(\frac{\pi xy}{2}\right) - (x^2 + y)\frac{\pi x}{2}\sin\left(\frac{\pi xy}{2}\right) \rightarrow f_y(x_0, y_0) = 2\pi$$

This gives the following equation for the plane:

$$z = 6\pi(x - 1) + 2\pi(y - 3).$$

The coordinates of the plane are $(0, 0, 0)$. Plugging this in the plane equation gives: $0 = -6\pi - 6\pi = -12\pi$ which is obviously wrong. The origin is then not a point of the plane.
3 Exercise 3-(20 points)

Let us consider the function \( f(x, y) = x^2y + x \). Compute \( \Delta f \) and find \( \epsilon_1 \) and \( \epsilon_2 \). Show the function is differentiable (explain why, using \( \epsilon_1 \) and \( \epsilon_2 \)). Find \( f_x(x, y) \) and \( f_y(x, y) \) from the expression of \( \Delta f \).

Solution:  Let us compute

\[
\Delta f = f(x + \Delta x, y + \Delta y) - f(x, y) = (x + \Delta x)^2(y + \Delta y) + x + \Delta x - (x^2y + x) \\
= (x^2 + (\Delta x)^2 + 2x\Delta x)(y + \Delta y) + x + \Delta x - (x^2y + x) \\
= x^2y + x^2\Delta y + y(\Delta x)^2 + \Delta y(\Delta x)^2 + 2xy\Delta x + 2x\Delta x\Delta y + x + \Delta x - (x^2y + x) \\
= (2xy + 1)\Delta x + x^2\Delta y + y(\Delta x)^2 + \Delta y(\Delta x)^2 + 2x\Delta x\Delta y.
\]

The function is differentiable if we can write the term \( y(\Delta x)^2 + \Delta y(\Delta x)^2 + 2x\Delta x\Delta y \) (that is all the terms that contain products of \( \Delta s \)) as \( \epsilon_1 \Delta x + \epsilon_2 \Delta y \) such that \( \lim \epsilon_1 = \lim \epsilon_2 = 0 \) when \( \Delta x \) and \( \Delta y \) go to 0. Regarding the equation above we can write

\[
y(\Delta x)^2 + \Delta y(\Delta x)^2 + 2x\Delta x\Delta y = \epsilon_1 \Delta x + \epsilon_2 \Delta y,
\]

with \( \epsilon_1 = y\Delta x + \Delta y\Delta x \) and \( \epsilon_2 = 2x\Delta x \). The function is differentiable.

We have

\[
\Delta f = (2xy + 1)\Delta x + x^2\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y.
\]

The term in front of \( \Delta x \) is then \( f_x(x, y) = 2xy + 1 \) and the term in front of \( \Delta y \) is then \( f_y(x, y) = x^2 \).
Let us consider the function \( f(x, y) = \frac{x^2 + y^2}{x^2 + y^2} \).

a) Write the differential \( df \) of the function \( f(x, y) \).

b) Knowing that \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \) and using chain rule: find the partial derivatives \( \frac{\partial f}{\partial r} \) and \( \frac{\partial f}{\partial \theta} \). (express the result as a function of \( r \) and \( \theta \)).

c) Compute the value of these partial derivatives at \( r = 1, \theta = \frac{\pi}{2} \).

Solution:

a) \[ df = f_x dx + f_y dy, \]

\[ f_x = \frac{1}{x^2 + y^2} \frac{1}{(x^2 + y^2)^2} \left( \frac{2x}{x^2 + y^2} - 2x + y \right) = \frac{y^2 - x^2 - 2xy}{(x^2 + y^2)^2}, \]

\[ f_y = \frac{1}{x^2 + y^2} \frac{1}{(x^2 + y^2)^2} \left( \frac{-2y}{x^2 + y^2} + 2y + x \right) = \frac{x^2 - y^2 - 2xy}{(x^2 + y^2)^2}. \]

This gives

\[ df = \frac{y^2 - x^2 - 2xy}{(x^2 + y^2)^2} dx + \frac{x^2 - y^2 - 2xy}{(x^2 + y^2)^2} dy. \]

b) The chain rule gives:

\[ \frac{\partial f}{\partial r} = f_x \frac{\partial x}{\partial r} + f_y \frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial f}{\partial \theta} = f_x \frac{\partial x}{\partial \theta} + f_y \frac{\partial y}{\partial \theta}. \]

Using \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \).

\[ \frac{\partial x}{\partial r} = \cos(\theta), \quad \frac{\partial y}{\partial r} = \sin(\theta), \]

\[ \frac{\partial x}{\partial \theta} = -r \sin(\theta), \quad \frac{\partial y}{\partial \theta} = r \cos(\theta). \]

and (remember that \( x^2 + y^2 = r^2 \))

\[ f_x = \frac{y^2 - x^2 - 2xy}{(x^2 + y^2)^2} \frac{\cos^2(\theta) - \sin^2(\theta) - 2 \cos(\theta) \sin(\theta)}{r^2}, \]

\[ f_y = \frac{x^2 - y^2 - 2xy}{(x^2 + y^2)^2} \frac{\sin^2(\theta) - \cos^2(\theta) - 2 \cos(\theta) \sin(\theta)}{r^2}. \]

Then

\[ \frac{\partial f}{\partial r} = \frac{\cos^2(\theta) - \sin^2(\theta) - 2 \cos(\theta) \sin(\theta)}{r^2} \cos(\theta) + \frac{\sin^2(\theta) - \cos^2(\theta) - 2 \cos(\theta) \sin(\theta)}{r^2} \sin(\theta), \]

\[ \frac{\partial f}{\partial \theta} = \frac{\cos^2(\theta) - \sin^2(\theta) - 2 \cos(\theta) \sin(\theta)}{r^2} (-r \sin(\theta)) + \frac{\sin^2(\theta) - \cos^2(\theta) - 2 \cos(\theta) \sin(\theta)}{r^2} (r \cos(\theta)). \]

c) Plugging \( r = 1 \) and \( \theta = \frac{\pi}{2} \) in the above expressions, we get

\[ \frac{\partial f}{\partial r} = 1, \]

\[ \frac{\partial f}{\partial \theta} = 1. \]
Exercise 5-(20 points)

Find the absolute maximum and minimum of the function \( f(x, y) = (x^2 + 1)(y^2 + \frac{1}{4}) \) on the domain \( D = [-2, 2] \times [-1, 1] \).

Hint: you can use the fact that the boundaries \( y = -1 \) and \( y = 1 \) are symmetric (so are \( x = -2 \) and \( x = 2 \)).

Solution: We need to first look for the critical points such that \( \nabla f = (f_x(x, y), f_y(x, y)) = 0 \). That is

\[
\begin{align*}
f_x(x, y) &= 2x(y^2 + \frac{1}{4}) = 0 \quad \rightarrow \quad x = 0, \\
f_y(x, y) &= 2y(x^2 + 1) = 0 \quad \rightarrow \quad y = 0.
\end{align*}
\]

There is only one critical point \((0, 0)\) and

\[
J = \begin{vmatrix}
f_{xx}(0, 0) & f_{xy}(0, 0) \\
f_{yx}(0, 0) & f_{yy}(0, 0)
\end{vmatrix} = \begin{vmatrix}
\frac{1}{2} & 0 \\
0 & 2
\end{vmatrix} = 1,
\]

\( J > 0 \) and \( f_{xx}(0, 0) = 1/2 > 0 \). The point \((0, 0)\) is a local minimum and \( f(0, 0) = \frac{1}{4} \).

The domain \( D \) has 4 boundaries, but since the boundary \( y = -1 \) is the same as \( y = 1 \) and \( x = -2 \) is the same as \( x = 2 \) (\( f(x, y) \) depends on \( x^2 \) and \( y^2 \), so the sign of \( x \) and \( y \) doesn’t matter).

Let us look at the boundary \( D_1 = \{-2 \leq x \leq 2, \ y = -1\} \). Then \( f(x, -1) = \frac{5}{4}(x^2 + 1) \). This is parabola that has a minimum at \( x = 0 \) (draw quickly a graph to see this or compute the derivative). Therefore the minimum on this boundary is \( f(0, -1) = \frac{5}{4} \) and the maximum must be either at \( x = -2 \) or \( x = 2 \) and \( f(2, -1) = f(-2, -1) = \frac{25}{4} \).

Let us look now at the boundary \( D_2 = \{x = 2, \ -1 \leq y \leq 1\} \). Then \( f(2, y) = 5(y^2 + \frac{1}{4}) \). This is also parabola that has a minimum at \( y = 0 \). Therefore the minimum on this boundary is \( f(2, 0) = \frac{5}{4} \) and the maximum must be either at \( y = -1 \) or \( y = 1 \) and \( f(2, -1) = f(2, -1) = \frac{25}{4} \).

Comparing all the maximums and minimums we found, we conclude that the absolute minimum occurs at \((0, 0)\) with the value \( f(0, 0) = \frac{1}{4} \) and the absolute maximum occurs at the points \((\pm 2, \pm 1)\) with the \( f(\pm 2, \pm 1) = \frac{25}{4} \).