Second midterm exam - Sample Test

Student name: 
Student ID: 

Calculators, book and notes are not allowed. 

**Explain all your answers.** Unsupported answers will not receive full credit.

1 Exercise 1 (15 points)

Find the critical point(s) of the function 

\[ f(x, y) = (x^2 + y^2)e^{-y}. \]

Are these points local maximum, local minimum or neither of those?

Let us solve:

\[ f_x(x, y) = 2xe^{-y} = 0 \rightarrow x = 0, \]
\[ f_y(x, y) = 2ye^{-y} - (x^2 + y^2)e^{-y} = 0 \rightarrow 2y - y^2 = 0. \]

The last equation gives either \( y = 0 \) or \( y = 2 \). There are 2 critical points \((0, 0)\) and \((0, 2)\). Let us compute now the matrix \( J \):

\[
J = \begin{vmatrix}
  f_{xx} & f_{xy} \\
  f_{yx} & f_{yy}
\end{vmatrix} = \begin{vmatrix}
  2e^{-y} & 0 \\
  0 & (2 - 4y + y^2)e^{-y}
\end{vmatrix}.
\]

(I kept \( y \) in the equation since \( y \) can take 2 values). For \( y = 0 \), \( J = 2 \). Since \( f_{xx}(0, 0) = 2 > 0 \). Therefore \((0, 0)\) is a local minimum.

For \( y = 2 \), \( J = -2e^{-4} < 0 \), therefore \((0, 2)\) is a saddle-point.

2 Exercise 2 (10 points)

What is the differential of \( f(x, y) \) ?

\[ f(x, y) = (x + 1) \cos(x + 2y). \]

we have \( f_x = \cos(x + 2y) - (x + 1) \sin(x + 2y) \) and \( f_y = -(x + 1) \sin(x + 2y) \). This gives
\[ df = \left( \cos(x + 2y) - (x + 1) \sin(x + 2y) \right)dx - 2(x + 1) \sin(x + 2y)dy. \]
3 Exercise 3-(25 points)

Let \( f(x, y) \) be the following function of 2 variables:

\[
f(x, y) = (x + 1) \cos(y).
\]

If \( x = se^t \) and \( y = se^{-t} \), find the partial derivatives \( \frac{\partial f}{\partial s}(x, y) \) and \( \frac{\partial f}{\partial t}(x, y) \), using the chain rule.

Find the value of \( \frac{\partial f}{\partial s}(x, y) \) and \( \frac{\partial f}{\partial t}(x, y) \) for \( s = 1 \) and \( t = 0 \).

we have \( f_x = \cos(y) = \cos(se^{-t}) \) and \( f_y = -(x + 1) \sin(y) = -(se^t + 1) \sin(se^{-t}) \). This gives

\[
df = f_x dx + f_y dy
\]

\[
\frac{\partial f}{\partial s}(x, y) = f_x \frac{\partial x}{\partial s} + f_y \frac{\partial y}{\partial s},
\]

\[
\frac{\partial f}{\partial t}(x, y) = f_x \frac{\partial x}{\partial t} + f_y \frac{\partial y}{\partial t}.
\]

We have \( \frac{\partial x}{\partial s} = e^t \), \( \frac{\partial x}{\partial t} = se^t \), \( \frac{\partial y}{\partial s} = e^{-t} \) and \( \frac{\partial y}{\partial t} = -se^{-t} \). At \( s = 1 \) and \( t = 0 \), this gives:

\[
\frac{\partial f}{\partial s} = \cos(1) - 2 \sin(1),
\]

\[
\frac{\partial f}{\partial t} = \cos(1) + 2 \sin(1).
\]
4 Exercise 4-(20 points)

What is the equation of the tangent plane of the hyperbolic paraboloid:

\[ z = \frac{x^2}{2} - \frac{y^2}{4}, \]

at the point \( (1, 2, -\frac{3}{2}) \). Does the plane pass through the origin?

We have \( f_x(x, y) = x \) \( \rightarrow f_x(1, 2) = 1 \) and \( f_y(x, y) = -\frac{y}{2} \) \( \rightarrow f_y(1, 2) = -1 \). The equation of the plane is then:

\[ z + \frac{1}{2} = (x - 1) - (y - 2). \]

To check if the plane passes through the origin, let us plug \( x = y = z = 0 \) in the above equation: this gives \( \frac{1}{2} = 1 \) which is obviously wrong. The origin is not a point of the plane.
5 Exercise 5-(30 points)

Find the absolute maximum and minimum of the function $f(x, y)$:

$$f(x, y) = (y^2 - 1) \left( x - \frac{1}{2} \right),$$

on the domain $D = \{0 \leq x \leq 1, \ 0 \leq y \leq x\}$.

Let us look for the critical points first:

$$f_x = y^2 - 1 = 0 \quad \rightarrow \quad y = \pm 1,$$

$$f_y = 2y(x - \frac{1}{2}) = 0.$$

We can disregard $y = -1$ since it’s not in $D$. If $y = 1$, we get $x = \frac{1}{2}$, This point is also not in the domain since $x < y$. There is no critical point in the domain $D$.

There are 3 boundaries for $D$: $B_1 = \{y = 0, \ 0 \leq x \leq 1\}$, $B_2 = \{x = 1, \ 0 \leq y \leq 1\}$ and $B_3 = \{y = x, \ 0 \leq x \leq 1\}$.

On the boundary $B_1$, $f(x, 0) = -(x - \frac{1}{2})$ which is a decreasing function. So the max of $f$ on $B_1$ is $f(0, 0) = \frac{1}{2}$ and the minimum of $f$ on $B_1$ is $f(1, 0) = -\frac{1}{2}$.

On the boundary $B_2$, $f(1, y) = \frac{1}{2}(y^2 - 1)$, the function is increasing on $0 \leq y \leq 1$ (you can draw a graph to see this). So the max of $f$ on $B_2$ is $f(1, 1) = 0$ and the minimum of $f$ on $B_2$ is $f(1, 0) = -\frac{1}{2}$.

On the boundary $B_3$, $f(x, x) = (x^2 - 1) \left( x - \frac{1}{2} \right) = x^3 - \frac{x^2}{2} - x + \frac{1}{2}$. By compute the first and second derivative, we can see the function has a minimum at $x \simeq .76$ and $f(.76, .76) \simeq -.11$. Also the max of $f$ on $B_3$ occurs at $(0, 0)$, which gives $f(0, 0) = \frac{1}{2}$.

Gathering all these results, we see that the absolute maximum is at $(0, 0)$ and $f(0, 0) = 1/2$ and the absolute minimum occurs at $(1, 0)$ and $f(1, 0) = -1/2$.  