Homework 5
Linear approximations - Tangent planes - Chain rule - Directional derivatives and gradient vectors.

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1 Exercises

1.1 Linear approximations - Tangent planes

1. Show that the function is differentiable by finding the values of $\epsilon_1$ and $\epsilon_2$ (see the definition of differentiability $\Delta f$ in your class notes).

$$f(x, y) = xy - y^2 + x.$$  

Can you find the expressions of $f_x(x, y)$ and $f_y(x, y)$ in $\Delta f$?

2. Write the differential of $f(x, y)$ at $(1, 2)$:

$$f(x, y) = xye^y.$$  

3. What is the equation of the tangent plane of the graph of $f(x, y)$ at the point $(2, -1, \frac{1}{2})$?

$$f(x, y) = \frac{1}{x^2 + 2y^2 + 1}$$

1.2 Chain rule

1. Find the derivative $\frac{df(x,y)}{dt}$ of $f(x, y)$:

$$f(x, y) = xy^2 + xy,$$

knowing $x = 2 + t^4$ and $y = 1 - t^3$.

2. Find the derivative $\frac{dg(x,y)}{dt}(x, y)$ of $g(x, y)$ at $t = 1$:

$$f(x, y) = \cos(x)\sin(y),$$

knowing $x = \pi t$ and $y = \sqrt{t}$.

3. Find the partial derivatives $\frac{df(x,y)}{dt}(x, y)$ and $\frac{df(x,y)}{ds}(x, y)$ of $f(x, y)$:

$$f(x, y) = \frac{x}{y},$$

knowing $x = se^t$ and $y = 1 - se^{-t}$.  

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4. Let \( f(x, y) \) be a differentiable function of 2 variables:
\[
f(x, y) = x^2 + y^2 + xy.
\]
If \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \), what are the partial derivatives \( \frac{\partial f}{\partial r}(x, y) \) and \( \frac{\partial f}{\partial \theta}(x, y) \)?

5. A 2 dimensional ellipsoid, centered at the origin and with radii 2 and 3, is described by the relation:
\[
\frac{x^2}{4} + \frac{y^2}{9} = 1.
\]
Using the implicit differentiation, find \( \frac{dy}{dx} \).

2 Directional derivatives and gradient vectors

1. Find the gradient vectors of the 2 following functions:
\[
f(x, y) = \cos(x)e^{2y},
g(x, y, z) = \frac{x}{z} \ln(y).
\]

2. Find the directions in which the directional derivative of \( f(x, y) = x^2 + \sin(xy) \) at the point \((1, 0)\) has the value 1.

3. What is the equation of the tangent plane of the hyperboloid of one sheet:
\[
x^2 + y^2 - z^2 = 1
\]
at the point \((1, 2, 2)\) ?

4. The equation of the cone is:
\[
x^2 + y^2 = z^2.
\]
Show that any tangent plane of the cone (at any point) passes through the origin.

3 Solutions

3.1 Linear approximations - Tangent planes

1. Let us compute the variation \( \Delta f = f(x + \Delta x, y + \Delta y) - f(x, y) \) of \( f(x, y) \):
\[
\Delta f = f(x + \Delta x, y + \Delta y) - f(x, y) \\
= (x + \Delta x)(y + \Delta y) - (y + \Delta y)^2 + x + \Delta x - (xy - y^2 + x).
\]

If we develop the different terms in the equation above, we see that some of them cancel out. Finally we get:
\[
\Delta f = x\Delta y + y\Delta x + \Delta x\Delta y - 2y\Delta y - \Delta y\Delta y + \Delta x.
\]

We can gather the terms like:
\[
\Delta f = (y + 1)\Delta x + (x - 2y)\Delta y + \Delta y\Delta x - \Delta y\Delta y.
\]

Let us define \( \epsilon_1 = \Delta y \) and \( \epsilon_2 = -\Delta y \). We can rewrite the above relation like:
\[
\Delta f = (y + 1)\Delta x + (x - 2y)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y.
\]
This expression proves that \( f(x, y) \) is differentiable since \( \epsilon_1 \) and \( \epsilon_2 \) go to 0 as \( (\Delta x, \Delta y) \to (0, 0) \). Moreover, we can recover the partial derivatives, which correspond to the factors in front of the \( \Delta x \) and \( \Delta y \) terms:

\[
f_x(x, y) = (y + 1), \quad f_y(x, y) = x - 2y.
\]

2. The differential of a function of 2 variables is:

\[
df(x, y) = f_x(x, y)dx + f_y(x, y)dy.
\]

In this case, the partial derivatives are:

\[
f_x(x, y) = ye^y, \quad f_y(x, y) = xe^y + xye^y = xe^y(y + 1).
\]

The differential of \( f(x, y) = xe^y \) is then:

\[
df(x, y) = ye^ydx + xe^y(y + 1)dy.
\]

3. The equation of the tangent plane of the graph \( z = f(x, y) \) at the point \( (x_0, y_0, z_0) \) is:

\[
z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(z - z_0).
\]

The partial derivatives of \( f(x, y) \) at \((2, -1)\) are:

\[
f_x(x, y) = -\frac{2x}{(x^2 + 2y^2 + 1)^2} \quad \rightarrow \quad f_x(2, -1) = -\frac{4}{49},
\]

\[
f_y(x, y) = -\frac{4y}{(x^2 + 2y^2 + 1)^2} \quad \rightarrow \quad f_y(2, -1) = \frac{4}{49}.
\]

Knowing that \( z_0 = \frac{1}{7} \), we can plug these results into the equation of the plane:

\[
z - \frac{1}{7} = -\frac{4}{49}(x - 2) + \frac{4}{49}(y + 1).
\]

### 3.2 Chain rule

1. Using the chain rule, the derivative of \( f(x, y) \) with respect to \( t \) is:

\[
\frac{df}{dt}(x, y) = f_x(x, y)\frac{dx}{dt} + f_y(x, y)\frac{dy}{dt}.
\]

The partial derivatives of \( f(x, y) \) are:

\[
f_x(x, y) = y^2 + y,
\]

\[
f_y(x, y) = 2xy + x.
\]

The derivatives of \( x \) and \( y \) with respect to \( t \) are:

\[
\frac{dx}{dt} = 4t^3,
\]

\[
\frac{dy}{dt} = -3t^2.
\]

Then the derivatives \( \frac{df}{dt} \) becomes:

\[
\frac{df}{dt}(x, y) = 4t^3f_x(x, y) - 3t^2f_y(x, y) = 4t^3(y^2 + y) - 3t^2(2xy + x).
\]

We can leave the equation like this, or replace \( x = 2 + t^4 \) and \( y = 1 - t^3 \), we get:

\[
\frac{df}{dt}(x, y) = 4t^3(2 - 3t^3 + t^6) - 3t^2(6 - 4t^3 + 3t^4 - 2t^7).
\]
2. Using the chain rule, the derivative of \( g(x, y) \) with respect to \( t \) is:

\[
\frac{dg}{dt}(x, y) = g_x(x, y) \frac{dx}{dt} + g_y(x, y) \frac{dy}{dt}.
\]

The partial derivatives of \( g(x, y) \) are:

\[
g_x(x, y) = -\sin(x) \sin(y),
\]

\[
g_y(x, y) = \cos(x) \cos(y).
\]

We want to find the differential at \( t = 1 \). \( t = 1 \) corresponds to the point \((\pi, 1)\) (since \( x = \pi t \) and \( y = \sqrt{t} \)). The partial derivatives at that point are then:

\[
g_x(\pi, 1) = -\sin(\pi) \sin(1) = 0,
\]

\[
g_y(\pi, 1) = \cos(\pi) \cos(1) = -\cos(1).
\]

The derivatives of \( x \) and \( y \) with respect to \( t \) at \( t = 1 \) are:

\[
\frac{dx}{dt} = \pi, \quad \text{at } t = 1: \quad \frac{dx}{dt} = \pi
\]

\[
\frac{dy}{dt} = \frac{1}{2\sqrt{t}}, \quad \text{at } t = 1: \quad \frac{dy}{dt} = \frac{1}{2}.
\]

The derivative of \( g(x, y) \) at \( t = 1 \) is then:

\[
\frac{dg}{dt}(x, y) = -\frac{\cos(1)}{2} \approx -0.27.
\]

3. Using the chain rule, the partial derivatives of \( f(x, y) \) with respect to \( s \) and \( t \) are:

\[
\frac{\partial f(x, y)}{\partial s} = f_x(x, y) \frac{\partial x}{\partial s} + f_y(x, y) \frac{\partial y}{\partial s},
\]

\[
\frac{\partial f(x, y)}{\partial t} = f_x(x, y) \frac{\partial x}{\partial t} + f_y(x, y) \frac{\partial y}{\partial t}.
\]

The partial derivatives \( f_x(x, y) \) and \( f_y(x, y) \) are:

\[
f_x(x, y) = \frac{1}{y},
\]

\[
f_y(x, y) = -\frac{x}{y^2}.
\]

The partial derivatives of \( x = se^t \) and \( y = 1 - se^{-t} \) with respect to \( s \) and \( t \) are:

\[
\frac{\partial x}{\partial s} = e^t, \quad \frac{\partial x}{\partial t} = se^t,
\]

\[
\frac{\partial y}{\partial s} = -e^{-t}, \quad \frac{\partial y}{\partial t} = se^{-t}.
\]

We get then for the partial derivatives of \( f(x, y) \) with respect to \( s \) and \( t \):

\[
\frac{\partial f(x, y)}{\partial s} = \frac{1}{y} e^t + \frac{x}{y^2} e^{-t},
\]

\[
\frac{\partial f(x, y)}{\partial t} = \frac{se^t}{y} - \frac{x}{y^2} se^{-t}.
\]
We can leave the equation like this or replace \( x = se^t \) and \( y = 1 - se^{-t} \):

\[
\frac{\partial f(x, y)}{\partial s} = \frac{e^t}{1 - se^{-t}} + \frac{s}{(1 - se^{-t})^2},
\]

\[
\frac{\partial f(x, y)}{\partial t} = \frac{se^t}{1 - se^{-t}} - \frac{s^2}{(1 - se^{-t})^2}.
\]

4. Using the chain rule, the partial derivatives of \( f(x, y) \) with respect to \( r \) and \( \theta \) are:

\[
\frac{\partial f(x, y)}{\partial r} = f_x(x, y) \frac{\partial x}{\partial r} + f_y(x, y) \frac{\partial y}{\partial r},
\]

\[
\frac{\partial f(x, y)}{\partial \theta} = f_x(x, y) \frac{\partial x}{\partial \theta} + f_y(x, y) \frac{\partial y}{\partial \theta}.
\]

The partial derivatives \( f_x(x, y) \) and \( f_y(x, y) \) are:

\[
f_x(x, y) = 2x + y,
\]

\[
f_y(x, y) = 2y + x.
\]

The partial derivatives of \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \) with respect to \( r \) and \( \theta \) are:

\[
\frac{\partial x}{\partial r} = \cos(\theta), \quad \frac{\partial x}{\partial \theta} = -r \sin(\theta),
\]

\[
\frac{\partial y}{\partial r} = \sin(\theta), \quad \frac{\partial y}{\partial \theta} = r \cos(\theta).
\]

We get then for the partial derivatives of \( f(x, y) \) with respect to \( r \) and \( \theta \):

\[
\frac{\partial f(x, y)}{\partial r} = (2x + y) \cos(\theta) + (2y + x) \sin(\theta),
\]

\[
\frac{\partial f(x, y)}{\partial \theta} = -(2x + y)r \sin(\theta) + (2x + y)r \cos(\theta).
\]

If we replace \( x \) by \( r \cos(\theta) \) and \( y \) by \( r \sin(\theta) \) (and using the fact that \( \cos^2(\theta) + \sin^2(\theta) = 1 \)), we finally get:

\[
\frac{\partial f(x, y)}{\partial r} = 2r + 2r \sin(\theta) \cos(\theta),
\]

\[
\frac{\partial f(x, y)}{\partial \theta} = r^2(\cos^2(\theta) - \sin^2(\theta)).
\]

5. We need first to rewrite the equation of the ellipsoid as:

\[
x^2 + y^2 = 4 + \frac{9}{2} - 1 = 0.
\]

This equation can be view as the level curve \( F(x, y) = 0 \) of the function of 2 variables \( F(x, y) = \frac{x^2}{4} + \frac{y^2}{9} - 1 \). Using the rule for the implicit differentiation, we have:

\[
\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}.
\]

The 2 partial derivatives of \( F(x, y) \) are:

\[
F_x(x, y) = \frac{x}{2},
\]

\[
F_y(x, y) = \frac{2y}{9}.
\]

We then have:

\[
\frac{dy}{dx} = -\frac{\frac{x}{2}}{\frac{2y}{9}} = -\frac{9x}{4y}.
\]
3.3 Directional derivatives and gradient vectors

1. The partial derivatives of $f(x, y)$ are:

\[ f_x(x, y) = -\sin(x)e^{2y}, \]
\[ f_y(x, y) = 2\cos(x)e^{2y}. \]

The gradient vector of $f(x, y)$ is then:

\[ \vec{\nabla} f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \langle -\sin(x)e^{2y}, 2\cos(x)e^{2y} \rangle. \]

The partial derivatives of $g(x, y, z)$ are:

\[ g_x(x, y, z) = \frac{1}{z}\ln(y), \]
\[ g_y(x, y, z) = \frac{x}{yz}, \]
\[ g_z(x, y, z) = -\frac{x}{z^2}\ln(y). \]

The gradient vector of $g(x, y, z)$ is then:

\[ \vec{\nabla} g(x, y, z) = \langle g_x(x, y, z), g_y(x, y, z), g_z(x, y, z) \rangle = \langle \frac{1}{z}\ln(y), \frac{x}{yz}, -\frac{x}{z^2}\ln(y) \rangle. \]

2. The partial derivatives of $f(x, y)$ are:

\[ f_x(x, y) = 2x + y\cos(xy), \]
\[ f_y(x, y) = x\cos(xy). \]

The gradient vector of $f(x, y)$ is:

\[ \vec{\nabla} f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \langle 2x + y\cos(xy), x\cos(xy) \rangle. \]

The directional derivative in the direction of the vector $\vec{u}$ is:

\[ D_{\vec{u}}f(x, y) = \vec{\nabla} f(x, y) \cdot \vec{u}. \]

Let us take a vector $\vec{u}$ of the form $\vec{u} = \langle a, b \rangle$. Moreover, we are interested in the directional derivative at $(1, 0)$. In that case, $\vec{\nabla} f(1, 0) = \langle 2, 1 \rangle$. The directional derivative at $(1, 0)$ is then:

\[ D_{\vec{u}}f(1, 0) = \langle 2, 1 \rangle \cdot \langle a, b \rangle = 2a + b. \]

We want it to be equal to 1. We then have the relation for the vector: $2a + b = 1$.

This implies that the directional derivative is 1 in the direction of any vector of the form $\vec{u} = \langle a, 1 - 2a \rangle$. We only consider unit vector. We need to choose $a$ such that $|\vec{u}| = 1$:

\[ |\vec{u}| = 1 \rightarrow a^2 + (1 - 2a)^2 = 1 \]
\[ \rightarrow 5a^2 = 4a. \]

There exist 2 solutions: either $a = 0$ or $a = \frac{4}{5}$.

The directional derivative of $f(x, y)$ is 1 in the direction of, either $\vec{u} = \langle 0, 1 \rangle$ or $\langle \frac{4}{5}, -\frac{3}{5} \rangle$. 

3. To find the equation of the tangent plane of the hyperboloid, let us write it first as a level curve:

\[ F(x, y, z) = 0, \quad \text{with} \quad F(x, y, z) = x^2 + y^2 - z^2 - 1. \]

We know that the gradient vector of \( F \) gives a normal vector of the tangent plane. The gradient vector of \( F(x, y, z) \) is:

\[ \nabla F(x, y, z) = \langle F_x(x, y, z), F_y(x, y, z), F_z(x, y, z) \rangle = \langle 2x, 2y, -2z \rangle. \]

At the point \((1,2,2)\), the gradient vector is:

\[ \nabla F(1, 2, 2) = \langle 2, 4, -4 \rangle. \]

We have a normal vector. We also know a point of the plane \( \vec{r}_0 = (1, 2, 2) \). Let us define the vector \( \vec{r} = \langle x, y, z \rangle \), the equation of the tangent plane at \((1, 2, 2)\) is then:

\[ \nabla F(1, 2, 2) \cdot (\vec{r} - \vec{r}_0) = 0 \quad \Rightarrow \quad \langle 2, 4, -4 \rangle \cdot \langle x - 1, y - 2, z - 2 \rangle = 0, \]

that is:

\[ 2(x - 1) + 4(y - 2) - 4(z - 2) = 0. \]

4. To find the equation of the tangent plane of the cone, let us write it as a level curve:

\[ F(x, y, z) = 0, \quad \text{with} \quad F(x, y, z) = x^2 + y^2 - z^2. \]

Here again, the gradient vector of \( F \) gives a normal vector of the tangent plane. The gradient vector of \( F(x, y, z) \) is:

\[ \nabla F(x, y, z) = \langle F_x(x, y, z), F_y(x, y, z), F_z(x, y, z) \rangle = \langle 2x, 2y, -2z \rangle. \]

Let us pick \( \vec{r}_0 = \langle x_0, y_0, z_0 \rangle \) as a point of the cone. The gradient vector at this point is \( \nabla F(x_0, y_0, z_0) = \langle 2x_0, 2y_0, -2z_0 \rangle \) and the equation of the tangent plane at \((x_0, y_0, z_0)\) is:

\[ \nabla F(x_0, y_0, z_0) \cdot (\vec{r} - \vec{r}_0) = 0 \quad \Rightarrow \quad \langle 2x_0, 2y_0, -2z_0 \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0, \]

with \( \vec{r} = \langle x, y, z \rangle \). In order to check if the plane passes through the origin, we have to check if the point \( \vec{r} = (0, 0, 0) = \vec{0} \) is a point of the plane:

\[ \nabla F(x_0, y_0, z_0) \cdot (\vec{0} - \vec{r}_0) = 0 \quad \Rightarrow \quad \langle 2x_0, 2y_0, -2z_0 \rangle \cdot \langle -x_0, -y_0, -z_0 \rangle = 0. \]

This corresponds to the equation \( 2x_0^2 + 2y_0^2 - 2z_0^2 = 0 \), which we can rewrite \( x_0^2 + y_0^2 - z_0^2 = 0 \). Since the point \((x_0, y_0, z_0)\) is by definition a point of the cone, it has to satisfy the equation of the cone \( x^2 + y^2 = z^2 \). This implies that \( x_0^2 + y_0^2 - z_0^2 \) is zero and then the origin is a point of the tangent plane.