First midterm exam - Math2D
Solutions

04/30/2007

Name:
Student ID:
Discussion Section:

Notes, books and calculators are not allowed.

**Explain all your answers.**
Any result given without explanation will not receive credit.

1 Exercise 1 (10 points)

Let $\vec{A} = \langle -1, 2, 1 \rangle$, $\vec{B} = \langle 1, 0, -2 \rangle$, $\vec{C} = \langle 4, 0, 4 \rangle$ and $\vec{D} = \langle -1, -1, 1 \rangle$. Compute $\vec{D} \cdot ((\vec{A} - 2\vec{B}) \times \vec{C})$.

**Solution:** We get $\vec{A} - 2\vec{B} = \langle -3, 2, 5 \rangle$. Then

$$
(\vec{A} - 2\vec{B}) \times \vec{C} = \langle -3, 2, 5 \rangle \times \langle 4, 0, 4 \rangle = \begin{vmatrix}
-3 & 4 \\
2 & 0 \\
5 & 4
\end{vmatrix} = 8 \begin{vmatrix}
2 & 0 \\
5 & 4
\end{vmatrix} = 8 \cdot 16 = 8 \cdot 32 = \langle 8, 32, -8 \rangle
$$

Then $\vec{D} \cdot ((\vec{A} - 2\vec{B}) \times \vec{C}) = \langle -1, -1, 1 \rangle \cdot \langle 8, 32, -8 \rangle = -8 - 32 - 8 = -48.$
Do the following limits exist? (justify your answer):

1) \( \lim_{(x,y) \to (0,0)} \frac{\sin(\sqrt{x^2 + y^2})}{4\sqrt{x^2 + y^2}} \)

2) \( \lim_{(x,y) \to (0,0)} \frac{ye^x}{3xy + 1} \)

3) \( \lim_{(x,y) \to (0,0)} \frac{xy}{3x^2 + y^2} \)

Hint: for the first one, you can use the following result: \( \lim_{z \to 0} \frac{\sin(z)}{z} = 1 \).

Solution:

1) Let us rewrite the limit in polar coordinates:

\[
\lim_{(x,y) \to (0,0)} \frac{\sin(\sqrt{x^2 + y^2})}{4\sqrt{x^2 + y^2}} = \lim_{r \to 0} \frac{\sin(r)}{4r} = \frac{1}{4},
\]

using the result given as a hint. Therefore the limit exists and equals 1/4.

2) The function is well-defined at (0, 0). Therefore the limit is simply the value of the function at (0, 0) which is \( \frac{ye^x}{3xy + 1} = \frac{0e^0}{0+1} = \frac{0}{1} = 0 \).

3) We can check the limits on different paths.

\[
\lim_{y \to 0} \lim_{x \to 0} \frac{xy}{3x^2 + y^2} = \lim_{y \to 0} 0 = 0,
\]

\[
\lim_{x \to 0} \lim_{y \to 0} \frac{xy}{3x^2 + y^2} = \lim_{x \to 0} 0 = 0,
\]

Let \( y = mx \),

\[
\lim_{x \to 0} \lim_{y \to 0} \frac{mx^2}{3x^2 + m^2x^2} = \lim_{x \to 0} \frac{m}{3 + m^2} = \frac{m}{3 + m^2}
\]

We get different limits depending on the value of \( m \). Therefore the limit doesn’t exist.
3 Exercise 3 (20 points)

Compute all the partial derivatives of \( f(x, y) \) and \( g(x, y) \), with

\[
\begin{align*}
    f(x, y) &= \frac{3x}{x + y}, \\
    g(x, y) &= 4 - e^{x^2y}.
\end{align*}
\]

Solution:

\[
\begin{align*}
    f_x(x, y) &= \frac{3}{x + y} - \frac{3x}{(x + y)^2} = \frac{3y}{(x + y)^2}, \\
    f_y(x, y) &= -\frac{3x}{(x + y)^2}, \\
    g_x(x, y) &= -2xye^{x^2y}, \\
    g_y(x, y) &= -x^2e^{x^2y}.
\end{align*}
\]
4 Exercise 4 (20 points)

Let \( A = (2, 0, 1) \) and \( B = (1, 1, 1) \) be two points of \( \mathbb{R}^3 \).

1. Find the symmetric equation of the line passing through \( A \) and \( B \).

2. Find the equation of the plane that passes through \( A \) and is perpendicular to line passing through \( A \) and \( B \).

Solution:
1) The vector \( \vec{AB} \) can be used as direction vector: \( \vec{AB} = \langle -1, 1, 0 \rangle \). Since the last coordinate is 0, it means the z-coordinate doesn’t vary on the line. Therefore one of symmetric equations is \( z = 1 \). The others are:

\[
\frac{x - 2}{-1} = \frac{y - 0}{1} \rightarrow 2 - x = y.
\]

2) Since the plane is perpendicular to the line, the direction vector \( \vec{AB} \) of the line can be used as a normal vector of the plane. Therefore the equation of the plane is \( \vec{AB} \cdot (\vec{r} - \vec{r}_0) = 0 \), with \( \vec{r} = \langle -1, 1, 0 \rangle \) and \( \vec{r}_0 = \vec{OA} = \langle 2, 0, 1 \rangle \) (we could also have used \( \vec{r}_0 = \vec{OB} \)). Then:

\[
\vec{AB} \cdot (\vec{r} - \vec{r}_0) = 0 \rightarrow -(x - 2) + y = 0
\]
5 Exercise 5 (20 points)

Let a curve be described by the following parametric equations:

\[ x = 5, \quad y = 3 + 2e^{2t}, \quad z = e^{2t}. \]

a) Find the parametric equation of the line tangent to the curve at \( t = 0 \).

b) What is the length of the curve between \( t = 0 \) and \( t = 1 \) ?

**Solution:**
a) We have \( \vec{r}(t) = (5, 3 + 2e^{2t}, e^{2t}). \) We can write the parametric equation of the tangent line at \( t = 0 \) as \( \vec{v}(q) = \vec{r}(0) + q\vec{r}'(0) \), with \( q \) the parameter. Since \( \vec{r}(0) = (5, 5, 1) \) and \( \vec{r}'(t) = (0, 4e^{2t}, 2e^{2t}) \), which gives \( \vec{r}'(0) = (0, 4, 2) \), we get

\[ \vec{v}(q) = (5, 5 + 4q, 1 + 2q). \]

b) The length is given by \( \int_0^1 |\vec{r}'(t)| \, dt \). We know \( \vec{r}'(t) = (0, 4e^{2t}, 2e^{2t}) \), then \( |\vec{r}'(t)| = \sqrt{0^2 + (4e^{2t})^2 + (2e^{2t})^2} = e^{2t}\sqrt{20} \). Then

\[ \int_0^1 |\vec{r}'(t)| \, dt = \int_0^1 e^{2t}\sqrt{20} \, dt = \left. \frac{\sqrt{20}}{2} e^{2t} \right|_0^1 = \frac{\sqrt{20}}{2} (e^2 - 1). \]