Second midterm exam

05/21/2007

Student name:
Student ID:
Discussion Section:

Calculators, book and notes are not allowed.

**Explain all your answers.** Unsupported answers will not receive full credit.

1 Exercise 1 (25 points)

Find the equation of the tangent plane to the function \( f(x, y) = \cos(x + \pi) + \frac{y}{x+y}e^{-2x} \) at \((0, 2)\). Does the plane pass through the origin?

**Solution:**

The equation of the plane is \( z - f(0, 2) = f_x(0, 2)x + f_y(0, 2)(y - 2) \), with

\[
\begin{align*}
    f_x(x, y) &= -\sin(x + \pi) - \frac{y}{(x+y)^2}e^{-2x} - \frac{2y}{x+y}e^{-2x}, \\
    f_y(x, y) &= \frac{1}{x+y}e^{-2x} - \frac{y}{(x+y)^2}e^{-2x}.
\end{align*}
\]

Then \( f_x(0, 2) = -\frac{5}{2} \) and \( f_y(0, 2) = 0 \). Also \( f(0, 2) = 0 \). Then the equation of the plane is \( z = -\frac{5}{2}x \).

We can see that \((0,0,0)\) is a solution of the equation of the plane. Therefore the plane passes through the origin.
2 Exercise 2 (25 points)

Consider the following equation:

\[ c \frac{\partial f(x, t)}{\partial x} = - \frac{\partial f(x, t)}{\partial t} \]

with \( c \) a constant. Show that \( f(x, t) = e^{-(x-ct)^2} \) is a solution of this equation.

Solution:

We have, for \( f(x, t) = e^{-(x-ct)^2} \)

\[
\frac{\partial f(x, t)}{\partial x} = -2(x - ct)e^{-(x-ct)^2},
\]

\[
\frac{\partial f(x, t)}{\partial t} = 2c(x - ct)e^{-(x-ct)^2}.
\]

Therefore \( c \frac{\partial f(x, t)}{\partial x} \) equals \( - \frac{\partial f(x, t)}{\partial t} \).
3 Exercise 3-(25 points)

Let \( f(x, y) \) be a function of 2 variables.

1. Write the general expression of the differential \( df \).

2. Assume now that \( f(x, y) \) is of the form \( f(x, y) = g(x - y) \), for some function \( g \). Let us introduce a new set of variables \( u = x + y \) and \( v = x - y \) (that is \( x = (u + v)/2 \) and \( y = (u - v)/2 \)). Using chain rule, show that

\[
\frac{\partial f}{\partial u} = \frac{1}{2} \frac{\partial f}{\partial x} + \frac{1}{2} \frac{\partial f}{\partial y}.
\]

3. Since \( f(x, y) = g(x - y) = g(v) \) depends only on \( v \), use the previous result to show that

\[
\frac{\partial f}{\partial x} = -\frac{\partial f}{\partial y}.
\]

Solution:

1. \( df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \).

2. Since \( x \) and \( y \) depend on \( u \) and \( v \), \( f(x, y) \) also depends on \( u \) and \( v \). We can then compute \( \frac{\partial f}{\partial u} \) using chain rule:

\[
\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} = \frac{1}{2} \frac{\partial f}{\partial x} + \frac{1}{2} \frac{\partial f}{\partial y},
\]

since \( \frac{\partial x}{\partial u} = \frac{\partial y}{\partial u} = 1/2 \) (using \( x = (u + v)/2 \) and \( y = (u - v)/2 \)).

3. Since \( f(x, y) \) does not depend on \( u \) (\( f(x, y) = g(v) \)), we then have

\[
0 = \frac{1}{2} \frac{\partial f}{\partial x} + \frac{1}{2} \frac{\partial f}{\partial y},
\]

which gives the result we need.

Remark: (not required in the exam) We can notice that this result gives also an answer to the previous exercise. Using \( y = ct \), we can see that \( c \frac{\partial f(x, t)}{\partial x} = -\frac{\partial f(x, t)}{\partial t} \) is the same as \( \frac{\partial f(x, y)}{\partial x} = -\frac{\partial f(x, y)}{\partial y} \). It means that \( f(x, y) \) must be a function of \( x - y \). Since \( f(x, t) = e^{-(x-ct)^2} = e^{-(x-y)^2} \) is a function of \( x - y \), it is a solution of the equation \( \frac{\partial f(x, y)}{\partial x} = -\frac{\partial f(x, y)}{\partial y} \).
4 Exercise 4-(25 points)

Consider the function \( f(x, y) = xy^2 + (x + \frac{9}{24})^2 \) on the domain \( x^2 + y^2 \leq 1 \).

1. Find all the critical point(s) on \( x^2 + y^2 \leq 1 \). Are these points local maximum, local minimum or neither?

2. Find the maximum and minimum at the boundary \( x^2 + y^2 = 1 \) (hint: use Lagrange Multiplier).

3. What are the absolute maximum and absolute minimum on \( x^2 + y^2 \leq 1 \).

Solution:

1. The critical points are given by \( \vec{\nabla} f(x, y) = 0 \). This gives the following 2 equations:
   \[
   f_x = y^2 + 2(x + \frac{9}{24}) = 0, \\
   f_y = 2xy = 0.
   \]
   The second equation gives 2 solutions \( x = 0 \) or \( y = 0 \). If \( x = 0 \), the first equation becomes \( y^2 + 2\frac{9}{24} = 0 \), which doesn’t have a solution. So \( x = 0 \) is not possible.
   If \( y = 0 \), the first equation gives \( 2(x + \frac{9}{24}) = 0 \), that is \( x = -\frac{9}{24} \). There is only one critical point \( \left(-\frac{9}{24}, 0\right) \).

   Then:
   \[
   \begin{vmatrix}
   f_{xx} & f_{xy} \\
   f_{yx} & f_{yy}
   \end{vmatrix} = \begin{vmatrix}
   2 & 2y \\
   2y & 2x
   \end{vmatrix} = 4x - 4y^2 = -\frac{9}{6} \text{ at } \left(-\frac{9}{24}, 0\right).
   \]
   We get a negative value. The point is then a saddle-point (neither max, nor min).

2. The max/min at the boundaries \( g(x, y) = x^2 + y^2 = 1 \) are given by \( \vec{\nabla} f(x, y) = \lambda \vec{\nabla} g(x, y) \):
   \[
   y^2 + 2(x + \frac{9}{24}) = 2\lambda x, \\
   2xy = 2\lambda y.
   \]
   The second equation gives 2 solutions: \( y = 0 \), which implies \( x = \pm 1 \) (through constraint \( x^2 + y^2 = 1 \)), or \( x = \lambda \). Therefore the first equation becomes (replacing \( y^2 \) by \( 1 - x^2 \), because of the constraint):
   \[
   1 - x^2 + 2\left(x + \frac{9}{24}\right) = 2x^2 \rightarrow -3x^2 + 2x + \frac{21}{12} = 0,
   \]
   which has 2 roots: \( x = \frac{7}{6} \) (not in the domain since \( x^2 + y^2 = 1 \) imposes \( -1 \leq x \leq 1 \)), and \( x = -\frac{1}{2} \). If \( x = -\frac{1}{2} \) then \( y = \pm \frac{\sqrt{3}}{2} \) (using \( x^2 + y^2 = 1 \)).
   We get 4 solutions on the boundaries: \((1,0), (-1,0), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, -\frac{\sqrt{3}}{2})\). The value of \( f \) at these points is
   \[
   f(1,0) = \left(\frac{33}{24}\right)^2, \\
   f(-1,0) = \left(\frac{15}{24}\right)^2, \\
   f(-\frac{1}{2}, \frac{\sqrt{3}}{2}) = f(-\frac{1}{2}, -\frac{\sqrt{3}}{2}) = -\frac{3}{8} + \left(\frac{3}{24}\right)^2.
   \]
The maximum is clearly at $(1, 0)$ and the minimum at $(-\frac{1}{2}, \pm \frac{\sqrt{3}}{2})$.

3. Since there is no critical point being max/min, the absolute and minimum are on the boundaries. We saw in question 2 that the max corresponds to the point $(1, 0)$ and the min to $(-\frac{1}{2}, \pm \frac{\sqrt{3}}{2})$. 