1. Exercise 6, section 3.1:
The set of all polynomials $P$ contains vectors of the form: $v \in P$, $v = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$, with real coefficients $a_0, a_1, \ldots$

it is very easy to see that, with the usual addition and scalar multiplication, the set $P$ is closed under these two operations and satisfies the axioms of the vectors (the proof is very straightforward).

**Remark:** Regarding the axioms of the vector space, we define the additive identity $0$ as being the polynomial whose coefficients are all zero $a_0 = a_1 = a_2 = \ldots = 0$. Also, for each $v \in P$, such that $v = a_0 + a_1 x + a_2 x^2 \ldots$, we define the additive inverse $-v$ as the polynomial $-v = (-a_0) + (-a_1)x + (-a_2)x^2 + \ldots$

2. Exercise 10, section 3.1:
The space $S$ given in this exercise is not a vector space. At least one of the axioms fails.

More precisely, the vector space does not have an additive identity $0$ such that, for any $v \in P$, we have $v + 0 = v$. We can easily see this: let $v \in S$ be a vector such that $v = (x_1, x_2)$ and $x_2 \neq 0$. Let us assume that $0 = (y_1, y_2)$ for given $y_1$ and $y_2$. This gives:

$$v + 0 = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, 0),$$

according to the definition of the addition $\oplus$. We see that, whatever $y_2$ is, the addition $v + 0$ will give a vector whose second component will always be 0. So the axiom $v + 0 = v$ will always fail if $x_2 \neq 0$. Therefore there is no additive identity.

3. Exercise 1(a), section 3.2:
Let $S = \{(x_1, x_2)^T \in \mathbb{R}^2 \mid x_1 + x_2 = 0\}$. We need to check that this set is closed under the addition and scalar multiplication.

Let $v = (x_1, x_2)^T$ be an element of $S$ and $\alpha$ is a scalar. Then $\alpha v = \alpha(x_1, x_2)^T = (\alpha x_1, \alpha x_2)^T$. This vector is still an element of $S$ since $(\alpha x_1) + (\alpha x_2) = \alpha(x_1 + x_2) = \alpha.0 = 0$ (because $v$ is an element of $S$).

Let $v = (x_1, x_2)^T$ and $u = (y_1, y_2)^T$ be two elements of $S$. Then $v + u = (x_1 + y_1, x_2 + y_2)^T$, which is still an element of $S$ since $(x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) = 0 + 0 = 0$ (because $v$ and $u$ were elements of $S$).

The set $S$ is closed under the addition and scalar multiplication. It is then a subspace of $\mathbb{R}^2$.

4. Exercise 1(b), section 3.2:
Let $S = \{(x_1, x_2)^T \in \mathbb{R}^2 \mid x_1 x_2 = 0\}$. We need to check that this set is closed under the addition and scalar multiplication.

Let $v = (x_1, x_2)^T$ and $u = (y_1, y_2)^T$ be two elements of $S$. Then $v + u = (x_1 + y_1, x_2 + y_2)^T$, which is not an element of $S$. We need the product of the components to be zero: $(x_1 + y_1)(x_2 + y_2) = 0$. All we know is that $x_1 x_2 = 0$ and $y_1 y_2 = 0$ (since $v$ and $u$ are elements of $S$). This leaves us with: $(x_1 + y_1)(x_2 + y_2) = x_1 y_2 + y_1 x_2$ which might not be
zero (for example, we can pick \(v = (0, 1)^T\) and \(u = (1, 0)^T\) being 2 elements of \(S\). In this case \(x_1y_2 + y_1x_2 = 1 \neq 0\). The set \(S\) is not closed under the addition and therefore is not a subspace of \(R^2\).

5. Exercise 3(a), section 3.2:
Let \(S\) be the set \(S = \{A \in R^{2\times2} \mid A\text{ is diagonal}\}\). We need to check that this set is closed under the addition and scalar multiplication.

Let \(A \in S\) and \(x\) be a scalar. \(A\) is a matrix of the form \(A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\). Then \(xA = x\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} xa_1 & 0 \\ 0 & xa_2 \end{pmatrix}\) which is still an element of \(S\).

Let \(A \in S\) and \(B \in S\). \(A\) is a matrix of the form \(A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\) and \(B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}\). Then \(A + B = \begin{pmatrix} a_1 + b_1 & 0 \\ 0 & a_2 + b_2 \end{pmatrix}\) which is still an element of \(S\).

\(S\) is closed under the addition and the scalar multiplication. Therefore \(S\) is a subspace of \(R^{2\times2}\).

6. Exercise 4(a), section 3.2:
The nullspace of the matrix \(A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\) is the set of vectors \(N(A) = \{v \in R^2 \mid Av = 0\}\).

It is given by the solution of the system of equations:

\[
\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]

There is only one solution to this system of equation, which is: \(v = (0, 0)^T = 0\). The nullspace of the matrix \(A\) is then simply \(N(A) = \{(0, 0)^T\}\).

**Remark:** We could have answered this question faster since \(\det(A) = 1 \neq 0\) therefore the matrix is not singular and the nullspace of \(A\) contains then only the vector 0.

7. Exercise 6(a), section 3.2:
Let \(S = \{f \in C[-1,1] \mid f(-1) = f(1)\}\). We need to check that the set \(S\) is closed under the addition and scalar multiplication.

Let \(f \in S\) and \(x\) be a scalar. The scalar multiplication gives \((xf)(x) = xf(x)\) for any \(x\) in \([-1, 1]\). Since \(f\) is continuous on \([-1, 1]\), \(xf\) is also continuous on that interval. Also we still have \(xf(1) = xf(-1)\). The set is closed under the scalar multiplication.

Let \(f \in S\) and \(g \in S\). The addition gives \((f + g)(x) = f(x) + g(x)\) for any \(x\) in \([-1, 1]\). Since \(f\) and \(g\) are continuous on \([-1,1]\), \(f + g\) is also continuous on the same interval. Also we still have \(f(-1) + g(-1) = f(1) + g(1)\). The set is closed under the addition. Therefore \(S\) is a subspace of \(C[-1,1]\).

8. Let \(S = \{f \in C[-1,1] \mid f\text{ is odd}\}\) \(\{(f\text{ odd on }[-1,1]\) means that \(f(x) = -f(-x)\) for any \(x \in [-1,1]\)\). The proof in this case is the same as the proof given in the previous exercise.
We can conclude that \(S\) is a subspace of \(C[-1,1]\).

9. Let \(S = \{f \in C[-1,1] \mid f\text{ nondecreasing on }[-1,1]\}\) \(\{(f\text{ nondecreasing on }[-1,1]\) means that, if \(x \geq y\) then \(f(x) \geq f(y)\) for any \(x\) and \(y\) in \([-1, 1]\)\).
We can easily see that the set is not closed under the scalar multiplication. For example, let \(f \in S\) and \(x\) be a scalar such that \(x < 0\). Since \(f\) is nondecreasing, we have \(f(x) \geq f(y)\), if \(x \geq y\). But then, if \(x < 0\), we get \(\alpha f(x) \leq \alpha f(y)\), if \(x \geq y\). So, if \(x < 0\), the function \(\alpha f\) is nonincreasing and can not be an element of \(S\).
The set \(S\) is not a subspace of \(C[-1,1]\).