1. Section 3.4, exercise 3:

(a) Let us consider the matrix \( A = (x_1, x_2) \) with \( x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, x_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \). The determinant is:

\[
\text{det}(A) = \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2.
\]

It is non-zero, so the vectors are linearly independent. Moreover, \( R^2 \) is a vector space of dimension 2. Then any basis of \( R^2 \) has 2 vectors. Therefore \( \{x_1, x_2\} \) is a basis for \( R^2 \).

(b) Since \( \{x_1, x_2\} \) is a basis for \( R^2 \) any additional vector would be, by definition, linearly dependent of the vectors of the basis. Then \( \{x_1, x_2, x_3\} \) are linearly dependent. (c) Considering (a), (b), we have \( \text{Span}(x_1, x_2, x_3) = \text{Span}(x_1, x_2) \) (since \( x_3 \) is linearly dependent of \( \{x_1, x_2\} \)). The set of vectors \( \{x_1, x_2\} \) being a basis of \( R^2 \), we have \( \text{Span}(x_1, x_2, x_3) = \text{Span}(x_1, x_2) = R^2 \) and therefore \( \text{Span}(x_1, x_2, x_3) \) is of dimension 2.

2. Section 3.4, exercise 4:

Let us consider the matrix \( A = (x_1, x_2, x_3) \) with \( x_1 = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}, x_2 = \begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix}, x_3 = \begin{pmatrix} -6 \\ 4 \\ -8 \end{pmatrix} \). The determinant is:

\[
\text{det}(A) = \begin{vmatrix} 3 & -3 & -6 \\ -2 & 1 & 4 \\ 4 & -4 & -8 \end{vmatrix} = 0.
\]

It is zero, so the vectors are linearly dependent: we can actually see that \( x_3 = 2x_2 \). Then \( \text{Span}(x_1, x_2, x_3) = \text{Span}(x_1, x_2) \). But \( x_1 \) and \( x_2 \) are also linearly dependent, since \( x_1 = -x_2 \). Then \( \text{Span}(x_1, x_2) = \text{Span}(x_1) \). The spanned set is then of dimension 1.

3. Section 3.4, exercise 7:

Let \( v \) be a vector of \( S \). We can write it as:

\[
v = (a + b, a - b + 2c, b, c)^T = a(1, 1, 0, 0)^T + b(1, -1, 1, 0)^T + c(0, 2, 0, 1)^T
\]

\[
= av_1 + bv_2 + cv_3,
\]

with \( v_1 = (1, 1, 0, 0)^T, v_2 = (1, -1, 1, 0)^T \) and \( v_3 = (0, 2, 0, 1)^T \). These vectors form a basis for \( S \) since any vector \( v \in S \) can be written as a linear combination of \( v_1, v_2, v_3 \). We can also prove that these vectors are linearly independent (since the solution of \( c_1v_1 + c_2v_2 + c_3v_3 = 0 \) implies \( c_1 = c_2 = c_3 = 0 \)). Then \( \{v_1, v_2, v_3\} \) form a basis of \( S \).
4. Section 3.4, exercise 8:
(a) \( \mathbb{R}^3 \) is a vector space of dimension 3. So two vectors \( x_1 \) and \( x_2 \) cannot span \( \mathbb{R}^3 \). Then \( \{x_1, x_2\} \) cannot be a basis for \( \mathbb{R}^3 \).
(b) The set \( \{x_1, x_2, x_3\} \) would form a basis if and only if the vectors are linearly independent.
(c) We can pick, for example, \( x_3 = (1, 0, 0)^T \). Then \( \{x_1, x_2, x_3\} \) are linearly independent and therefore form a basis.

5. Section 3.4, exercise 11:
Let \( v \) be a vector of \( S \). We can write \( v \) as:
\[
v = ax^2 + bx + 2a + 3b = a(x^2 + 2) + b(x + 3) = av_1 + bv_2,
\]
with \( v_1 = x^2 + 2 \) and \( v_2 = x + 3 \). The vectors \( v_1 \) and \( v_2 \) span by definition \( S \) (since the vectors of \( S \) are written as a combination of \( v_1 \) and \( v_2 \)). They are also linearly independent (for example, the Wronskian would be \( W = 2 \) at \( x = 0 \)). Therefore, they form a basis for \( S \).

6. Section 3.5, exercise 1(b):
We can see that \( u_1 = e_1 + 2e_2 \) and \( u_2 = 2e_1 + 5e_2 \). The transition matrix from \( U = [u_1, u_2] \) to \( E = [e_1, e_2] \) is
\[
P_{U \rightarrow E} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.
\]

7. Section 3.5, exercise 5:
(a) The transition matrix from \( U = [u_1, u_2, u_3] \) to \( E = [e_1, e_2, e_3] \) is:
\[
P_{U \rightarrow E} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}.
\]
Then the transition matrix from \( E \) to \( U \) is:
\[
P_{E \rightarrow U} = P_{U \rightarrow E}^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.
\]
(b) The coordinates in basis \( U \) are given by:
(i) \( P_{E \rightarrow U}(3, 2, 5)^T = (1, -4, 3)^T \),
(ii) \( P_{E \rightarrow U}(1, 1, 2)^T = (0, -1, 1)^T \),
(iii) \( P_{E \rightarrow U}(2, 3, 2)^T = (2, 2, -1)^T \).

8. Section 3.5, exercise 9:
(a) We can notice that \( 2x - 1 = 2(x) - (1) \) and \( 2x + 1 = 2(x) + (1) \). The transition matrix from \( A = [2x - 1, 2x + 1] \) to \( B = [x, 1] \) is:
\[
P_{A \rightarrow B} = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}.
\]
(b) The transition matrix from \( B \) to \( A \) is then:
\[
P_{B \rightarrow A} = P_{A \rightarrow B}^{-1} = \frac{1}{4} \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}.
\]
9. Section 3.6, exercise 1(a):
   (i) The row space of the matrix is \( \text{Span}(v_1, v_2, v_3) \) with \( v_1 = (1, 3, 2), v_2 = (2, 1, 4) \) and 
       \( v_3 = (4, 7, 8) \). But \( v_3 = 2v_1 + v_2 \). Therefore the vectors are not linearly independent and 
       \( \text{Span}(v_1, v_2, v_3) = \text{Span}(v_1, v_2) \). Since \( v_1 \) and \( v_2 \) are independent, they form a basis for 
       the row space.
   (ii) The column space of the matrix is \( \text{Span}(u_1, u_2, u_3) \) with \( u_1 = (1, 2, 4)^T, u_2 = (3, 1, 7)^T \) 
       and \( u_3 = (2, 4, 8)^T \). But \( u_3 = 2u_1 \). Therefore the vectors are not linearly independent and 
       \( \text{Span}(u_1, u_2, u_3) = \text{Span}(u_1, u_2) \). Since \( u_1 \) and \( u_2 \) are independent, they form a basis for 
       the column space.
   (iii) The row space has a dimension 2, so the rank of the matrix is 2. We should expect 
       the dimension of the nullspace to be 3-2=1 (the matrix being 3x3 matrix). We need to 
       solve \( Av = 0 \) with \( A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{pmatrix} \). This gives:

\[
\begin{pmatrix}
1 & 3 & 2 \\
2 & 1 & 4 \\
4 & 7 & 8
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 3 & 2 \\
2 & 1 & 4 \\
0 & 0 & 0
\end{pmatrix},
\rightarrow
\begin{pmatrix}
1 & 3 & 2 \\
0 & -5 & 0 \\
0 & -5 & 0
\end{pmatrix}
\]

we can drop the last row since we proved (see question (i)) that the last row vector was 
   a linear combination of the others. We then have 2 equations for 3 unknowns 
   \( x_1, x_2, x_3 \). We can decide that \( x_3 \) is a free parameter. We then get that \( x_2 = 0 \) and \( x_1 = -2x_3 \). The 
   nullspace of \( A \) is then given by

\[
N(A) = \{ v \in \mathbb{R}^3, v = x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} , x_3 \in \mathbb{R} \} = \text{Span}\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}
\]

10. Section 3.6, exercise 1(b):
   (i) The row space of the matrix is \( \text{Span}(v_1, v_2, v_3) \) with \( v_1 = (-3, 1, 3, 4), v_2 = (1, 2, -1, -2) \) 
       and \( v_3 = (-3, 8, 4, 2) \). The vectors are linearly independent. We can prove this by solving 
       the equation: \( c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \). This gives

\[
\begin{pmatrix}
-3 & 1 & -3 \\
1 & 2 & 8 \\
3 & -1 & 4 \\
4 & -2 & 2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-3 & 1 & -3 \\
1 & 2 & 8 \\
3 & -1 & 4 \\
0 & 0 & 0
\end{pmatrix},
\]

The matrix having 4 row vectors in \( \mathbb{R}^3 \), they must be linearly dependent (\( \mathbb{R}^3 \) is a space 
   of dimension 3, so any set of more than 3 vectors is linearly dependent). We can pick the 
   last row vector as a combination of the 3 others and replace the row by zero in the matrix. 
   We just have to solve now:

\[
\begin{pmatrix}
-3 & 1 & -3 \\
1 & 2 & 8 \\
3 & -1 & 4 \\
0 & 0 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
-3 & 1 & 3 \\
1 & 2 & 8 \\
3 & -1 & 4 \\
0 & 0 & 0
\end{pmatrix}.
\]

But the matrix has a determinant non zero. So the solution is \( c_1 = c_2 = c_3 = 0 \) and the 
   vectors are linearly independent and form a basis for the row space. The dimension of the 
   row space is then 3.
   (ii) The column space of the matrix is \( \text{Span}(v_1, v_2, v_3, v_4) \) with \( v_1 = (-3, 1, -3)^T, v_2 = 

\]
(1, 2, 8)^T$, $v_3 = (3, -1, 4)^T$ and $v_4 = (4, -2, 2)^T$. The vectors are linearly dependent, since we have 4 vectors in $R^3$ (being a space of dimension 3). Then $Span(v_1, v_2, v_3, v_4) = Span(v_1, v_2, v_3)$ (I took $v_4$ out of the spanning set. Any other vector $v_1$, $v_2$ or $v_3$ would have worked too). We can see that $v_1$, $v_2$ and $v_3$ are independent since the determinant of the matrix:

\[
\begin{vmatrix}
-3 & 1 & 3 \\
1 & 2 & -1 \\
-3 & 8 & 4
\end{vmatrix} = -10
\]

is non zero. The vectors $v_1$, $v_2$ and $v_3$ form then a basis for the column space.

(iii) To find the nullspace, we need to solve the following equation:

\[
\left( \begin{array}{cccc|c}
-3 & 1 & 3 & 4 & 0 \\
1 & 2 & -1 & -2 & 0 \\
-3 & 8 & 4 & 2 & 0 \\
\end{array} \right) \rightarrow \left( \begin{array}{cccc|c}
-3 & 1 & 3 & 4 & 0 \\
1 & 2 & -1 & -2 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{array} \right) \\
\rightarrow \left( \begin{array}{cccc|c}
-3 & 15 & 0 & 0 & 0 \\
0 & 7 & 0 & -2 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{array} \right),
\]

after using several row operations. This gives that: \(x_3 = 0, x_1 = 5x_2, x_4 = \frac{7}{2}x_2\). We can pick $x_2$ as a free parameter. The nullspace is then:

\[
N(A) = \{ v \in R^3, v = x_2 \begin{pmatrix}
5 \\
1 \\
\frac{7}{2}
\end{pmatrix}, x_2 \in R \} = Span \left\{ \begin{pmatrix}
5 \\
1 \\
\frac{7}{2}
\end{pmatrix} \right\}
\]

11. Section 3.6, exercise 4(c):

The column space of $A$ is $Span(v_1, v_2)$ with $v_1 = (2, 3)^T$, $v_2 = (1, 4)^T$. We can easily check that they are linearly independent. We can also see that $b = 2v_1$. So the vector is in the column space of $A$. The system is then consistent (we can even add that, since the vectors are linearly independent, there exists only one solution, being $v = (2, 0)^T$).

12. Section 3.6, exercise 4(c):

The column space of $A$ is $Span(v_1, v_2, v_3)$ with $v_1 = (1, 1, 1)^T$, $v_2 = (1, 1, 1)^T$, and $v_3 = (2, 2, 2)^T$. Since $v_1 = v_2 = \frac{1}{2}v_3$, we have $Span(v_1, v_2, v_3) = Span(v_1)$. The vector $b$ is not in the column space of $A$. The system is not consistent.