Homework 5
Solutions

1. Section 5.2 exercise 1(a):
The transpose of the matrix $A$ is:

\[ A^T = \begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix}. \]

The range of $A^T$ is given by all the vectors $A^T \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)$, for any $x_1, x_2 \in \mathbb{R}$. We see that it corresponds to vectors of the form:

\[ A^T \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \begin{pmatrix} 3x_1 + 6x_2^2 \\ 4x_1 + 8x_2 \end{pmatrix} = (x_1 + 2x_2) \begin{pmatrix} 3 \\ 4 \end{pmatrix}. \]

Then $R(A^T) = Span\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix} \}$. To find the nullspace of $A$, we can either look for all the vectors such that $Av = 0$ or use the fact that $N(A) = R(A^T)^\perp$. We are then looking for vectors $v$ of $\mathbb{R}^2$ such that $\langle v, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \rangle = 0$. This gives $v = a \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, with $a$ being any real constant. Therefore $N(A) = Span\{ \begin{pmatrix} -4 \\ 3 \end{pmatrix} \}$.

In the same way, we can prove that $R(A) = Span\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \}$ and $N(A^T) = Span\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \}$.

2. Section 5.2, exercise 2:

Let $S = Span\{v_1\}$ with $v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

(a) The subspace $S^\perp$ is given by all the vectors $v \in \mathbb{R}^3$ such that $\langle v, v_1 \rangle = 0$. This gives vectors of the form $v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ such that $x_2 = x_1 + x_2$. In other words, $v = x_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. That is: $S^\perp = Span\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \}$. The 2 vectors in the spanning set give a basis (they are linearly independent, so they must form a basis of the spanning set).

(b) The 2 vectors of $S^\perp$ describe a plane in $\mathbb{R}^3$. The vectors in $S$ being orthogonal to it, they are then parallel to the normal vector of the plane.

3. Section 5.2, exercise 3:

(a) The 2 vectors $x$ and $y$ span the set $c_1 x + c_2 y$. In other words, $S = R(A^T)$ where $A^T$ is the
matrix whose column vectors are $x$ and $y$. Therefore $S^\perp$ is given by $S^\perp = R(A^T)^\perp = N(A)$.

(b) Using the above result, we know that $S^\perp = N(A)$ where

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}.$$ 

We can find then $N(A) = \text{Span}\{ \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} \}$. 

4. Section 5.2, exercise 6:
If $(3, 1, 2)$ is in the row space of a matrix $A$, then $(3, 1, 2)^T$ must be in the column space of $A^T$. This implies that $(3, 1, 2)^T$ is in $R(A^T)$. We know that $N(A) = R(A^T)^\perp$. So any vector of $N(A)$ is orthogonal to any vector of $R(A^T)$. We can easily see that $(3, 1, 2)^T$ and $(2, 1, 1)^T$ are not orthogonal. Therefore it’s impossible that $(2, 1, 1)^T$ is in $N(A)$ if $(3, 1, 2)$ is in the row space of $A$.

5. Section 5.2, exercise 14:
(a) if $v \in N(B)$, we have then $Bv = 0$, and this implies that $Cv = ABv = A(Bv) = A0 = 0$. Therefore $v$ is also in $N(C)$. If all the elements of $N(B)$ are also elements of $N(C)$, we can conclude that $N(B)$ is a subset of $N(C)$. To make sure it is a also subspace of $N(C)$, we should verify the closure relations (it is very easy to check that it is closed under the addition and scalar multiplication).

(b) Let $v \in N(C)^\perp$. This implies that $Cv \neq 0$. Therefore $Cv = ABv \neq 0$. The vector $v$ can not be in $N(B)$ (otherwise $ABv = 0$), so $v \in N(B)^\perp$. Then, any element of $N(C)^\perp$ is also an element of $N(B)^\perp$. $N(C)^\perp$ is then a subset $N(B)^\perp$ (once again, the closure relation are very easy to check, so we can conclude that $N(C)^\perp$ is a subspace of $N(B)^\perp$.

Since $N(C)^\perp = R(C^T)$ and $N(B)^\perp = R(B^T)$, we can conclude that $R(C^T)$ is a subspace of $R(B^T)$.

6. Section 5.4, exercise 2:
(a) $\cos(\theta) = \frac{|x \cdot y|}{\|x\| \|y\|} = \frac{12}{\sqrt{3} \sqrt{2}} = \frac{\sqrt{6}}{2}$, this gives $\theta = \pi/4$.

(b) $p = \frac{\langle x, y \rangle}{\langle y, y \rangle} y = \frac{12}{72} y = \frac{1}{6} y$.

(c) $x - p = x - \frac{1}{6} y$. Then $\langle x - p, p \rangle = \frac{1}{6} \langle x, y \rangle - \frac{1}{36} \langle y, y \rangle = \frac{12}{6} - \frac{72}{36} = 0$.

(d) $\|x - p\| = \sqrt{2}$, $\|x\| = 2$ and $\|p\| = \sqrt{2}$. We then have $\|p\|^2 + \|x - p\|^2 = \|x\|^2$.

7. Section 5.4, exercise 8:
(a) $\cos(\theta) = \frac{\langle 1, x \rangle}{\|1\| \|x\|} = \frac{\sqrt{3}}{3}$, this gives $\theta = \pi/6$. (we need to compute first $\|1\| = 1$, $\|x\| = \frac{1}{\sqrt{3}}$ and $\langle 1, x \rangle = \frac{1}{2}$).

(b) $p = \frac{\langle 1, x \rangle}{\langle x, x \rangle} x = \frac{2}{3} x$. We can check that $\langle 1 - p, p \rangle = \langle 1 - \frac{3}{2} x, \frac{2}{3} x \rangle = 0$.

(c) $\|1\| = 1$, $\|p\| = \frac{\sqrt{3}}{3}$ and $\|1 - p\| = \frac{1}{2}$. We then have $\|p\|^2 + \|1 - p\|^2 = \|1\|^2$.

8. Section 5.4, exercise 10:
The inner product of $x$ and $x^2$ will give

$$\langle x, x^2 \rangle = \sum_{i=1}^{5} x_i x_i^2 = \sum_{i=1}^{5} \frac{(i - 3)^3}{8} = \frac{1}{8} (-8 - 10 + 1 + 8) = 0.$$ 

They are orthogonal.

9. Section 5.4, exercise 14:
Let $v$ and $w$ be 2 vectors in $R^n$, and $\alpha \in R$. We need to verify all the properties of the
norm:

(1) it is obvious to see that $\|v\|_\infty \geq 0$ by definition. If $\|v\|_\infty = 0$, this implies that the maximum of the absolute value of the component of $v$ is 0. The only solution is then $v = 0$. Therefore $\|v\|_\infty = 0 \iff v = 0$.

(2) $\|\alpha v\|_\infty = \max_{1 \leq i \leq n} |\alpha v_i| = \max_{1 \leq i \leq n} |\alpha| |v_i| = |\alpha| \max_{1 \leq i \leq n} |v_i| = |\alpha| \|v\|_\infty$ since $\alpha$ is a constant.

(3) We can use the property $|a + b| \leq |a| + |b|$ for any real numbers $a$ and $b$. We then have

$\|v + w\|_\infty = \max_{1 \leq i \leq n} |v_i + w_i| \leq \max_{1 \leq i \leq n} |v_i| + |w_i| \leq \|v\|_\infty + \|w\|_\infty$.

Therefore $\|x\|_\infty$ defines a norm in $\mathbb{R}^n$.

10. Section 5.4, exercise 20:

Let $x = (x_1, x_2)^T$. Let us assume that $\|x\|_2 \leq \|x\|_1$. This implies $\|x\|_2^2 \leq \|x\|_1^2$. It gives:

$x_1^2 + x_2^2 \leq (|x_1| + |x_2|)^2$. Since $(|x_1| + |x_2|)^2 = x_1^2 + x_2^2 + 2|x_1||x_2|$, we see that $\|x\|_2 \leq \|x\|_1$ is true.