Homework 7
Solutions

1. Section 5.5, exercise 1(c):
   Let \(u_1 = (1, -1)^T\) and \(u_2 = (1, 1)^T\). We can easily check that \(\langle u_i, u_j \rangle = \delta_{i,j}\) for \(i, j = 1, 2\) (it gives 4 relations). The basis is then an orthonormal basis.

2. Section 5.5, exercise 1(c):
   Let \(u_1 = (\sqrt{3}/2, 1/2)^T\) and \(u_2 = (-1/2, \sqrt{3}/2)^T\). We can easily check that \(\langle u_i, u_j \rangle = \delta_{i,j}\) for \(i, j = 1, 2\) (it gives 4 relations). The basis is then an orthonormal basis.

3. Section 5.5, exercise 2:
   \begin{itemize}
   \item We need to check that \(\langle u_i, u_j \rangle = \delta_{i,j}\) for \(i, j = 1, 2, 3\). It gives 6 relations. They show that \(\{u_1, u_2, u_3\}\) is an orthonormal basis.
   \item Since \(\{u_1, u_2, u_3\}\) is basis, we have \(x = c_1u_1 + c_2u_2 + c_3u_3\) and the coefficients are given by \(c_i = \langle x, u_i \rangle\). We get then \(c_1 = -\frac{2}{3\sqrt{2}}, c_2 = \frac{5}{3}\) and \(c_3 = 0\). According to the Parseval theorem, \(\|x\|^2 = c_1^2 + c_2^2 + c_3^2 = 27/9 = 3\). Then \(\|x\| = \sqrt{3}\).
   \end{itemize}

4. Section 5.5, exercise 7:
   If \(x \perp u_2\), then \(c_2 = 0\). If \(\langle x, u_1 \rangle = 4\) then \(c_1 = 4\). Using Parseval theorem, we have \(\|x\|^2 = c_1^2 + c_2^2 + c_3^2 = 16 + c_3^2 = 25\). Then \(c_3^2 = 9\), that is \(c_3 = \pm 3\).

5. Section 5.5, exercise 8:
   Since \(\cos(x)\) and \(\sin(x)\) form an orthonormal set in \(C[-\pi, \pi]\), we get:
   \[\langle f, g \rangle = \langle 3\cos(x) + 2\sin(x), \cos(x) - \sin(x) \rangle = 3 - 2 = 1.\]

6. Section 5.5, exercise 27:
   The set \(S\) is of the form \(S = \{u, v_1, ..., v_n, w_1, ..., w_n\}\). with \(v_k = \cos(kx)\) and \(w_k = \sin(kx)\). We need to check that \(\langle u, u \rangle = \langle v_k, v_k \rangle = \langle w_k, w_k \rangle = 1\) for any \(k\) and also that each vector is orthogonal to the others. Using the trigonometric relations: \(\cos^2(y) = (1 + \cos(2y))/2\) and \(\sin^2(y) = (1 - \cos(2y))/2\):
   \[
   \langle u, u \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} dx = 1,
   \]
   \[
   \langle v_k, v_k \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(kx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left( \frac{1 + \cos(2kx)}{2} \right) dx = 1,
   \]
   \[
   \langle w_k, w_k \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(kx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left( \frac{1 - \cos(2kx)}{2} \right) dx = 1.
   \]
   The vectors are all unit vectors. We need to check that each vector is orthogonal to the others. Let \(i, j = 1, ..., n\) and \(i \neq j\). Using the trigonometric relations \(\cos(ix)\cos(jx) = \)

\[
\begin{align*}
\langle u, v \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\cos(ix) dx}{\sqrt{2}} = \frac{1}{i\pi \sqrt{2}} \sin(ix) \bigg|_{-\pi}^{\pi} = 0, \\
\langle u, w_i \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin(ix) dx}{\sqrt{2}} = 0 \\
\langle v_i, v_j \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(ix) \cos(jx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cos((i-j)x) + \cos((i+j)x)) dx = 0, \\
\langle w_i, w_j \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(ix) \sin(jx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cos((i-j)x) - \cos((i+j)x)) dx = 0, \\
\langle v_i, w_j \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(ix) \sin(jx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\sin((i-j)x) + \sin((i+j)x)) dx = 0.
\end{align*}
\]

The set is then an orthonormal set.

7. Section 5.5, exercise 29:
Since the set \( S \) is an orthonormal set, all the vectors in \( S_1 \) are orthogonal to all the vectors in \( S_2 \) (and conversely, all the vectors in \( S_2 \) are orthogonal to all the vectors in \( S_1 \)). We then have \( S_1 \perp S_2 \).

8. Section 5.6, exercise 1(a):
The range of \( A \) is given by \( R(A) = \text{Span}\{u_1, u_2\} \), with \( u_1 = (-1, 1)^T \) and \( u_2 = (3, 5)^T \). Let us construct an orthonormal basis using \( \{u_1, u_2\} \).
Let \( v_1 = \frac{u_1}{||u_1||} \). Since \( ||u_1|| = \sqrt{2} \). Therefore \( v_1 = \frac{1}{\sqrt{2}}(-1, 1)^T \).
let \( v_2 = u_2 - \langle u_2, v_1 \rangle v_1 \). We have \( \langle u_2, v_1 \rangle = -2/\sqrt{2} \). Then \( v_2 = (2, 6)^T \). We get \( v_2 = \frac{v_2}{||v_2||} \) \( (1, 3)^T / \sqrt{10} \). Then \( \{v_1, v_2\} \) is an orthonormal basis for \( R(A) \).

9. Section 5.6, exercise 3:
Let \( u_1 = (1, 2, -2)^T \), \( u_2 = (4, 3, 2)^T \) and \( u_3 = (1, 2, 1)^T \).
Let \( v_1 = \frac{u_1}{||u_1||} \) and \( ||u_1|| = 3 \). Then \( v_1 = \frac{1}{3}(1, 2, -2)^T \).
Let \( v_2 = u_2 - \langle u_2, v_1 \rangle v_1 \) and \( \langle u_2, v_1 \rangle = 2 \). Then \( v_2 = \frac{5}{3}(2, 1, 2)^T \). This gives \( ||v_2|| = 5 \).
Then \( v_2 = \frac{v_2}{||v_2||} = \frac{1}{3}(2, 1, 2)^T \).
Let \( v_3 = u_3 - \langle u_3, v_1 \rangle v_1 - \langle u_3, v_2 \rangle v_2 \) and \( \langle u_3, v_1 \rangle = 1 \) and \( \langle u_3, v_2 \rangle = 2 \). Then \( v_3 = \frac{1}{3}(-2, 2, 3)^T \). Then \( v_3 = \frac{v_3}{||v_3||} \) \( \sqrt{17}/3 \).
The set \( \{v_1, v_2, v_3\} \) forms an orthonormal basis for \( R^3 \).

10. Section 5.6, exercise 4:
Let \( v_1 = \frac{1}{||1||} \) and \( ||1|| = \sqrt{2} \). Then \( v_1 = \frac{1}{\sqrt{2}} \).
Let \( v_2 = \frac{x - \langle x, v_1 \rangle v_1}{||x - \langle x, v_1 \rangle v_1||} = \frac{1}{\sqrt{2}} x \).
Let \( v_3 = \frac{x^2 - \langle x^2, v_1 \rangle v_1 - \langle x^2, v_2 \rangle v_2}{||x^2 - \langle x^2, v_1 \rangle v_1 - \langle x^2, v_2 \rangle v_2||} = \frac{1}{3} x^2 - \frac{1}{3} \).
We have \( ||v_3|| = \sqrt{\frac{8}{45}} \). We define then \( v_3 = \frac{v_3}{||v_3||} \).
Then \( \{v_1, v_2, v_3\} \) is an orthonormal basis for the subspace spanned by \( 1, x \) and \( x^2 \).

11. Section 5.6, exercise 8:
Let \( v_1 = \frac{x}{||x||} \), and \( ||x|| = 5 \). Then \( v_1 = (4, 2, 2, 1)^T / 5 \).
Let \( v_2 = x_2 - \langle x_2, v_1 \rangle \) and \( \langle x_2, v_1 \rangle = 2 \). Then \( v_2 = \frac{1}{5}(2, -4, -4, 8)^T \). And \( ||v_2|| = 2 \). Then \( v_2 = \frac{v_2}{||v_2||} \) \( (2, -4, -4, 8)^T \).
Let \( \tilde{v}_3 = x_3 - \langle x_3, v_1 \rangle v_1 - \langle x_3, v_2 \rangle v_2 \) and \( \langle x_3, v_1 \rangle = 1 \) and \( \langle x_3, v_2 \rangle = 1 \). Then \( \tilde{v}_3 = (0, 1, -1, 0)^T \). Since \( \|\tilde{v}_3\| = \sqrt{2} \), we have \( v_3 = \frac{1}{\sqrt{2}}(0, 1, -1, 0)^T \).

The set \( \{v_1, v_2, v_3\} \) is an orthonormal basis for the subspace spanned by \( x_1, x_2 \) and \( x_3 \).

12. Section 5.6, exercise 11:
The Gram-Schmidt process would fail in the case. We can construct 2 orthonormal vectors from \( v_1 \) and \( v_2 \). Let us call them respectively \( x_1 \) and \( x_2 \). However to construct the third vector from \( v_3 \), we need to subtract to \( v_3 \) its projection onto the subspace spanned by \( x_1 \) and \( x_2 \) (remember that, using Gram-Schmidt, we get something of the form \( \tilde{x}_3 = v_3 - \langle v_3, x_1 \rangle x_1 - \langle v_3, x_2 \rangle x_2 \)). The set spanned by \( x_1 \) and \( x_2 \) is the same as the set spanned by \( v_1 \) and \( v_2 \) (since \( x_1 \) and \( x_2 \) are just combinations of them). Therefore \( v_3 \) belongs to \( \text{Span}(x_1, x_2) \). The Gram-Schmidt process would give the vector 0 for the last vector.