Homework 6
Key answers

June 11, 2005

1. (2 points) (Not taken from the book) Rewrite the differential equation as a system of first-order differential equation:

$y^{(3)} - 3y = e^t.$

**Solution:** Let us introduce the following variables $x_1 = y$, $x_2 = y'$ and $x_3 = y''$. We then have:

$$
\begin{align*}
x_1' &= x_2 \\
x_2' &= x_3 \\
x_3' &= 3x_1 + e^t
\end{align*}
$$

2. (2 points) (Not taken from the book) Write the system of differential equations in the form \( \vec{x}' = A\vec{x} \) (\( A \) being a matrix):

$$
\begin{align*}
x_1' &= x_1 + x_2 - x_3 \\
x_2' &= 3x_1 - x_2 + 4x_3 \\
x_3' &= -x_1 - x_2
\end{align*}
$$

**Solution:** We can rewrite the system in the form \( \vec{x}' = A\vec{x} \) with

$$
\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},
$$

and

$$
A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \\ -1 & -1 & 0 \end{pmatrix}.
$$

3. (4 points) (from ch. 3.4, ex 4) Find a basis for the solutions of the differential equation:

$$
\vec{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \vec{x}.
$$

**Solution:** We can rewrite the above equation as a system of 3 equations:

\begin{align*}
x_1' &= x_1, \\
x_2' &= x_1 + x_2, \\
x_3' &= x_1 + x_2 + x_3.
\end{align*}
The first equation can be solved directly and gives \( x_1 = c_1 e^t \) with \( c_1 \) a constant. Then the second equation becomes
\[
x'_2 = x_2 + c_1 e^t.
\]
It is a first order non homogeneous equation whose solution is \( x_2 = c_2 e^t + c_1 t e^t \), with \( c_2 \) a constant. The third equation is then
\[
x'_3 = x_3 + c_1 (1 + t) e^t + c_2 e^t,
\]
It is a first order non homogeneous equation whose solution is \( x_2 = c_3 e^t + c_2 t e^t + c_1 (t + t^2/2) e^t \), with \( c_3 \) a constant. This gives the solution:
\[
\vec{x} = \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} = \begin{pmatrix}
c_1 e^t \\
c_2 e^t + c_1 t e^t \\
c_3 e^t + c_2 t e^t + c_1 (t + t^2/2) e^t
\end{pmatrix},
\]
which can written as:
\[
\vec{x} = c_1 e^t \begin{pmatrix}
1 \\
t \\
t + \frac{t^2}{2}
\end{pmatrix} + c_2 e^t \begin{pmatrix}
0 \\
1 \\
t
\end{pmatrix} + c_3 e^t \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}.
\]
We can see 3 vectors that form a basis (you can also easily check that they are linearly independent):
\[
\vec{x}^1 = e^t \begin{pmatrix}
1 \\
t \\
t + \frac{t^2}{2}
\end{pmatrix}, \quad \vec{x}^2 = e^t \begin{pmatrix}
0 \\
1 \\
t
\end{pmatrix}, \quad \vec{x}^3 = e^t \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}.
\]

4. (4 points) (from ch. 3.4, ex 6) Determine whether the given vectors are a basis for the solutions of the differential equations
\[
\vec{x}' = \begin{pmatrix}
4 & -2 & 2 \\
-1 & 3 & 1 \\
1 & -1 & 5
\end{pmatrix}\vec{x}, \quad \vec{x}^1 = \begin{pmatrix}
e^{2t} \\
e^{4t} \\
0
\end{pmatrix}, \quad \vec{x}^2 = \begin{pmatrix}
e^{2t} \\
e^{4t} \\
0
\end{pmatrix}, \quad \vec{x}^3 = \begin{pmatrix}
e^{6t} \\
e^{6t} \\
0
\end{pmatrix}.
\]

Solution: We can easily check that \( \vec{x}^1, \vec{x}^2 \) and \( \vec{x}^3 \) are solutions of \( \vec{x}' = A\vec{x} \) (just check that \( \vec{x}^1' = A\vec{x}^1 \), etc...). Now, we have to make sure that they are linearly independant. Let us consider:
\[
c_1 \vec{x}^1 + c_2 \vec{x}^2 + c_3 \vec{x}^3 = \vec{0}.
\]
At \( t = 0 \), this gives:
\[
c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{0},
\]
which we can rewrite as:
\[
c_1 + c_3 = 0, \\
c_1 + c_2 = 0, \\
c_2 + c_3 = 0.
\]
Which gives \( c_3 = -c_1, \ c_2 = -c_1 \) and then \(-2c_1 = 0\). So \( c_1 = c_2 = c_3 = 0 \). They are linearly independant and therefore form a basis.
5. (4 points) (from ch. 3.8, ex 2) Find the general solution of the differential equation:

\[ \vec{x}' = \begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix} \vec{x}. \]

**Solution:** The matrix has 2 eigenvalues of multiplicity 1 \( \lambda_1 = -1 \) and \( \lambda_2 = 2 \). The corresponding eigenvectors are:

\[ \vec{v}^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{v}^2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}. \]

These gives the general solution:

\[ \vec{x} = c_1 e^{-t} \vec{v}^1 + c_2 e^{2t} \vec{v}^2. \]

6. (4 points) (from ch. 3.8, ex 12) Solve the initial-value problem

\[ \vec{x}' = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 2 & 1 \\ 4 & 1 & -3 \end{pmatrix} \vec{x}. \]

with the condition

\[ \vec{x}(0) = \begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix}. \]

**Solution:** The eigenvalues are given by:

\[ \lambda^3 - 2\lambda^2 - \lambda + 2 = 0. \]

The polynomial has three roots of multiplicity 1: \( \lambda_1 = 1, \ \lambda_2 = -1 \) and \( \lambda_3 = 2 \). The corresponding eigenvectors are:

\[ \vec{v}^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{v}^2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{2}{3} \\ 1 \end{pmatrix}, \quad \vec{v}^3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \]

The general solution is then

\[ \vec{x} = c_1 e^t \vec{v}^1 + c_2 e^{-t} \vec{v}^2 + c_3 e^{2t} \vec{v}^3. \]

Taking into account the initial condition, we find that \( c_1 = 9, \ c_2 = -\frac{4}{3}, \ c_3 = \frac{4}{3}. \)