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A note on fluxes in six-dimensional string theory backgrounds

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Abstract

We study the structure of warped compactifications of type IIB string theory to six space–time dimensions. We find that the most general four-manifold describing the internal dimensions is conformal to a Kähler manifold, in contrast with the heterotic case where the four-manifold must be conformally Calabi–Yau.

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1. Introduction

Flux compactifications of string theory have attracted much attention recently because in such backgrounds, many longstanding questions concerning the connection of string theory to the real world, are put in a new perspective. To name a few of their properties, moduli fields can be stabilized at the string tree level and therefore in a calculable manner [1,4,9,15,18,21]. Supersymmetry can be broken without inducing a large cosmological constant [2,13,18], and also a large hierarchy can be induced in a natural way [13]. In a different context, flux backgrounds have been shown to provide gravity dual descriptions to confining gauge theories (see for example [19,20]).

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Because of their importance for phenomenology, most of the recent work has been concerned with fluxes in four-dimensional space–times. This paper deals with the less studied case of flux backgrounds in six dimensions. We will see that in six dimensions, flux backgrounds are consistent solutions of string theory and therefore interesting in their own right. Moreover, since the internal manifolds are four-dimensional, the types of flux solutions are very much constrained.

For heterotic theory, flux compactifications to six dimensions were first considered almost twenty years ago [23] where a general analysis of string theory background to $10-2n$ dimensions with tensor fields acquiring an expectation value was given. However, it was not until quite recently that explicit examples of this construction were found and the issue of moduli stabilization could be addressed (see for example [3,7,9]). In [23] it was noticed that backgrounds with torsion in the connection appear in a natural way with the torsion being turned on by the three-form tensor field H satisfying

$$dH = \operatorname{tr} R \wedge R - \frac{1}{30} \operatorname{Tr} F \wedge F. \tag{1.1}$$

In order to preserve supersymmetry, the internal manifold has to be a complex n-manifold and the fundamental form $J_{a\bar{b}} = i g_{a\bar{b}}$ has to satisfy

$$\partial\bar{\partial} J = \frac{i}{30} \operatorname{Tr} F \wedge F - i \operatorname{tr} R \wedge R \quad \text{and} \quad d^\dagger J = i(\partial - \bar{\partial}) \log\|\omega\|. \tag{1.2}$$

Here ω is a holomorphic $(n, 0)$ form. The Yang–Mills field is required to satisfy the Donaldson–Uhlenbeck–Yau equation in a torsional background. These are the only conditions that have to be satisfied and many flux compactifications to four dimensions have been described in the literature.

Specializing to compactifications to six-dimensional Minkowski space–time, the allowed type of internal four-manifold is restricted. Indeed, as we will show, supersymmetry can only be preserved if on the internal manifold, there exists a spinor that is covariantly constant with respect to a conformally rescaled metric, i.e., a spinor which satisfies

$$\nabla'_m \varepsilon = 0, \tag{1.3}$$

where the covariant derivative is defined with the spin connection of the rescaled metric. The space–time metric will then be conformal to a Calabi–Yau two-fold (K3 or T^4).

In this paper we will study the compactification of type IIB supergravity to six dimensions in the presence of brane sources. We study the constraints on the space–time manifold imposed by non-vanishing fluxes which can be a zero-form and a three-form tensor field. The two tensor fields can be complex as opposed to the heterotic case in which they are real. We will see that the most general four-manifold describing the internal dimensions is conformal to a Kähler manifold, in contrast with the heterotic case where the four-manifold must be conformally Calabi–Yau.

2. Type IIB string theory compactified to six dimensions

The ten-dimensional type IIB supersymmetry transformations are [1]

$$\begin{aligned}\delta\psi_M &= \frac{1}{\kappa}\left(\nabla_M - \frac{i}{2}Q_M\right)\varepsilon + \frac{i}{480}\Gamma^{N_1\ldots N_5}F_{N_1\ldots N_5}\Gamma_M\varepsilon \\ &\quad - \frac{1}{96}\left(\Gamma_M{}^{PQR}G_{PQR} - 9\Gamma^{QR}G_{MQR}\right)B^{(10)\star}\varepsilon^\star, \\ \delta\lambda &= \frac{1}{\kappa}\Gamma^M P_M B^{(10)\star}\varepsilon^\star + \frac{1}{24}\Gamma^{MNP}G_{MNP}\varepsilon. \end{aligned} \tag{2.1}$$

Here $B^{(10)}$ denotes the ten-dimensional complex conjugation matrix. This factor does not conventionally appear in the supersymmetry transformations of the type IIB theory because usually the Majorana basis is chosen where $B^{(10)} = 1$. But as will be apparent, this is not an appropriate basis for dimensionally reducing the fermionic variables to $6 + 4$ dimensions. The ten-dimensional supersymmetry parameter is complex and satisfies the Weyl condition

$$\Gamma^{(10)}\varepsilon = \varepsilon. \tag{2.2}$$

We will consider configurations with a six-dimensional Poincaré invariance. The line element is of the form

$$ds^2 = e^{2D}\eta_{\mu\nu}\,dx^\mu\,dx^\nu + e^{-6D}g_{mn}\,dy^m\,dy^n, \tag{2.3}$$

where Latin indices denote the internal four-dimensional coordinates while Greek indices denote the six-dimensional Minkowski space–time coordinates. Moreover, $D = D(y)$ is the warp factor depending on the coordinates of the internal manifold only. We have arranged the powers of the warp factor for later convenience.

In constructing a space–time with six-dimensional Poincaré invariance, we set to zero the following components of the tensor fields

$$P_\mu = Q_\mu = G_{\mu MN} = F_{N_1\ldots N_5} = 0. \tag{2.4}$$

As a result supersymmetric configurations satisfy

$$\begin{aligned} &\left(\nabla_m - \frac{i}{2}Q_m\right)\varepsilon - \frac{\kappa}{24}e^{6D}\Gamma_m{}^{pqr}G_{pqr}B^{(10)\star}\varepsilon^\star = 0, \\ &\partial\!\!\!/ D\varepsilon = \frac{\kappa}{8}e^{6D}G\!\!\!/ B^{(10)\star}\varepsilon^\star, \\ &P\!\!\!/ B^{(10)\star}\varepsilon^\star = -\frac{\kappa}{4}e^{6D}G\!\!\!/ \varepsilon, \end{aligned} \tag{2.5}$$

where we have rescaled the spinor ε according to $\varepsilon \to e^{-3D/2}\varepsilon$. As is implied by the second equation of (2.5), a non-constant warp factor $D(y)$ requires at least one component of G_{mnp} being non-zero.

[1] Our notation mostly follows those of [22] except we use a mostly positive signature convention for the metric. See the appendices for a list of our conventions.

Next we decompose the ten-dimensional spinors. We shall represent an anti-commuting six-dimensional spinor as a pair of Weyl spinors, ξ_i, for $i = 1, 2$, satisfying the symplectic Majorana–Weyl condition

$$\begin{aligned} &\Gamma^{(6)}\xi_i^{\pm} = \pm\xi_i^{\pm}, \\ &\varepsilon_{ij} B^{(6)\star}\left(\xi_j^{\pm}\right)^{\star} = \xi_i^{\pm}. \end{aligned} \tag{2.6}$$

Here $\pm$ indicates the six-dimensional chirality. In four dimensions we will only impose the Weyl condition. The commuting four-dimensional Euclidean spinors are written similarly as

$$\begin{aligned} &\Gamma^{(4)}\eta_i^{\pm} = \pm\eta_i^{\pm}, \\ &\varepsilon_{ij} B^{(4)\star}\left(\eta_j^{\pm}\right)^{\star} = \tilde{\eta}_i^{\pm}. \end{aligned} \tag{2.7}$$

However, we do not a priori impose the condition $\tilde{\eta}_i = \eta_i$.

Under the decomposition of the Lorentz algebra $SO(9,1) \to SO(5,1) \times SO(4)$, a positive chirality spinor decomposes according to

$$\mathbf{16}_+ \to (\mathbf{4}_+, \mathbf{2}_+) + (\mathbf{4}_-, \mathbf{2}_-).$$

For the supersymmetry parameter ε this decomposition can be written as

$$\varepsilon = \varepsilon^{ij}\left(\xi_i^+ \otimes \eta_j^+ + \xi_i^- \otimes \eta_j^-\right). \tag{2.8}$$

First note that the decomposition is invariant under an $SU(2)$ transformation acting on the spinor labels i and j. Secondly, if we had set $\tilde{\eta}_i = \eta_i$, then $B^{(10)\star}\varepsilon^\star = \varepsilon$ and thus ε becomes a ten-dimensional Majorana–Weyl spinor. Indeed, any condition relating η_1 and η_2 reduces the number of spinor degrees of freedom by $1/2$. Now, inserting (2.8) into (2.5) we see that we obtain two independent conditions for each of the six-dimensional chiralities. So we will relabel the spinors according to $\xi_i^+ = \xi_i$ and $\eta_j^+ = \eta_j$ and work only with a spinor of positive six-dimensional chirality. We thus set

$$\varepsilon = \varepsilon^{ij}\xi_i \otimes \eta_j \quad \text{and} \quad B^{(10)\star}\varepsilon^\star = \varepsilon^{ij}\xi_i \otimes \tilde{\eta}_j. \tag{2.9}$$

Inserting the above decomposition into the supersymmetry variations we obtain that supersymmetry can only be preserved if and only if the following conditions are satisfied

$$\left(\nabla_m - \frac{i}{2}Q_m\right)\eta_j + g_m\tilde{\eta}_j = 0, \tag{2.10}$$

$$\partial\!\!\!/ D\eta_j = -\frac{1}{2}g\!\!\!/\tilde{\eta}_j, \tag{2.11}$$

$$P\!\!\!/\tilde{\eta}_j = g\!\!\!/\eta_j. \tag{2.12}$$

Here we have introduced the dualized one-form field $g = -\frac{\kappa}{4}e^{6D}(\star G)$, or equivalently,

$$G_{mnp} = -\frac{4}{\kappa}e^{-6D}\varepsilon_{mnp}{}^{q}g_q. \tag{2.13}$$

In the following we will use the conditions (2.10)–(2.12) to determine the form of the supersymmetric background. First we note that by using (2.7) we obtain the following result

$$\eta^{\dagger i}\tilde{\eta}_j = \frac{1}{2}\delta^i{}_j v, \tag{2.14}$$

where $\eta^\dagger = (\eta^\star)^T$ and $v = \eta^{\dagger k}\tilde{\eta}_k$ is a complex function. The covariant derivative on the bilinear spinor is then

$$\begin{aligned}\nabla_m\big(\eta^{\dagger i}\eta_j\big) &= -\frac{1}{2}\delta^i{}_j\big(g_m v + g_m^\star v^\star\big)\\ &= \delta^i{}_j\big(\eta^{\dagger k}\gamma_m \partial\!\!\!/ D\eta_k + \eta^{\dagger k}\partial\!\!\!/ D\gamma_m\eta_k\big)\\ &= 2\delta^i{}_j\partial_m D\eta^{\dagger k}\eta_k,\end{aligned} \tag{2.15}$$

where in the second line, we have noted the following relation

$$\begin{aligned}g_m\eta^{\dagger k}\tilde{\eta}_k &= \frac{1}{2}\eta^{\dagger k}(\gamma_m g\!\!\!/ + g\!\!\!/\gamma_m)\tilde{\eta}_k\\ &= -\eta^{\dagger k}\gamma_m\partial\!\!\!/ D\eta_k - \tilde{\eta}^{\dagger k}\partial\!\!\!/ D\gamma_m\tilde{\eta}_k\\ &= -2\eta^{\dagger k}\gamma_m\partial\!\!\!/ D\eta_k\end{aligned} \tag{2.16}$$

obtained utilizing (2.11) and its complex conjugated form. Solving (2.15), we find

$$\eta^{\dagger i}\eta_j = \frac{1}{2}e^{4(D+D_0)}\delta^i{}_j + A^i{}_j, \tag{2.17}$$

where D_0 is a normalization constant and $A^i{}_j$ is a constant traceless Hermitian matrix. We can diagonalize on the $SU(2)$ indices so that

$$A^i{}_j = \frac{1}{2}e^{4A}(\sigma_3)^i{}_j. \tag{2.18}$$

Therefore a nonzero $A^i{}_j$ is equivalent to η_1 and η_2 having different normalizations. If present, an additional constraint

$$D > A - D_0, \tag{2.19}$$

must be imposed so that $\eta^{\dagger 2}\eta_2 > 0$. This constraint effectively sets a minimum value for D.[2]

We now renormalize the spinors by defining

$$\begin{aligned}\eta_1 &= \alpha_+\lambda_1, &\quad \tilde{\eta}_1 &= \alpha_-\tilde{\lambda}_1,\\ \eta_2 &= \alpha_-\lambda_2, &\quad \tilde{\eta}_2 &= \alpha_+\tilde{\lambda}_2,\end{aligned} \tag{2.20}$$

where

$$\alpha_\pm^2 = \frac{1}{2}\big(e^{4(D+D_0)} \pm e^{4A}\big). \tag{2.21}$$

The normalized spinors λ_i then satisfy

$$\lambda^{\dagger i}\lambda_j = \delta^i_j. \tag{2.22}$$

[2] For another scenario where a bound for the warped factor occurs, see [8].

An additional constraint on λ_i comes from the relation

$$\lambda^{\dagger i}\tilde{\lambda}_j = e^{i\varphi}\delta^i_j, \tag{2.23}$$

where φ is in general a y-dependent function. This results from (2.14) and the constraint $|\lambda^{\dagger i}\tilde{\lambda}_j|^2 = \delta^i_j$, which can be derived by applying the Fierz identity and Eq. (A.12). Noting that λ_i are two-component spinors, (2.22) and (2.23) together imply that

$$\tilde{\lambda}_i = \varepsilon_{ij} B^{(4)\star}(\lambda_j)^\star = e^{i\varphi}\lambda_i. \tag{2.24}$$

The supersymmetry conditions (2.10)–(2.12) then become

$$\begin{aligned}
\left(\nabla_m - \frac{i}{2}Q_m + \partial_m \ln\alpha_+\right)\lambda_1 &= -\frac{\alpha_-}{\alpha_+} g_m e^{i\varphi}\lambda_1, \\
\left(\nabla_m - \frac{i}{2}Q_m + \partial_m \ln\alpha_-\right)\lambda_2 &= -\frac{\alpha_+}{\alpha_-} g_m e^{i\varphi}\lambda_2,
\end{aligned} \tag{2.25}$$

$$\partial\!\!\!/ D\lambda_1 = -\frac{1}{2}\frac{\alpha_-}{\alpha_+} g\!\!\!/ e^{i\varphi}\lambda_1, \qquad \partial\!\!\!/ D\lambda_2 = -\frac{1}{2}\frac{\alpha_+}{\alpha_-} g\!\!\!/ e^{i\varphi}\lambda_2, \tag{2.26}$$

$$P\!\!\!/ e^{i\varphi}\lambda_1 = \frac{\alpha_+}{\alpha_-} g\!\!\!/\lambda_1, \qquad P\!\!\!/ e^{i\varphi}\lambda_2 = \frac{\alpha_-}{\alpha_+} g\!\!\!/\lambda_2. \tag{2.27}$$

Let us clarify the relationships of the various fields above. First, by considering the expression $(g_m e^{i\varphi}\lambda^{\dagger 1}\lambda_1 + g^\star_m e^{-i\varphi}\lambda^{\dagger 1}\lambda_1)$ and performing a calculation similar to (2.16), we obtain the relation

$$g_m e^{i\varphi} + g^\star_m e^{-i\varphi} = -2\partial_m D\left(\frac{\alpha_+}{\alpha_-} + \frac{\alpha_-}{\alpha_+}\right) = -2\frac{\alpha_+}{\alpha_-}\partial_m \ln\alpha_+ = -2\frac{\alpha_-}{\alpha_+}\partial_m \ln\alpha_-. \tag{2.28}$$

The above relation can also be derived simply from the constraint $\nabla_m(\lambda^{\dagger i}\lambda_j) = 0$. Second, note that (2.24) explicitly relates λ_1 with λ_2 up to a phase factor φ that must be determined. We can find the variation of φ by requiring that the two equations of (2.25) are equivalent given (2.24). This results in

$$\partial_m\varphi = -Q_m + \frac{1}{2i}\left(\frac{\alpha_+}{\alpha_-} + \frac{\alpha_-}{\alpha_+}\right)\left(g_m e^{i\varphi} - g^\star_m e^{-i\varphi}\right), \tag{2.29}$$

having applied (2.28) in the calculation. Now, with (2.28) and (2.29), we can simplify (2.25) further to

$$\begin{aligned}
\nabla_m\lambda_1 &= \left(-\frac{i}{2}\partial_m\varphi + \frac{i}{2}W_m\right)\lambda_1, \\
\nabla_m\lambda_2 &= \left(-\frac{i}{2}\partial_m\varphi - \frac{i}{2}W_m\right)\lambda_2,
\end{aligned} \tag{2.30}$$

where

$$W_m = \frac{1}{2i}\left(\frac{\alpha_+}{\alpha_-} - \frac{\alpha_-}{\alpha_+}\right)\left(g_m e^{i\varphi} - g^\star_m e^{-i\varphi}\right). \tag{2.31}$$

We now proceed to discuss the complex structure of the four-manifold. Consider the triplet of almost complex structures

$$(J_A)_m{}^n = \frac{i}{2}(\sigma_A)^j{}_i \lambda^{\dagger i} \gamma_m{}^n \lambda_j \tag{2.32}$$

for $A = 1, 2, 3$. Using (A.12), it can be shown that

$$(J_A)_m{}^n (J_B)_n{}^p = -\delta_{AB}\delta_m{}^p + \varepsilon_{AB}{}^C (J_C)_m{}^p. \tag{2.33}$$

Furthermore, (2.32) and (2.33) imply the metric g_{mn} on the four-manifold is Hermitian with respect to each of the above almost complex structures; that is,

$$(J_A)_m{}^k (J_A)_n{}^l g_{kl} = g_{mn} \quad \text{for } A = 1, 2, 3. \tag{2.34}$$

The above three equations taken together define an almost hyper-Kähler structure on the four-manifold (see for example [17]). To be specific, we will take $J = J_3$ as the almost complex structure on the four-manifold. With respect to J, we have the following (p, q)-forms

$$\begin{aligned}
\Omega_{mn} &= (J_2 + iJ_1)_{mn}, \quad && (2,0),\\
J_{mn} &= (J_3)_{mn}, && (1,1),\\
\bar{\Omega}_{mn} &= (J_2 - iJ_1)_{mn}, && (0,2).
\end{aligned} \tag{2.35}$$

The covariant derivatives of the Hermitian form, J, and the complex 2-form, Ω, can be easily obtained using (2.30). We find

$$\begin{aligned}
\nabla_p J_{mn} &= 0,\\
\nabla_p \Omega_{mn} &= -iW_p \Omega_{mn},
\end{aligned} \tag{2.36}$$

where $\Omega_{mn} = -\lambda^{\dagger 1} \gamma_{mn} \lambda_2$.[3] From (2.36), we conclude that the four-manifold is not only complex, but also Kähler. However, the manifold is Calabi–Yau if and only if $\nabla_p J_{mn} = \nabla_p \Omega_{mn} = 0$ and in this case the manifold has a hyper-Kähler structure. This condition is satisfied when $W_m = 0$.

The presence of a complex structure allows us to introduce holomorphic and anti-holomorphic coordinates, $a, b, \ldots$ and $\bar{a}, \bar{b}, \ldots$, and take $J_a{}^b = i\delta_a{}^b$ and $J_{\bar{a}}{}^{\bar{b}} = -i\delta_{\bar{a}}{}^{\bar{b}}$. The Kähler form is related to the metric by

$$J_{a\bar{b}} = i g_{a\bar{b}},$$

which implies

$$\gamma_a \lambda_1 = \gamma^{\bar{a}} \lambda_1 = 0 \quad \text{and} \quad \gamma_{\bar{a}} \lambda_2 = \gamma^a \lambda_2 = 0. \tag{2.37}$$

They can be applied to (2.26)–(2.27) to give

$$g_a = -2\frac{\alpha_+}{\alpha_-} e^{-i\varphi} \partial_a D, \qquad g_{\bar{a}} = -2\frac{\alpha_-}{\alpha_+} e^{-i\varphi} \partial_{\bar{a}} D, \tag{2.38}$$

[3] We point out that in the intrinsic torsion classification of supersymmetric compactifications (see for example [7,12,16]), (2.36) implies that the only nonzero torsional class [11] is $(\mathcal{W}_5)_m = J_m{}^n W_n$.

$$P_a = -2\left(\frac{\alpha_+}{\alpha_-}\right)^2 e^{-2i\varphi}\partial_a D, \qquad P_{\bar a} = -2\left(\frac{\alpha_-}{\alpha_+}\right)^2 e^{-2i\varphi}\partial_{\bar a} D. \tag{2.39}$$

Using both equations in (2.38), we obtain

$$g_a e^{i\varphi} - g_a^\star e^{-i\varphi} = -2\left(\frac{\alpha_+}{\alpha_-} - \frac{\alpha_-}{\alpha_+}\right)\partial_a D. \tag{2.40}$$

Therefore, if $\alpha_+ \neq \alpha_-$ and $\partial_m D \neq 0$ in a region on the internal four-manifold then $\mathrm{Im}[g_m e^{i\varphi}] \neq 0$ and the background geometry will be conformal to a Kähler but non-Calabi–Yau manifold. The precise relation between α_+ and α_- is determined by the constant A which is not fixed by the present analysis.

In the following we consider the special class of solutions with

$$C_0 = 0 \quad \text{and} \quad \mathrm{Im}\big[g_m e^{i\varphi}\big] = 0, \tag{2.41}$$

where C_0 is the R–R 1-form and the second condition implying $\alpha_+ = \alpha_-$. From (B.2), we know that in such backgrounds,

$$Q_m = 0 \quad \text{and} \quad P_m = \frac{1}{2}\partial_m\phi. \tag{2.42}$$

With (2.29)–(2.31), these conditions imply

$$\partial_m\varphi = 0, \qquad \nabla_m\lambda_1 = \nabla_m\lambda_2 = 0, \qquad W_m = 0, \tag{2.43}$$

or that the four-manifold is conformally Calabi–Yau. Furthermore, with P_m real, (2.39) implies solutions only for $\varphi = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. We therefore have

$$\begin{aligned} &g_m = -2e^{-i\varphi}\partial_m D, \\ &P_m = \frac{1}{2}\partial_m\phi = -2e^{-2i\varphi}\partial_m D, \quad \text{for } \varphi = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}. \end{aligned} \tag{2.44}$$

The sourceless Bianchi identity for G_{mnp} is

$$\partial_{[m}G_{npq]} = -P_{[m}G^\star_{npq]}. \tag{2.45}$$

With $G_{npq} \sim e^{-6D}\varepsilon_{npq}{}^r g_r$ as in (2.13), (2.45) implies

$$\begin{aligned} 0 &= -\partial_{[m}\big(e^{-8D}\varepsilon_{npq]}{}^r\partial_r D\big) \\ &= -\partial_{[m}\big(\tilde\varepsilon_{npq]}{}^r\partial_r D\big) \\ &= \tilde\Box D, \end{aligned} \tag{2.46}$$

where in the second line, we have rescaled the metric to $\tilde g_{mn} = e^{-8D}g_{mn}$, which leads to $\tilde\varepsilon_{mnpq}\tilde g^{qr} = e^{-8D}\varepsilon_{mnpq}g^{qr}$. For $\varphi = 0, \pi$, $\tilde g_{mn}$ corresponds to the metric of the four-manifold in the string frame. Here, g_m is purely real and only the NS–NS 2-form B_{mn} is turned on (see (B.3)). This is indeed the result for NS 5-branes [6]. As for $\varphi = \pi/2, 3\pi/2$, g_m is purely imaginary and signifies the presence of D5- and/or $\bar{\text{D}}$5-branes. With sources, (2.46) gets modified to

$$\tilde\Box D = \star\rho_5, \tag{2.47}$$

where ρ_5 is the density distribution of the 5-branes.

The above special class of type IIB solution are in a sense two copies of analogous six-dimensional heterotic backgrounds. It is straightforward to show that the four-manifold must be conformally Calabi–Yau in the heterotic case. In order to see this consider the heterotic dilatino and gravitino supersymmetry constraints in the string frame[4]

$$\delta\lambda = \Gamma^M \partial_M \phi \varepsilon - \frac{1}{6} H_{PQR} \Gamma^{PQR} \varepsilon = 0, \tag{2.48}$$

$$\delta\psi_M = \nabla_M \varepsilon - \frac{1}{4} H_{MPQ} \Gamma^{PQ} \varepsilon = 0. \tag{2.49}$$

As shown in [23], the string frame metric must take the form

$$ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu + g_{mn}(y)\, dy^m\, dy^n. \tag{2.50}$$

We shall take $\phi = \phi(y)$ and let the only non-zero components of the three-form take the form $H_{mnp} = \varepsilon_{mnp}{}^q h_q(y)$. The supersymmetry transformations become

$$\delta\lambda = \Gamma^m \big(\partial_m \phi - h_m \Gamma^{(4)}\big)\varepsilon = 0, \tag{2.51}$$

$$\delta\psi_m = \left(\nabla_m - \frac{1}{2} \Gamma_m{}^r h_r \Gamma^{(4)}\right)\varepsilon = 0. \tag{2.52}$$

We now rescale the internal metric, according to $g_{mn} = e^{2\phi} g'_{mn}$ and find

$$\nabla'_m \varepsilon = 0. \tag{2.53}$$

Therefore, g'_{mn} can only be a Calabi–Yau metric and the internal four-manifold is conformally Calabi–Yau.

3. Conclusion and outlook

In this paper we have discussed the compactification of type IIB string theory to six dimensions. With non-vanishing fluxes, we have translated the conditions for unbroken supersymmetry into conditions on the background geometry and the tensor fields. Our work can be viewed as a step towards a complete classification of string theory vacua in six dimensions.

There are several open questions which we will leave for future work. First, we have focused on six-dimensional space–times with a vanishing cosmological constant. In this case we have seen that the conditions for unbroken supersymmetry factorize into two independent conditions involving spinors of a definite six-dimensional chirality only. An immediate open question is if unbroken supersymmetry implies a vanishing of the six-dimensional cosmological constant. If AdS_6 backgrounds exist they would have the interesting property that the equations involving spinors of different six-dimensional chirality do not decouple. Such a property has been found in supergravity backgrounds like for example in [20].

[4] We use the standard notation in the NS5-brane literature [5] which differs from that in [23] by a rescaling of the dilaton. Also, the string metric is related to Einstein metric by $g^E_{MN} = e^{-\phi/2} g^S_{MN}$.

Another open question concerns compactifications of the heterotic string to six dimensions. One of the most exciting string theory developments in the last few years is the statistical approach to string theory compactifications initiated by Douglas and collaborators [10]. This approach opens the door to the possibility of making real world predictions, maybe for the scale of supersymmetry breaking or for the possibility that large extra dimensions appear in nature. It would certainly be very interesting to generalize this statistical approach to compactifications of the heterotic string. Studying the distribution and number of flux vacua of the heterotic string compactified on a six-dimensional torsional background certainly sounds like a formidable task since not much is known about vector bundles on manifolds with torsion. But it would be an interesting problem to study the number of heterotic flux vacua in six dimensions. The background geometry can only be conformal to a Calabi–Yau two-fold (either K3 or T^4). Moreover, the fluxes are gauge fields and are constrained to satisfy the Donaldson–Uhlenbeck–Yau equations

$$F_{ab} = F_{\bar{a}\bar{b}} = F_{a\bar{b}} J^{a\bar{b}} = 0. \tag{3.1}$$

In summary flux backgrounds in six dimensions are simple enough that the background geometry can be described in a concrete way yet complicated enough to capture many interesting properties. We will return to the issues raised above in future works.

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Appendix A

Our notation and conventions are as follows (a good reference about spinors and properties of the Clifford algebra is [24]).

- The different types of indices that we use are:

$M, N, \ldots$	are ten-dimensional Lorentzian indices,
$A, B, \ldots$	are ten-dimensional tangent space indices,
$\mu, \nu, \ldots$	are six-dimensional Lorentzian indices,
$m, n, \ldots$	are real indices of the Euclidean submanifold,
$a, b, \ldots$ and $\bar{a}, \bar{b}, \ldots$	are complex indices of the Euclidean submanifold.

In addition, we use $i, j, k, l = 1, 2$ as $SU(2)$ indices labeling different spinors and not their components. Moreover, the coordinates of the external space are denoted by $x = (x^0, x^1, \ldots, x^5)$ while $y = (x^6, x^7, x^8, x^9)$ denotes the coordinates of the four-manifold.

• We follow the mostly positive signature for our metrics. The gamma-matrices Γ^A are Hermitian, for $A = 1, \dots, 9$ while Γ^0 is anti-Hermitian. They satisfy

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}, \tag{A.1}$$

where η^{AB} has the signature $(-, +, \dots, +)$. We decompose the 10d gamma matrices as follows.

$$\begin{aligned} &\Gamma^A = \gamma^A \otimes \Gamma^{(4)}, \quad A = 0, \dots, 5, \\ &\Gamma^A = I^{(6)} \otimes \gamma^A, \quad A = 6, \dots, 9, \end{aligned} \tag{A.2}$$

where $I^{(6)}$ is the 6d identity matrix and $\Gamma^{(4)}$ is the 4d chirality matrix. An explicit representation is given by

$$\begin{aligned} &\gamma^0 = i\sigma_2 \otimes \sigma_3 \otimes \sigma_3, \qquad \gamma^2 = I \otimes \sigma_1 \otimes \sigma_3, \qquad \gamma^4 = I \otimes I \otimes \sigma_1, \\ &\gamma^1 = \sigma_1 \otimes \sigma_3 \otimes \sigma_3, \qquad \gamma^3 = I \otimes \sigma_2 \otimes \sigma_3, \qquad \gamma^5 = I \otimes I \otimes \sigma_2, \end{aligned}$$

$$\begin{aligned} &\gamma^6 = \sigma_1 \otimes \sigma_3, \qquad \gamma^8 = I \otimes \sigma_1, \\ &\gamma^7 = \sigma_2 \otimes \sigma_3, \qquad \gamma^9 = I \otimes \sigma_2, \end{aligned}$$

where σ_i are the Pauli matrices and I is the 2×2 identity matrix. The 10d chirality matrix is

$$\begin{aligned} \Gamma^{(10)} &= \Gamma^0 \Gamma^1 \cdots \Gamma^9 \\ &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3, \end{aligned} \tag{A.3}$$

and can be written as $\Gamma^{(10)} = \Gamma^{(6)} \otimes \Gamma^{(4)}$ where the 6d and 4d chirality matrices are defined as

$$\begin{aligned} &\Gamma^{(6)} = -\gamma^0 \gamma^1 \cdots \gamma^5 = \sigma_3 \otimes \sigma_3 \otimes \sigma_3, \\ &\Gamma^{(4)} = -\gamma^6 \gamma^7 \gamma^8 \gamma^9 = \sigma_3 \otimes \sigma_3. \end{aligned} \tag{A.4}$$

• The complex conjugation matrix, $B^{(10)}$, satisfies

$$B^{(10)} \Gamma^A \left(B^{(10)}\right)^{-1} = -\Gamma^{A\star} \quad \text{and} \quad B^{(10)} \Sigma^{AB} \left(B^{(10)}\right)^{-1} = -\Sigma^{AB\star}, \tag{A.5}$$

where $\Sigma^{AB} = -\frac{i}{4}[\Gamma^A . \Gamma^B]$ are Lorentz generators. In 10d, $B^{(10)}$ is explicitly

$$B^{(10)} = \Gamma^{(10)} \Gamma^3 \Gamma^5 \Gamma^7 \Gamma^9 = \sigma_3 \otimes \sigma_1 \otimes i\sigma_2 \otimes \sigma_1 \otimes i\sigma_2. \tag{A.6}$$

Note that $B^{(10)\star} B^{(10)} = 1$. Furthermore, we can decompose $B^{(10)} = B^{(6)} \otimes B^{(4)}$ with

$$\begin{aligned} &B^{(6)} = \Gamma^{(6)} \gamma^3 \gamma^5 = \sigma_3 \otimes \sigma_1 \otimes -i\sigma_2, \\ &B^{(4)} = \Gamma^{(4)} \gamma^7 \gamma^9 = \sigma_1 \otimes -i\sigma_2. \end{aligned} \tag{A.7}$$

Notice that $B^{(6)\star} B^{(6)} = B^{(4)\star} B^{(4)} = -1$, and in particular, $B^{(4)T} = -B^{(4)}$.

• With coordinate indices, $\varepsilon_{m_1 \dots m_d}$ denotes the Levi-Civita tensor. In particular, $\varepsilon_{6789} = \sqrt{|g|}$ with the metric referring to that in (2.3). However, with indices labeling spinors, ε_{ij} is defined with the values $\varepsilon_{12} = \varepsilon^{12} = 1$.

- We have also defined

$$\not{H} = \frac{1}{n!} H_{N_1 \dots N_n} \Gamma^{N_1 \dots N_n}. \tag{A.8}$$

- Some useful gamma matrix identities are

$$\gamma_{mn} \Gamma^{(4)} = \frac{1}{2} \varepsilon_{mnpq} \gamma^{pq}, \qquad \gamma_m \Gamma^{(4)} = -\frac{1}{6} \varepsilon_{mnpq} \gamma^{npq}, \tag{A.9}$$

$$\begin{aligned} &[\gamma_{mn}, \gamma^r] = -4\delta^r{}_{[m} \gamma_{n]}, && \{\gamma_{mn}, \gamma^r\} = 2\gamma_{mn}{}^r, \\ &[\gamma_{mnp}, \gamma^r] = 2\gamma_{mnp}{}^r, && \{\gamma_{mnp}, \gamma^r\} = 6\delta^r{}_{[m} \gamma_{np]}, \\ &[\gamma_{mn}, \gamma^{pq}] = -4\delta_{[m}{}^{[p} \gamma_{n]}{}^{q]}, && \{\gamma_{mn}, \gamma^{pq}\} = 2\gamma_{mn}{}^{pq} - 4\delta_{[mn]}{}^{pq}. \end{aligned} \tag{A.10}$$

- The 4d Fierz identity

$$\chi \psi^\dagger = \frac{1}{4} \sum_{n=0}^{4} \frac{1}{n!} \Gamma^{c_n \dots c_1} \psi^\dagger \Gamma_{c_1 \dots c_n} \chi, \tag{A.11}$$

can be used together with (A.10) to derive the following useful formula

$$\begin{aligned} \lambda^{\dagger i} \gamma_m{}^n \lambda_j \lambda^{\dagger k} \gamma_n{}^p \lambda_l = {} & \lambda^{\dagger i} \gamma_m{}^p \lambda_l \lambda^{\dagger k} \lambda_j - \lambda^{\dagger i} \lambda_l \lambda^{\dagger k} \gamma_m{}^p \lambda_j \\ & + \delta_m{}^p \left(2\lambda^{\dagger i} \lambda_l \lambda^{\dagger k} \lambda_j - \lambda^{\dagger i} \lambda_j \lambda^{\dagger k} \lambda_l\right). \end{aligned} \tag{A.12}$$

Appendix B

Some formulas and definitions of the fields in type IIB supergravity [22].

$$\begin{aligned} &\tau = \tau_1 + i\tau_2 = C_0 + i e^{-\phi}, \qquad P_M = f^2 \partial_M B, \\ &B = \frac{1 + i\tau}{1 - i\tau}, \qquad Q_M = f^2 \operatorname{Im}(B \partial_M B^*), \\ &f = (1 - B^* B)^{-1/2}, \qquad G_{MNP} = f(F_{MNP} - B F^*_{MNP}). \end{aligned} \tag{B.1}$$

F_{MNP} is related to the NS–NS and R–R two-forms by $F_3 = \frac{g}{\kappa}(dB_2 + i\, dC_2)$ where $g^2 = 2\kappa^2/((2\pi)^7 \alpha'^4)$ [14]. If we take $\tau_1 = C_0 = 0$, then the formulas reduce to

$$\begin{aligned} &B = \tanh \frac{\phi}{2}, \qquad P_M = -\frac{\partial_M \tau_2}{2\tau_2} = \frac{1}{2} \partial_M \phi, \\ &f = \frac{1 + \tau_2}{2\sqrt{\tau_2}} = \cosh \frac{\phi}{2}, \qquad Q_M = \frac{-1}{2\tau_2} \operatorname{Im}\left[\partial_M \tau_2 \left(\frac{1 - \tau_2}{1 + \tau_2}\right)\right] = 0, \end{aligned} \tag{B.2}$$

$$\begin{aligned} G_{MNP} &= \frac{g}{\kappa}\left(f(1 - B)\, dB_2 + i f(1 + B)\, dC_2\right) \\ &= \frac{g}{\kappa}\left(e^{-\phi/2} dB_2 + i e^{\phi/2} dC_2\right). \end{aligned} \tag{B.3}$$

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