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Moduli space of torsional manifolds

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Abstract

We characterize the geometric moduli of non-Kähler manifolds with torsion. Heterotic supersymmetric flux compactifications require that the six-dimensional internal manifold be balanced, the gauge bundle be Hermitian Yang–Mills, and also the anomaly cancellation be satisfied. We perform the linearized variation of these constraints to derive the defining equations for the local moduli. We explicitly determine the metric deformations of the smooth flux solution corresponding to a torus bundle over $K3$.

1. Introduction

Ever since the discovery of Calabi–Yau compactifications [1], string theorists have tried to make the connection to the Minimal Supersymmetric Standard Model (MSSM) and grand unified theories (GUT). This turned out to be a difficult problem, as many times "exotic particles" appear along the way. These are particles that play no role in the current version of the MSSM.[1] Recently [2,3] have made a rather interesting proposal for three generation models without exotics in the context of Calabi–Yau compactifications of the heterotic string.[2]

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[1] It is, of course, possible that additional particles not known at present might be discovered, leading to an extension of the MSSM.

[2] String duality implies that in principle one could get realistic models in the context of type II theories. A concrete proposal has been made recently in terms of a D3-brane in the presence of a dP_8 singularity [4]. Alternatively, one could use intersecting D-brane models. For a review see [5].

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Even though these models have some rather interesting features, it is not possible to predict with them the values of the coupling constants of the Standard Model, because compactifications on conventional Calabi–Yau compactifications lead to unfixed moduli, and therefore additional massless scalars. This issue can only be addressed in the context of flux compactifications, which are known to lift the moduli [6,7].

If flux compactifications are considered in the context of the heterotic theory, the resulting internal geometry is a non-Kähler manifold with torsion [8–10]. Simple examples of such compactifications were constructed in [11,12] in the orbifold limit and a smooth compactification was constructed in [13,14] in terms of a T^2 bundle over $K3$. See [15–18] for some related works. It would be extremely exciting to construct a torsional manifold with all the features of the MSSM. At present, we are not yet at such a state. Many properties of Calabi–Yau manifolds are not shared by non-Kähler manifolds with torsion, so that well-known aspects of Calabi–Yau manifolds need to be rederived for these manifolds.

One of the important open questions is to understand how to characterize the scalar massless fields, in other words, the moduli space of heterotic flux compactifications. We investigate this question by analyzing the local moduli space emerging in such compactifications from a spacetime approach. A massless scalar field in the effective four-dimensional theory emerges for each independent modulus of the background geometry. Thus, the dimension of the moduli space corresponds to the number of massless scalar fields in the theory. In our analysis, we restrict to supersymmetric deformations, as we expect the analysis of the supersymmetry constraints to be easier than the analysis of the equations of motion. While the later equations are corrected by R^2 terms, the form of the supersymmetry transformations is not modified to R^2 order, as long as the heterotic anomaly cancellation condition is imposed [19]. That a solution of both the supersymmetry constraints and the modified Bianchi identity is also a solution to the equations of motion has been shown in [20,21].

Unlike the Calabi–Yau case, the supersymmetry constraint equations in general non-linearly couple the various fields and thus the analysis even at the linearized variation level is non-trivial. As an example of our general analysis, we shall give the description of the scalar metric moduli for the smooth solution of a T^2 bundle over $K3$ presented in [13,14]. It is an interesting question to understand whether the massless moduli found in our approach are lifted by higher order terms in the low energy effective action. For conventional Calabi–Yau compactifications it is known that moduli fields appearing in the leading order equations will remain massless even if higher order corrections are taken into account [22,23]. In our case, such an analysis has not been performed yet from the spacetime point of view, though the question can be answered from the world-sheet approach recently developed in [18]. In this work, a gauged linear sigma model was constructed which in the IR flows to an interacting conformal field theory. The analysis of the linear model indicates that massless fields emerging at leading order in α' will remain massless, even if corrections to the spacetime action are taken into account.

This paper is organized as follows. In Section 2, we perform the linear variation of the supersymmetry constraints. In Section 3, we analyze the variation of the T^2 bundle over $K3$ solution and discuss its local moduli space. In Section 4, conclusions and future directions are presented. In Appendix A, we clarify some of mathematical notations that we used.

2. Determining equations for the moduli fields

The non-Kähler manifolds with torsion $\mathcal{M}$ that we are interested in are complex manifolds described in terms of a Hermitian form which is related to the metric

$$J = i g_{a\bar{b}}\, dz^a \wedge d\bar{z}^{\bar{b}}, \tag{2.1}$$

and a no-where vanishing holomorphic three-form

$$d\Omega = 0, \tag{2.2}$$

satisfying $J \wedge \Omega = 0$. The geometry can be deformed by either deforming the Hermitian form or deforming the complex structure of $\mathcal{M}$. We are interested in deformations that preserve the supersymmetry constraints as well as the anomaly cancellation condition.

$\mathcal{N} = 1$ supersymmetry for heterotic flux compactifications to four spacetime dimensions imposes three conditions: the internal geometry has to be conformally balanced, the gauge bundle satisfies the Hermitian Yang–Mills equation, and the H-flux satisfies the anomaly cancellation condition. Explicitly, they are [13,14]

$$d\big(\|\Omega\|_J\, J \wedge J\big) = 0, \tag{2.3}$$

$$F^{(2,0)} = F^{(0,2)} = 0, \qquad F_{mn} J^{mn} = 0, \tag{2.4}$$

$$2i\partial\bar{\partial} J = \frac{\alpha'}{4}\big[\mathrm{tr}(R \wedge R) - \mathrm{tr}(F \wedge F)\big]. \tag{2.5}$$

Above, we have replaced the two standard background fields—the three-form H and the dilaton field ϕ—with the required supersymmetric relations

$$H = i(\bar{\partial} - \partial) J, \tag{2.6}$$

$$\|\Omega\|_J = e^{-2(\phi+\phi_0)}. \tag{2.7}$$

Doing so allows us to consider the constraint equations solely in terms of the geometrical data (J, Ω) and the gauge bundle.

Deformations of the metric that are of pure type, i.e. $(0,2)$ or $(2,0)$, describe deformations of the complex structure

$$\Omega_{ab}{}^{\bar{d}}\, \delta g_{\bar{d}\bar{c}}\, dz^a \wedge dz^b \wedge d\bar{z}^{\bar{c}}, \tag{2.8}$$

while deformations of mixed type, i.e. of type $(1,1)$, describe deformations of the Hermitian form

$$i\delta g_{a\bar{b}}\, dz^a \wedge d\bar{z}^{\bar{b}}. \tag{2.9}$$

We analyze below the linear variation of the three constraint equations (2.3)–(2.5) with respect to a background solution. For simplicity, we shall keep the complex structure of the six-dimensional internal geometry fixed. For the moduli space of Calabi–Yau compactifications, it turns out that the Kähler and complex structure deformations decouple from one another [24]. It would be interesting to determine whether some decoupling still persists in the non-Kähler case and more generally how the Hermitian and complex structure deformations are coupled. We will leave this more general analysis for future work.

2.1. Conformally balanced condition

We consider the linear variation of the conformally balanced condition (2.3). We shall vary the metric or Hermitian form $J_{a\bar{b}} = i g_{a\bar{b}}$ while holding fixed the complex structure. Let

$$J'_{a\bar{b}} = J_{a\bar{b}} + \delta J_{a\bar{b}}, \tag{2.10}$$

then we have to first order in δJ,

$$J' \wedge J' = J \wedge J + 2J \wedge \delta J, \tag{2.11}$$

$$\begin{aligned}\|\Omega\|^2_{J'} &= \frac{|g_{a\bar{b}}|}{|g'_{a\bar{b}}|}\|\Omega\|^2_J = \frac{|g_{a\bar{b}}|}{|g_{a\bar{b}}|(1+g^{c\bar{d}}\delta g_{c\bar{d}})}\|\Omega\|^2_J \\ &= \big(1 - g^{c\bar{d}}\delta g_{c\bar{d}}\big)\|\Omega\|^2_J.\end{aligned} \tag{2.12}$$

Note that (2.7) with (2.12) imply the dilaton variation

$$\delta\phi = \frac{1}{4}g^{a\bar{b}}\delta g_{a\bar{b}} = \frac{1}{8}J^{mn}\delta J_{mn}. \tag{2.13}$$

The linear variation of the conformally balanced condition can be written as

$$d\big(\|\Omega\|_{J'}J' \wedge J'\big) = d\big(\|\Omega\|_J J \wedge J + 2\delta\rho\big) = 0, \tag{2.14}$$

where $\delta\rho$ is a four-form given by

$$\delta\rho = \|\Omega\|_J\left[J \wedge \delta J - \frac{1}{8}(J \wedge J)J^{mn}\delta J_{mn}\right]. \tag{2.15}$$

We can invert (2.15) and express δJ in terms of $\delta\rho$. To do this, we note that any $(2,2)$-form, ω_4, can be Lefschetz decomposed as follows

$$\omega_4 = L\Lambda\omega_4 - \frac{1}{4}L^2\Lambda^2\omega_4, \tag{2.16}$$

where the Lefschetz operator L and its adjoint Λ have the following action on exterior forms

$$\begin{aligned} L&: \quad \omega \to J \wedge \omega, \\ \Lambda&: \quad \omega \to J \llcorner \omega.\end{aligned} \tag{2.17}$$

Comparing (2.15) with (2.16), we find the relation

$$\delta J_{mn} = \frac{1}{2\|\Omega\|_J}\delta\rho_{mnrs}J^{rs}. \tag{2.18}$$

From the linear variation of Eq. (2.14), we observe that the allowed deformations (i.e. which preserve the conformally balanced condition) satisfy $d\delta\rho = 0$. Eq. (2.18) implies that any variation of the Hermitian metric can be expressed in terms of a variation by a closed $(2,2)$-form. Equivalently, we can also express the linear variation condition directly for the Hermitian metric as

$$d^*\left[\delta J' - \frac{1}{4}J\big(J^{mn}\delta J'_{mn}\big)\right] = 0, \tag{2.19}$$

where $\delta J' = \|\Omega\|_J\delta J$.

Note that δJ variations that are equivalent to a coordinate transformation (i.e. a diffeomorphism) are physically unobservable and must therefore be quotient out. Under an infinitesimal coordinate transformation

$$y'^m = y^m + v^m(y), \tag{2.20}$$

the variation of a p-form ω_p is given by the Lie derivative

$$\delta\omega_p = -\mathcal{L}_v\omega_p = -\big[i_v(d\omega_p) + d(i_v\omega_p)\big], \tag{2.21}$$

where $v = v^m \partial_m$ is a vector field and i_v denotes the interior product. For the conformally balanced four-form, a coordinate transformation results in

$$\mathcal{L}_v\big(\|\Omega\|_J J \wedge J\big) = d\big[i_v\big(\|\Omega\|_J J \wedge J\big)\big]. \tag{2.22}$$

We can thus identify, as physically not relevant, $\delta\rho$ variations that are exterior derivatives of a non-primitive three-form

$$\delta\rho \sim d\big(\|\Omega\|_J \beta \wedge J\big), \tag{2.23}$$

where $\beta_m = v^n J_{nm}$. Using (2.18), this corresponds to deformations of the Hermitian form

$$\delta J \sim \frac{1}{\|\Omega\|_J} \Lambda d\big(\|\Omega\|_J \beta \wedge J\big). \tag{2.24}$$

Let us now interpret the content of the above variation formulas. By the identification of (2.18), variations of the Hermitian metric that preserve the conformally balanced condition can be parametrized by closed $(2,2)$-forms. Moreover, modding out by diffeomorphisms results in the cohomology[3]

$$\frac{\ker(d) \cap \Lambda^{2,2}}{d(\beta \wedge J)}. \tag{2.25}$$

Thus, the space of conformally balanced metrics is equivalent to the space of closed $(2,2)$-forms modded out by those which are exterior derivatives of a non-primitive three-form. But notice that exact forms which are exterior derivative of a primitive three-form are not quotient out. Hence, if there exists such a primitive three-form, ω_3^0, then the space of balanced metrics is infinite-dimensional. This is because $d(f\omega_3^0)$ where f is any real function would be closed but not modded out.

The cohomology of (2.25) can also be expressed directly in terms of $(1,1)$-forms. From (2.19), every co-closed $(1,1)$-form defines a metric deformation preserving the conformally balanced condition. To see this explicitly, we note that any $(1,1)$-form can be Lefschetz decomposed as follows

$$\begin{aligned} C_{mn} &= (C_0)_{mn} + \frac{1}{6} J_{mn} J^{rs} C_{rs} \\ &\equiv (C_0)_{mn} + \frac{1}{3} J_{mn} C_\Lambda, \end{aligned} \tag{2.26}$$

where C_0 denotes the primitive part and $C_\Lambda = \frac{1}{2} J^{rs} C_{rs}$ encodes the non-primitivity of C_{mn}. We can therefore re-express (2.19) as

$$\begin{aligned} 0 &= d^*\left(\delta J' - \frac{1}{2} J \delta J'_\Lambda\right) \\ &= d^*\left(\delta J'_0 - \frac{1}{6} J \delta J'_\Lambda\right) \\ &= d^* C, \end{aligned} \tag{2.27}$$

[3] Note that complex structures are also defined up to diffeomorphism. So any diffeomorphism generated by a real vector field will keep the complex structure in the same equivalence class.

where we have defined a new $(1,1)$-form $C = C_0 + \frac{1}{3}JC_\Lambda$ with $C_0 = \delta J'_0$ and $C_\Lambda = -\frac{1}{2}\delta J'_\Lambda$. Furthermore, variations associated with diffeomorphisms can be written as

$$\delta J' \sim \Lambda d\big(\|\Omega\|_J \beta \wedge J\big), \tag{2.28}$$

so that we have

$$\left(\delta J' - \frac{1}{2} J \delta J'_\Lambda\right) \sim d^*(\tilde{\beta}' \wedge J), \tag{2.29}$$

where $\tilde{\beta}'_m = J_m{}^n \beta_n \|\Omega\|_J$. Eqs. (2.27) and (2.28) together imply the cohomology

$$\frac{\ker(d^*) \cap \Lambda^{1,1}}{d^*(\beta \wedge J)}. \tag{2.30}$$

Therefore, the local moduli space can also be described as spanning all co-closed $(1,1)$-forms modulo those which are d^* of non-primitive three-forms. This space is isomorphic to that of (2.25) and is in general infinite-dimensional. We have however yet to consider the two other supersymmetry constraints. Imposing them, especially the anomaly cancellation condition, will greatly reduce the number of allowed deformations and render the moduli space finite-dimensional. This can be seen clearly in the T^2 bundle over $K3$ example discussed in the next section.

Finally, let us point out that if we had taken into consideration variations of the complex structure, then a δJ variation will in general include also a $(2,0)$ and a $(0,2)$ part. Nevertheless, $J + \delta J$ must still be a $(1,1)$-form with respect to the deformed complex structure as is required by supersymmetry.

2.2. Hermitian Yang–Mills condition

Any variation of the Hermitian gauge connection with the complex structure held fixed will preserve the holomorphic condition $F^{(2,0)} = F^{(0,2)} = 0$. As for the primitivity condition $F_{mn}J^{mn} = 0$, we shall vary its equivalent form

$$0 = \delta(F \wedge J \wedge J) = \delta F \wedge J^2 + 2F \wedge J \wedge \delta J. \tag{2.31}$$

The Hermitian field strength F can be written as

$$F_{\bar{a}b} = \bar{\partial}_{\bar{a}} A_b = \bar{\partial}_{\bar{a}}\big(h^{\alpha\bar{\beta}} \partial_b h_{\bar{\beta}\gamma}\big) = \bar{\partial}_{\bar{a}}\big(\bar{h}^{-1}\partial_b \bar{h}\big), \tag{2.32}$$

where α, $\bar{\beta}$, γ are gauge indices and $\bar{h} = h_{\bar{\beta}\alpha}$ is the transpose of the Hermitian metric on the gauge bundle. Under the variation, $\bar{h}' = \bar{h} + \delta\bar{h}$, the gauge field varies as

$$\begin{aligned} \delta A = A' - A &= \bar{h}^{-1}\partial(\delta\bar{h}) + \delta\bar{h}^{-1}\partial\bar{h} \\ &= \bar{h}^{-1}\partial\big[\bar{h}\big(\bar{h}^{-1}\delta\bar{h}\big)\big] - \bar{h}^{-1}\delta\bar{h}\big(\bar{h}^{-1}\partial\bar{h}\big) \\ &= \partial\big(\bar{h}^{-1}\delta\bar{h}\big) + A\big(\bar{h}^{-1}\delta\bar{h}\big) - \big(\bar{h}^{-1}\delta\bar{h}\big)A \\ &\equiv D^A\big(\bar{h}^{-1}\delta\bar{h}\big). \end{aligned} \tag{2.33}$$

This implies that the field strength varies as $\delta F = \bar{\partial}(D^A(\bar{h}^{-1}\delta\bar{h}))$. Inserting into (2.31), we obtain

$$0 = \bar{\partial}\big(D^A\big(\bar{h}^{-1}\delta\bar{h}\big)\big) \wedge J^2 + 2F \wedge J \wedge \delta J. \tag{2.34}$$

This gives the constraint relation between the variations of the Hermitian form and the gauge field. The pair $(\delta J, \delta h)$ will be further constrained when inserted into the anomaly cancellation condition as we now show.

2.3. Anomaly cancellation condition

We can write the variation of the anomaly cancellation equation as

$$2i\partial\bar{\partial}\delta J = \frac{\alpha'}{2}\big(\mathrm{tr}\big[R(g)\wedge\delta R(g)\big] - \mathrm{tr}\big[F(h)\wedge\delta F(h)\big]\big). \tag{2.35}$$

The left-hand side is a $\partial\bar{\partial}$ of a $(1,1)$-form, so we should write the variation of the right-hand side of the equation similarly. With the curvature defined using the Hermitian connection, we can write the variation using the Bott–Chern form [25,26]. For two Hermitian metrics (g_1, g_0) that are smoothly connected by a path parameterized by a parameter $t \in [0,1]$, the difference of the first Pontryagin classes is given by the Bott–Chern form

$$\mathrm{tr}[R_1\wedge R_1] - \mathrm{tr}[R_0\wedge R_0] = 2i\partial\bar{\partial}BC_2(g_1, g_0), \tag{2.36}$$

where

$$BC_2(g_1, g_0) = 2i\int_0^1 \mathrm{tr}\big[R_t\bar{g}_t^{-1}\dot{\bar{g}}_t\big]\,dt, \tag{2.37}$$

and $\bar{g} = g_{\bar{a}b}$ denotes the transpose of the Hermitian metric, the dot denotes the derivative with respect to t, and the "tr" in (2.37) traces over only the holomorphic indices.[4] We now use the Bott–Chern formula to obtain the variation. Let

$$g_t = g + t(g' - g) = g + t\delta g, \tag{2.38}$$

where $t\in[0,1]$ and in particular $g_0 = g$ and $g_1 = g'$. Then to first order in δg, we have

$$\delta\big(\mathrm{tr}[R\wedge R]\big) = 2\,\mathrm{tr}[R\wedge\delta R] = -4\partial\bar{\partial}\big(\mathrm{tr}\big[R\bar{g}^{-1}\delta\bar{g}\big]\big), \tag{2.39}$$

where the trace can be more simply written in components as

$$\mathrm{tr}\big[R\bar{g}^{-1}\delta\bar{g}\big]_{a\bar{b}} = -iR_{a\bar{b}}{}^{c\bar{d}}\delta J_{c\bar{d}}. \tag{2.40}$$

With (2.39), the linear variation of the anomaly equation (2.41) becomes

$$2i\partial\bar{\partial}\delta J = -\alpha'\partial\bar{\partial}\big(\mathrm{tr}\big[R\bar{g}^{-1}\delta\bar{g}\big] - \mathrm{tr}\big[F\bar{h}^{-1}\delta\bar{h}\big]\big). \tag{2.41}$$

By factoring out the $2i\partial\bar{\partial}$ derivatives, the anomaly condition can be equivalently expressed as

$$\delta J - i\frac{\alpha'}{2}\big(\mathrm{tr}\big[R\bar{g}^{-1}\delta\bar{g}\big] - \mathrm{tr}\big[F\bar{h}^{-1}\delta\bar{h}\big]\big) = \gamma, \tag{2.42}$$

where γ is a $\partial\bar{\partial}$ closed $(1,1)$-form.

Note that for the special case where either the gauge bundle is trivial (i.e. $F = 0$) or $\delta h = 0$, there is a simple relationship between δJ and γ. The anomaly variation with (2.40) inserted

[4] Note that the Bott–Chern form is defined only up to ∂ and $\bar{\partial}$ exact terms.

into (2.42) becomes

$$\delta J_{a\bar{b}} - \frac{\alpha'}{2} R_{a\bar{b}}{}^{c\bar{d}} \delta J_{c\bar{d}} = \gamma_{a\bar{b}}. \tag{2.43}$$

Grouping the two Hermitian indices $(a\bar{b})$ as a single index, we can solve for δJ by inverting the above equation and obtain

$$\delta J = (1-M)^{-1}\gamma, \tag{2.44}$$

where the curvature is encoded in the matrix $M_{a\bar{b}}{}^{c\bar{d}} = \frac{\alpha'}{2} R_{a\bar{b}}{}^{c\bar{d}}$. As long as $(1-M)$ is invertible, we see that δJ is parametrized by the space of $\partial\bar{\partial}$-closed $(1,1)$-forms γ. Modding out by diffeomorphism equivalence, we can obtain a cohomology associated with the anomaly equation of the form

$$\frac{\ker(\partial\bar{\partial}) \cap \Lambda^{1,1}}{\gamma_{\text{diff}}}, \tag{2.45}$$

where

$$\gamma_{\text{diff}} = (1-M)\delta J_{\text{diff}} = \frac{1}{\|\Omega\|_J}(1-M)\Lambda d\big(\|\Omega\|_J \beta \wedge J\big), \tag{2.46}$$

and δJ_{diff} is the variation of the Hermitian form corresponding to diffeomorphism given in (2.24).

To summarize, we list the three linear variation conditions with complex structure fixed.

$$d\left(\|\Omega\|_J\left[2J\wedge\delta J - \frac{1}{4}(J\wedge J)J^{mn}\delta J_{mn}\right]\right) = 0, \tag{2.47}$$

$$\bar{\partial}\big(D^A\big(\bar{h}^{-1}\delta\bar{h}\big)\big)\wedge J^2 + 2F\wedge J\wedge\delta J = 0, \tag{2.48}$$

$$\partial\bar{\partial}\left(\delta J - i\frac{\alpha'}{2}\big[\mathrm{tr}\big[R\bar{g}^{-1}\delta\bar{g}\big] - \mathrm{tr}\big[F\bar{h}^{-1}\delta\bar{h}\big]\big]\right) = 0. \tag{2.49}$$

In the next section, we will write down explicit deformations that satisfy the above equations for the T^2 bundle over $K3$ flux background.

3. T^2 bundle over $K3$ solution

The metric of the T^2 bundle over $K3$ solution [13,14] has the form

$$\begin{aligned} ds^2 &= e^{2\phi}\, ds^2_{K3} + (dx+\alpha_1)^2 + (dy+\alpha_2)^2 \\ &= e^{2\phi}\, ds^2_{K3} + |dz^3+\alpha|^2, \end{aligned} \tag{3.1}$$

where $\theta = dz^3 + \alpha$ is a $(1,0)$-form and $\alpha = \alpha_1 + i\alpha_2$. The twisting of the T^2 is encoded in the two-form defined on the base $K3$,

$$\omega = \omega_1 + i\omega_2 = d\theta = \omega_S^{(2,0)} + \omega_A^{(1,1)}, \tag{3.2}$$

which is required to be primitive

$$\omega\wedge J_{K3} = 0, \tag{3.3}$$

and obeys the quantization condition

$$\tilde{\omega}_i = \frac{\omega_i}{2\pi\sqrt{\alpha'}} \in H^2(K3,\mathbb{Z}). \tag{3.4}$$

With this metric ansatz, the anomaly cancellation equation reduces to a highly non-linear second-order differential equation for the dilaton ϕ. Importantly, a necessary condition for the existence of a solution for ϕ is that the background satisfies the topological condition

$$\int_{K3} \left(\|\tilde{\omega}_S\|^2 + \|\tilde{\omega}_A\|^2\right) \frac{J_{K3} \wedge J_{K3}}{2!} + \frac{1}{16\pi^2} \int_{K3} \mathrm{tr}\, F \wedge F = 24. \tag{3.5}$$

If this condition is satisfied, then the analysis of Fu and Yau [13] guarantees the existence of a smooth solution for ϕ that solves the differential equation of anomaly cancellation.

3.1. Equations for the moduli

For expressing the constraint equations of the allowed deformations, we first write down more explicitly the Hermitian metric. Note that the conventions we follow here are that $J_{a\bar{b}} = i g_{a\bar{b}}$ and $ds^2 = 2 g_{a\bar{b}}\, dz^a\, d\bar{z}^{\bar{b}}$. The Hermitian two-form can be expressed simply as

$$J = e^{2\phi} J_{K3} + \frac{i}{2} \theta \wedge \bar{\theta}, \tag{3.6}$$

and we write the corresponding metric as

$$g_{a\bar{b}} = \frac{1}{2} \begin{pmatrix} 2g' + \mathcal{B}\mathcal{B}^* & \mathcal{B} \\ \mathcal{B}^* & 1 \end{pmatrix}, \tag{3.7}$$

where $g'_{i\bar{j}} = e^{2\phi} g_{K3}$ is the base $K3$ metric with the $e^{2\phi}$ warp factor included, $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$ is a column vector with entries locally given by $\alpha = \mathcal{B}_1\, dz^1 + \mathcal{B}_2\, dz^2$, and $\mathcal{B}^* = \mathcal{B}^\dagger$.

An allowed deformation of the conformally balanced condition must satisfy the requirement that the four-form (2.15)

$$\begin{aligned} \delta\rho &= \|\Omega\|_J \left[J \wedge \delta J - \frac{1}{8} (J \wedge J) J^{mn} \delta J_{mn} \right] \\ &= J_{K3} \wedge \delta J + \frac{i}{2} e^{-2\phi} \theta \wedge \bar{\theta} \wedge \delta J - \frac{1}{8} \left(e^{2\phi} J_{K3} \wedge J_{K3} + i J_{K3} \wedge \theta \wedge \bar{\theta}\right) J^{mn} \delta J_{mn}, \end{aligned} \tag{3.8}$$

is d-closed.

As for the anomaly condition, we shall work with the constraint given in the form of (2.49) (with trivial gauge bundle)

$$\partial\bar{\partial}\left(\delta J - i \frac{\alpha'}{2} \mathrm{tr}\left[R \bar{g}^{-1} \delta\bar{g}\right] \right) = 0. \tag{3.9}$$

The curvature term can be written out explicitly as

$$\mathrm{tr}\left[R \bar{g}^{-1} \delta\bar{g}\right] = i \left(\tilde{R}^{\bar{j} i} \delta J_{i\bar{j}} + \tilde{R}^{\bar{j} 3} \delta J_{3\bar{j}} + \tilde{R}^{\bar{3} i} \delta J_{i\bar{3}} + \tilde{R}^{\bar{3} 3} \delta J_{3\bar{3}} \right), \tag{3.10}$$

where

$$\tilde{R}^{\bar{j} i} = -g'^{-1} R' - \frac{1}{2} \left(g'^{-1} \bar{\partial} \mathcal{B}\right) \left(\partial \mathcal{B}^* g'^{-1}\right), \tag{3.11}$$

$$\tilde{R}^{\bar{j} 3} = g'^{-1} R' \mathcal{B} + \partial\left(g'^{-1} \bar{\partial} \mathcal{B}\right) - \frac{1}{2} \left(g'^{1} \bar{\partial} \mathcal{B}\right) \left(\partial \mathcal{B}^* g'^{-1}\right) \mathcal{B}, \tag{3.12}$$

$$\tilde{R}^{\bar{3}i} = \mathcal{B}^* g'^{-1} R' - \bar{\partial}\big(\partial \mathcal{B}^* g'^{-1}\big) + \frac{1}{2}\mathcal{B}^*\big(g'^{-1}\bar{\partial}\mathcal{B}\big)\big(\partial\mathcal{B}^* g'^{-1}\big), \tag{3.13}$$

$$\tilde{R}^{\bar{3}3} = -\mathcal{B}^* g'^{-1} R'\mathcal{B} + \frac{1}{2}\mathcal{B}^* g'^{-1}\bar{\partial}\mathcal{B}\big(\partial\mathcal{B}^* g'^{-1}\big)\mathcal{B} + \bar{\partial}\big(\partial\mathcal{B}^* g'^{-1}\big)\mathcal{B} - \mathcal{B}^*\partial\big(g'^{-1}\bar{\partial}\mathcal{B}\big) - \partial\mathcal{B}^*\big(g'^{-1}\bar{\partial}\mathcal{B}\big), \tag{3.14}$$

and $R' = \bar{\partial}(\bar{g'}^{-1}\partial\bar{g'})$ is the curvature tensor of $K3$ with respect to the g' metric. Note that the $\tilde{R}^{\bar{b}a}$ are two-forms with components only on the coordinates of $K3$.

Below, we shall analyze the infinitesimal deformations of the T^2 bundle over $K3$ model with trivial gauge bundle. For this type of model, the topological constraint (3.5) is satisfied purely by the curvature of the T^2 twist. (See Section 5.2 in [14] for explicit examples.) We shall discuss the variation of the three components of the metric—the dilaton conformal factor, the $K3$ base, and the T^2 bundle—separately below. We will show that the moduli given below satisfy both the conformally balanced and anomaly cancellation condition. For the trivial bundle case, the Hermitian Yang–Mills condition does not place any constraint on the deformations. Finally, we will also discuss the variation of the complex structure in this model.

3.2. Deformation of the dilaton

The dilaton is associated to the warp factor of the $K3$ base. Thus, varying the dilaton corresponds to varying the local scale of the $K3$. The deformation of the Hermitian form due to the variation of the dilaton is

$$\delta J = 2\delta\phi e^{2\phi} J_{K3}, \tag{3.15}$$

where $\delta\phi$ depends only on the $K3$ coordinates. This is consistent with the dilaton variation condition $\delta\phi = (1/8)J^{mn}\delta J_{mn}$ of Eq. (2.13). As for the conformally balanced condition, it in fact does not place any constraint on the dilaton. The metric variation (3.15) when inserted into (3.8) gives the four-form

$$\delta\rho = e^{2\phi} J_{K3} \wedge J_{K3}\delta\phi, \tag{3.16}$$

which is indeed d-closed for any real function $\delta\phi$ on the base $K3$. Since the space of real function is infinite-dimensional, the dimensionality of the deformation space is also infinite if only the conformally balanced condition is considered.

Imposing anomaly cancellation condition will however make the deformation space finite. Anomaly cancellation (3.9) imposes the condition

$$\partial\bar{\partial}\left(\left[2e^{2\phi}J_{K3} - i\frac{\alpha'}{2}e^{-2\phi}\operatorname{tr}\big[\bar{\partial}\mathcal{B}\wedge\partial\mathcal{B}^* g_{K3}^{-1}\big] + 4\bar{\partial}\partial\phi\right]\delta\phi\right) = 0, \tag{3.17}$$

where we have used (3.11). The analysis of Fu and Yau [13] guarantees only a one-parameter family of solutions parametrized by the normalization

$$A = \left(\int_{K3} e^{-8\phi}\frac{J_{K3}\wedge J_{K3}}{2!}\right)^{1/4}, \tag{3.18}$$

as long as the topological condition (3.5) is satisfied and also $A \ll 1$. (See [14] for a discussion of the physical implications of the $A \ll 1$ bound.) The variation of the dilaton can thus be parametrized by the value of A.[5]

3.3. Deformations of the K3 metric

The metric moduli of the $K3$ are associated with deformations of the Hermitian form J_{K3} such that the curvatures of the T^2 bundle, ω_i for $i = 1, 2$, remain primitive (3.3). This implies that the allowed variation of δJ_{K3} satisfies

$$\omega_i \wedge \delta J_{K3} + \delta\omega_i \wedge J_{K3} = 0, \quad i = 1, 2. \tag{3.19}$$

Hence, of the 20 possible $h^{1,1}$ Kähler deformations of $K3$, only the subset that satisfies (3.19) is allowed.

First, consider the case where $\delta\omega_i = 0$. We then have the condition

$$\omega_i \wedge \delta J_{K3} = 0, \tag{3.20}$$

which must be satisfied locally at every point on $K3$. With the curvature form ω containing a $(1, 1)$ part, (3.20) is a very strong condition that in general can only be satisfied by a variation proportional to the Hermitian form, $\delta J_{K3} \sim J_{K3}$. But this would then be the modulus identified above as associated with the dilaton (3.15).

More generally, we can have $\delta\omega_i = i\partial\bar{\partial} f_i$, where f_i for $i = 1, 2$ are functions on the base $K3$. This form of $\delta\omega_i$ is required so that the variation does not change the $H^2(K3)$ integral class of ω_i as required by the quantization of (3.14). Let $\delta J_{K3} = \eta \in H^{1,1}(K3)$ and not proportional to J_{K3}, then the variation (3.19) corresponds to

$$\begin{aligned} 0 &= \omega_i \wedge \eta + i\partial\bar{\partial} f_i \wedge J_{K3} \\ &= \left(f_i' - \Delta f_i\right) \frac{J_{K3} \wedge J_{K3}}{2}. \end{aligned} \tag{3.21}$$

Here, we have replaced $\omega_i \wedge \eta = f_i' \frac{J_{K3} \wedge J_{K3}}{2}$ noting that the exterior product of two $(1, 1)$-forms on the base must be a function times the volume form of the $K3$. Now, the sufficient condition that a solution for f_i exists is that

$$\int_{K3} f_i' \frac{J_{K3} \wedge J_{K3}}{2} = \int_{K3} \omega_i \wedge \eta = 0. \tag{3.22}$$

But this is related to the requirement that the intersection numbers are zero. The intersection numbers of $K3$ are defined to be

$$d_{IJ} = \int_{K3} \tilde{\omega}_I \wedge \tilde{\omega}_J, \tag{3.23}$$

[5] Rigorously, one should be able to show that there does not exist a dilaton variation that satisfies (3.17) and leaves the normalization A unchanged. Regardless, the finite-dimensionality of the deformation space is ensured if one assumes the elliptic condition required by Fu and Yau [13] to solve the anomaly cancellation equation for ϕ.

where $\tilde{\omega}_I$, $I = 1, \ldots, 22$, denotes a basis of $H^2(K3, \mathbb{Z})$. The matrix d_{IJ} is the metric of the even self-dual lattice with Lorentzian signature $(3, 19)$ given by

$$(-E_8) \oplus (-E_8) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.24}$$

where

$$E_8 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \tag{3.25}$$

is the Cartan matrix of E_8 Lie algebra. Thus we see that a variation of $\delta J_{K3} = \eta$ is allowed as long as the intersection numbers of η with ω_i are zero. This implies at least that $\eta \neq \omega_1, \omega_2$.

The above variations of the Kähler form on the $K3$ require the metric variations

$$\begin{aligned} \delta J &= e^{2\phi}\eta + \frac{i}{2}(\delta\theta \wedge \bar{\theta} + \theta \wedge \delta\bar{\theta}) + 2\delta\phi e^{2\phi} J_{K3}, \\ \delta\rho &= \frac{i}{2}\big(\theta \wedge \bar{\theta} \wedge \eta + J_{K3} \wedge (\delta\theta \wedge \bar{\theta} + \theta \wedge \delta\bar{\theta})\big) + e^{2\phi} J_{K3} \wedge J_{K3}\delta\phi, \end{aligned} \tag{3.26}$$

where $\delta\theta = -i\partial(f_1 + if_2)$. One can check that the above $\delta\rho$ is closed when (3.21) is satisfied. We note that the additional variation of the dilaton in (3.26) is needed in order to satisfy the anomaly condition. With it, the analysis of Fu and Yau [13] then guarantees the existence of a solution for $\delta\phi$ for each consistent pair $(\eta, \delta\theta)$. Therefore, δJ variations in (3.26) satisfying (3.22) are indeed moduli.

3.4. Deformation of the T^2 bundle

We now consider the variation of the size of the T^2 bundle. This is an allowed variation of the conformally balanced condition since the metric variation

$$\delta J = \frac{i}{2}\epsilon\theta \wedge \bar{\theta}, \tag{3.27}$$

results in the closed four-form

$$\delta\rho = \epsilon\left(-\frac{1}{4}e^{2\phi} J_{K3} \wedge J_{K3} + \frac{i}{4}\theta \wedge \bar{\theta} \wedge J_{K3}\right), \tag{3.28}$$

where ϵ is a constant infinitesimal parameter. But we must also check the anomaly condition. The variation of the curvature term can be calculated using (3.11)–(3.14) and we obtain

$$\mathrm{tr}\big[R\bar{g}^{-1}\delta\bar{g}\big] = \frac{1}{2}\epsilon\, \mathrm{tr}\big[\bar{\partial}\mathcal{B} \wedge \partial\mathcal{B}^* g'^{-1}\big]. \tag{3.29}$$

The anomaly condition (3.9) therefore becomes

$$
\begin{aligned}
0 &= i\partial\bar\partial\left(\delta J - i\frac{\alpha'}{2}\,\mathrm{tr}\big[R\bar g^{-1}\delta\bar g\big]\right)\\
&= -\frac{1}{2}\epsilon\partial\bar\partial\left(\theta\wedge\bar\theta - \frac{\alpha'}{2}\,\mathrm{tr}\big[\bar\partial\mathcal{B}\wedge\partial\mathcal{B}^* g'^{-1}\big]\right)\\
&= \frac{1}{2}\epsilon\left(\|\omega\|^2\frac{J_{K3}^2}{2} + \frac{\alpha'}{2}\partial\bar\partial\,\mathrm{tr}\big[\bar\partial\mathcal{B}\wedge\partial\mathcal{B}^* g'^{-1}\big]\right),
\end{aligned}
\tag{3.30}
$$

but this cannot hold true. To see this, we can integrate the last line over the base $K3$. The first term gives a positive contribution while the second term integrates to zero. Here, we have used the fact that the two-form $\mathrm{tr}[\bar\partial\mathcal{B}\wedge\partial\mathcal{B}^* g'^{-1}]$ in the second term is well-defined and has dependence only on the base $K3$ as was shown in [13] (see Lemma 10 on page 11). Thus, the size of the torus cannot be continuously varied as it is fixed by the anomaly condition.

With the size of the torus fixed, it is evident that there cannot be any overall radial moduli $\delta J = \epsilon J$ for this model, as has also been noted previously in [21,27,28]. Actually, it is true in general that the anomaly cancellation forbids an overall constant radial modulus for any heterotic compactification with non-zero H-flux. The reason is simply that $\mathrm{tr}[R\wedge R]$ is invariant under constant scaling of the metric since the Riemann tensor, $R_{mn}{}^p{}_q$, is scale invariant. However $dH = 2i\partial\bar\partial J$ depends on J and cannot be scale invariant. Hence, the overall scale is not a modulus.

To summarize, the T^2 bundle over $K3$ model has a dilaton modulus and also moduli associated with the Kähler moduli of the base $K3$. The number of moduli in particular depends on the curvature of the T^2 twist, ω. The size of the T^2 is however fixed and hence there is no overall radial modulus in the model.

3.5. Fixing the complex structure

We have mostly taken the complex structure to be fixed in analyzing the moduli. But for the T^2 bundle over $K3$ solution, the complex structures are rather transparent and we can describe how they can be fixed. To begin, the complex structures are simply those on the $K3$ plus that on the T^2. For the T^2, its complex structure determines the integral first Chern class quantization condition (3.4) for ω_1 and ω_2. For an arbitrary torus complex structure $\tau = \tau_1 + i\tau_2$, the quantization conditions depend on τ and takes the form

$$
\frac{1}{2\pi\sqrt{\alpha'}}\int_\Gamma\left(\omega_1 - \frac{\tau_1}{\tau_2}\omega_2\right)\in\mathbb{Z},\qquad \frac{1}{2\pi\sqrt{\alpha'}\tau_2}\int_\Gamma\omega_2\in\mathbb{Z},
\tag{3.31}
$$

where $\Gamma\in H_2(K3,\mathbb{Z})$ is any two-cycle on $K3$. Therefore, fixing $\omega = \omega_1 + i\omega_2$ effectively fixes τ. And even if we were to allow ω to vary infinitesimally, the complex structure integrability condition $\omega = \omega_1 + i\omega_2 \in \Lambda^{(2,0)}(K3)\oplus\Lambda^{(1,1)}(K3)$ and the topological condition (3.5) must be imposed. All together, these strong conditions generically fix the T^2 complex structure moduli. Note also that the condition $\omega\in H^{(1,1)}(K3,\mathbb{Z}) = H^{(1,1)}(K3)\cap H^2(K3,\mathbb{Z})$ also strongly constrains the complex structure of the $K3$ since the dimension of $H^{(1,1)}(K3,\mathbb{Z})$ do vary with the complex structure of $K3$.[6]

The complex structures of $K3$ can also be fixed if the T^2 twist ω contains a $(2,0)$ self-dual part, $\omega^{(2,0)} = k\Omega_{K3}$, which up to a constant k must be proportional to the holomorphic

[6] That the T^2 complex structures are fixed has also be noted from the gauged linear sigma model point of view in [18].

$(2,0)$-form of $K3$. The above mentioned quantization condition for the $(2,0)$ part then takes the form (for $\tau = i$)

$$\frac{k}{2\pi\sqrt{\alpha'}}\int_{\Gamma}\Omega_{K3}\in\mathbb{Z}, \tag{3.32}$$

which defines the periods of the holomorphic $(2,0)$-form on the $K3$. These periods specify the complex structures chosen on $K3$, and the quantization condition thus fixes the complex structures on $K3$.

4. Conclusions and open questions

In this paper, we have derived the defining equations for the local moduli of supersymmetric heterotic flux compactifications. The defining equations were derived by performing a linear variation of the supersymmetry constraints obeyed by such compactifications. We further analyzed the corresponding geometric moduli spaces and discussed the particular example of a T^2 bundle over $K3$ in detail. This T^2 bundle over $K3$ solution is special in that in it is dual to M- or F-theory on $K3\times K3$. Notice that under infinitesimal deformations, the manifold $K3\times K3$ remains $K3\times K3$. Thus, the corresponding heterotic T^2 bundle over $K3$ dual must also be locally unique; that is, it remains a T^2 bundle over $K3$ under infinitesimal variation.

In much of our analysis, we have set the gauge bundle to be trivial. For the T^2 bundle over the $K3$ case, the non-trivial, non-$U(1)$ bundle are simply the stable bundles on $K3$ lifted to the six-dimensional space. The moduli space then corresponds to the space of $K3$ stable bundle. The dimension of this moduli space M is given by the Mukai formula [29]

$$\dim M = 2rc_2(E) - (r-1)c_1^2(E) - 2r^2 + 2, \tag{4.1}$$

where r is the rank of the bundle (i.e. the dimension of the fiber), and $(c_1(E), c_2(E))$ are the first and second Chern number of the gauge bundle E. It would be interesting to understand the moduli space of stable gauge bundle in general.

There are a number of interesting open questions. First, in our analysis we have kept for simplicity the complex structure fixed. It is well known that for Calabi–Yau compactifications the moduli space is a direct product of complex structure and Kähler structure deformations. For non-Kähler manifolds with torsion, this likely is not the case and it would be interesting to allow for a simultaneous variation of the complex structure and the Hermitian form.

It would be interesting to analyze the geometry of the moduli space and to determine if powerful tools such as the well-known "special geometry" of Calabi–Yau compactifications [30] can be derived in this case.

Furthermore, counting techniques for moduli fields need to be developed and we expect that the number of moduli can be characterized in terms of an index or some topological invariants of the manifold.

Finally, it would be interesting to analyze the moduli space from the world-sheet approach using the recently constructed gauged linear sigma model [18]. Moduli fields will correspond to the marginal deformations of the IR conformal field theory.

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Appendix A

We summarize our notation and conventions.

- Our index conventions are as follows: $m, n, p, q, \ldots$ denote real six-dimensional coordinates, $a, b, c, \ldots$ and $\bar{a}, \bar{b}, \bar{c}, \ldots$ denote six-dimensional complex coordinates, and $i, j, k, \ldots$ and $\bar{i}, \bar{j}, \bar{k}, \ldots$ denote four-dimensional complex coordinates on the base $K3$.
- The gauge field A_m and field strength F_{mn} take values in the $SO(32)$ or $E_8 \times E_8$ Lie-algebra with the generators being anti-Hermitian.
- The Riemann tensor is defined as follows

$$R_{mn}{}^p{}_q = \partial_m \Gamma_n{}^p{}_q - \partial_n \Gamma_m{}^p{}_q + \Gamma_m{}^p{}_r \Gamma_n{}^r{}_q - \Gamma_n{}^p{}_r \Gamma_m{}^r{}_q.$$

  With a Hermitian metric g with components $g_{a\bar{b}}$, we write the Hermitian curvature two-form as $R = \bar{\partial}[\bar{g}^{-1}\partial\bar{g}] = \bar{\partial}[(\partial g)g^{-1}]$ where $\bar{g}$ is the transposed of g with components $g_{\bar{b}a}$. Explicitly, in components, we write

$$R_{\bar{a}b}{}^c{}_d = \bar{\partial}_{\bar{a}}\left[g^{c\bar{d}}\partial g_{\bar{d}d}\right] = \bar{\partial}_{\bar{a}}\left[(\partial_b g_{d\bar{d}})g^{\bar{d}c}\right].$$

- We follow the convention standard in the mathematics literature for the Hodge star operator. For example, $(\star H)_{mnp} = \frac{1}{3!} H_{rst}\epsilon^{rst}{}_{mnp}$ with ϵ_{mnprst} being the Levi-Civita tensor.
- We use the definition for $\|\Omega\|_J^2$:

$$\Omega \wedge \star\bar{\Omega} = \|\Omega\|_J^2 \frac{J^3}{3!}.$$

- For a vector field, $v = v^m \partial_m$, the interior product acting on a p-form with components $\alpha_{m_1 m_2 \ldots m_p}$ is just

$$(i_v \alpha)_{m_2 m_3 \ldots m_p} = v^{m_1} \alpha_{m_1 m_2 \ldots m_p}.$$

- Given a Hermitian form J, the adjoint of the Lefschetz operator Λ acting on a p-form with components $\alpha_{m_1 m_2 \ldots m_p}$ is

$$(\Lambda\alpha)_{m_3 m_4 \ldots m_p} = \frac{1}{2!} J^{m_1 m_2} \alpha_{m_1 m_2 m_3 m_4 \ldots m_p}.$$

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