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Symplectic Morse theory and Witten deformation

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Abstract

On symplectic manifolds, we introduce a Morse-type complex with elements generated by pairs of critical points of a Morse function. The differential of the complex consists of gradient flows and an integration of the symplectic structure over spaces of gradient flow lines. Using results from the Witten deformation method, we prove that the cohomology of this complex is independent of both the Riemannian metric and the Morse function used to define the complex and is in fact isomorphic to the cohomology of differential forms of Tsai, Tseng and Yau (TTY). We also obtain Morse-type inequalities that bound the dimensions of the TTY cohomologies by the number of Morse critical points and the interaction of symplectic structure with the critical points.

Contents

1 Introduction

The Morse complex, also referred to as the Morse–Witten or Smale–Thom complex, captures the information of the standard homology groups of a closed manifold M by means of a Morse function f, i.e. a function whose Hessian at each critical point is non-degenerate, and

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a Riemannian metric g. The elements of the complex $C^k(M, f)$ are generated by the critical points $q \in Crit(f)$ of the Morse function f, and grouped together by their index, $k = ind(q)$, the number of negative eigenvalues of the Hessian matrix of f at q. The differential of the complex requires the use of the metric and is given by the gradient flow, $-\nabla f$, from one critical point to another. We will assume throughout the paper that the gradient flow satisfy the Morse-Smale transversality condition, that is, the stable and unstable manifolds of any two critical points intersect transversely. The homology of the Morse complex is well-known to be isomorphic to the standard homology, and therefore, independent of the choice of the Morse function f and the metric g. As a corollary of this isomorphism, the Morse inequalities bound the Betti numbers of M in terms of the number of critical points of the Morse function.

We are interested here to consider Morse theory in the presence of a symplectic structure, that is, on a symplectic manifold (M^{2n}, ω). On the cohomology side, besides the de Rham cohomology, Tsai, Tseng and Yau (TTY) [11, 13, 14] found novel symplectic cohomologies of differential forms. These cohomologies, which we will call TTY cohomologies and labelled by $F^pH(M, \omega)$, with $p = 0, 1, \ldots, n-1$, have interesting properties. For one, they can in general vary with the symplectic structure as seen in explicit examples of a six-dimensional nilmanifold [14] and of a three-ball product with a three-torus, $B^3 \times T^3$ [12]. These cohomologies have also been used to distinguish inequivalent symplectic structures on open 4-manifolds [7]. Of particular relevance here, when the symplectic structure is integral class $[\omega] \in H^2(M, \mathbb{Z})$, Tanaka and Tseng pointed out that the TTY cohomologies are isomorphic to the de Rham cohomologies of a higher dimensional sphere bundle over the symplectic manifold [10]. Specifically, denote by E_p the odd-dimensional sphere bundle $S^{2p+1} \to E_p \to M$ with Euler class ω^{p+1}, then $F^pH(M, \omega) \cong H_{dR}(E_p)$. Certainly, on this sphere bundle, which is a smooth manifold, we can bound the dimensions of the de Rham cohomology $H_{dR}(E_p)$ and hence, $F^pH(M, \omega)$, by Morse or Morse–Bott inequalities.

To simplify the discussion, we will focus mostly in this paper on the $p = 0$ TTY cohomology, $PH(M)$, called *primitive* cohomology, and introduced in [14]. (The case of $p \geq 1$ can be straightforwardly generalized from the $p = 0$ case and will be described explicitly in the concluding section of this paper.) By [10], when ω is an integral class, $PH(M)$ are isomorphic to the de Rham cohomology of the prequantum circle bundle X, i.e. a circle bundle with Euler class given by ω. A bound on the dimension $PH(M) \cong H_{dR}(X)$ can be obtained by taking a Morse function f on M and pulling it back to the circle bundle, which makes $\pi^* f$ a Morse–Bott function. Denote by $b_k^\omega = \dim PH^k(M)$. The Morse–Bott inequalities for a circle bundle states the existence of a polynomial $Q(s)$ with positive coefficients such that

$$(1+s) \sum_{k=0}^{2n} m_k \, s^k = \sum_{k=0}^{2n+1} b_k^\omega \, s^k \; + \; (1+s) Q(s)$$

where m_k denotes the number of critical points with index equal to k. Specifically, this gives the strong inequalities

$$\sum_{i=0}^{k} (-1)^{k-i} b_i^\omega \leq \sum_{i=0}^{k} (-1)^{k-i} (m_i + m_{i-1}) = m_k \tag{1.1}$$

and the weak inequalities

$$b_k^\omega \leq m_k + m_{k-1} \, . \tag{1.2}$$

Though these Morse–Bott inequalities (1.1)–(1.2) assume ω is an integral class, they in fact hold true for any symplectic structure. We recall the algebraic relation for the primitive cohomologies in [11].

$$PH^k(M) \cong \operatorname{coker}\left[\omega : H^{k-2}_{dR}(M) \to H^k_{dR}(M)\right] \oplus \ker\left[\omega : H^{k-1}_{dR}(M) \to H^{k+1}_{dR}(M)\right] \tag{1.3}$$

which immediately gives the weak inequalities of (1.2) just by bounding the dimensions of $H^k_{dR}(M)$ and $H^{k-1}_{dR}(M)$ by the number of Morse critical points, m_k and m_{k-1}, respectively. The strong inequalities can be similarly attained by applying the rank-nullity theorem. Hence, we find that the Morse–Bott inequalities only provide a rough estimate for b^ω_k. Moreover, note that the b^ω_k's on the left-hand-side of (1.1)–(1.2) generally depend on ω, while the m_k's on the right hand side do not. These observations make us ask two questions:

1. Can we find a Morse-type complex that incorporate the symplectic structure ω explicitly and whose cohomology matches that of the TTY cohomology?
2. Can we bound $b^\omega_k = \dim PH^k(M, \omega)$ by inequalities that in general vary with ω?

In this paper, we answer both questions in the affirmative. For the first, we introduce a symplectic Morse complex on (M^{2n}, ω) defined by a Morse-Smale pair (f, g) on M whose cohomology are isomorphic to the TTY primitive cohomology. Our symplectic Morse complex is motivated by the result of Tanaka–Tseng [10] which relates the cochain complex that underlies the TTY cohomologies with the cone complex of the wedge product map $\omega^{p+1} : \Omega^\bullet(M) \to \Omega^{\bullet+2p+2}(M)$ on the space of differential forms. Let us recall the definition of a cone on the de Rham complex with respect to ω^{p+1}. Again, for simplicity, we will focus on the case of $p = 0$.

Definition 1.1 Let (M^{2n}, ω) be a symplectic manifold. We define the **de Rham cone complex** of ω, $\mathrm{Cone}(\omega) = (\Omega^\bullet(M) \oplus \theta\, \Omega^{\bullet-1}(M),\ d_C)$:

$$\cdots \xrightarrow{d_C} \Omega^k(M) \oplus \theta\, \Omega^{k-1}(M) \xrightarrow{d_C} \Omega^{k+1}(M) \oplus \theta\, \Omega^k(M) \xrightarrow{d_C} \cdots$$

where θ is a formal parameter of degree one and the differential d_C can be expressed in matrix form as

$$d_C = \begin{pmatrix} d & \omega \\ 0 & -d \end{pmatrix} \tag{1.4}$$

with d the standard exterior derivative and ω acting by wedge product.

Note that the d-closedness of ω together with the Leibniz product rule ensures that $d_C\, d_C = 0$. Also, if we formally define $d\theta = \omega$, then d_C is just the exterior derivative acting on $\Omega^\bullet(M) \oplus \theta\, \Omega^{\bullet-1}(M)$. Of interest, Tanaka–Tseng [10] proved the isomorphism of this cone cohomology with the TTY cohomology:

$$H(\mathrm{Cone}(\omega)) \cong PH(M, \omega)\,. \tag{1.5}$$

Motivated by the relationship between de Rham complex and the Morse cochain complex over $\mathbb{R}$, we define in the following a *cone* Morse complex with respect to ω also over $\mathbb{R}$.

Definition 1.2 Let (M^{2n}, ω) be a symplectic manifold equipped with a Riemannian metric g and a Morse function f satisfying the Morse–Smale transversality condition. Let $C^k(M, f)$ be the $\mathbb{R}$-module with generators the critical points of f with index k. We define the **cone Morse cochain complex** of ω, $\mathrm{Cone}(c(\omega)) = (C^\bullet(M, f) \oplus C^{\bullet-1}(M, f),\ \partial_C)$:

$$\cdots \xrightarrow{\partial_C} C^k(M, f) \oplus C^{k-1}(M, f) \xrightarrow{\partial_C} C^{k+1}(M, f) \oplus C^k(M, f) \xrightarrow{\partial_C} \cdots$$

with

$$\partial_C = \begin{pmatrix} \partial & c(\omega) \\ 0 & -\partial \end{pmatrix}. \tag{1.6}$$

Here, ∂ is the standard Morse cochain differential defined by gradient flow, and $c(\omega) : C^k(M, f) \to C^{k+2}(M, f)$ acting on a critical point of index k is defined to be

$$c(\omega)\, q_k = \sum_{ind(r)=k+2} \left(\int_{\overline{\mathcal{M}(r_{k+2}, q_k)}} \omega \right) r_{k+2} \tag{1.7}$$

where $\mathcal{M}(r_{k+2}, q_k)$ is the two-dimensional subspace of M consisting of all flow lines from the index $k+2$ critical point, r_{k+2} , to q_k.

Notice that the elements of the Morse cone complex, $\mathrm{Cone}^k(c(\omega)) = C^k(M, f) \oplus C^{k-1}(M, f)$, can be generated by *pairs* of critical points of index k and $k-1$. The differential ∂_C consists of the standard Morse differential ∂ from gradient flow coupled with the $c(\omega)$ map which involves an integration of ω over the space of gradient flow lines. This type of maps has appeared in Austin-Braam [1] and Viterbo [15] to define a cup product on Morse cohomology. It satisfies the following commuting relation:

$$\partial\,(c(\omega)) = c(\omega)\partial. \tag{1.8}$$

(This relation follows from a more general Leibniz type formula. See [5, Appendix A].) With (1.8) and $\partial\partial = 0$, they together imply $\partial_C\, \partial_C = 0$. The cone Morse complex hence gives the following cohomology for $k = 0, 1, \ldots, 2n+1$,

$$H^k(\mathrm{Cone}(c(\omega))) = \frac{\ker \partial_C \cap \mathrm{Cone}^k(c(\omega))}{\mathrm{im}\, \partial_C \cap \mathrm{Cone}^k(c(\omega))}.$$

To go further and also achieve stronger inequalities than (1.1)–(1.2), we proceed in Sect. 2 to combine the results from the Witten deformation of the de Rham complex [2, 3, 17], together with relations from homological algebra to prove the following theorem which answers our first question.

Theorem 1.3 *Let* (M^{2n}, ω) *be a closed symplectic manifold. The cohomology of symplectic Morse complex is isomorphic to the TTY cohomology, i.e.* $H(\mathrm{Cone}(c(\omega))) \cong PH(M)$.

Theorem 1.3 importantly shows that the cohomology of the cone Morse complex is independent of the choice of both the Morse function and the Riemannian metric used to define $\mathrm{Cone}(c(\omega))$. Moreover, the dependence on the symplectic structure is explicit in the differential ∂_C which involves the integration of ω over flow lines between pairs of critical points of the Morse function. For the second question, we use the isomorphism of Theorem 1.3 to obtain Morse-type inequalities for $H(\mathrm{Cone}(\omega)) \cong PH(M)$. With m_k denoting the number of Morse critical points with index k, we are able to prove the following:

Theorem 1.4 *Let* (M, ω, f, g) *be a closed, symplectic manifold with the Morse function* f *and the Riemannian metric* g *satisfying the Morse-Smale transversality condition. Then there exists a polynomial* $Q(s)$ *with non-negative integer coefficients such that*

$$(1+s)\sum_{k=0}^{2n} m_k\, s^k - (s+s^2)\sum_{k=0}^{2n-2} \nu_k\, s^k = \sum_{k=0}^{2n+1} b_k^{\omega}\, s^k \,+\, (1+s)\, Q(s), \tag{1.9}$$

where $b_k^{\omega} = \dim H^k(\mathrm{Cone}(\omega))$ *and* $\nu_k = \mathrm{rank}\,\big(c(\omega) : C^k(M, f) \to C^{k+2}(M, f)\big)$.

Alternatively, we have the following Morse-type inequalities:
(A) Weak cone Morse inequalities

$$b_k^\omega \le m_k - \nu_{k-2} + m_{k-1} - \nu_{k-1}\,, \qquad k = 0, \dots, 2n+1; \tag{1.10}$$

(B) Strong cone Morse inequalities

$$\sum_{i=0}^{k} (-1)^{k-i} b_i^\omega \le m_k - \nu_{k-1}, \qquad k = 0, \dots, 2n+1. \tag{1.11}$$

Furthermore, the above inequalities become equalities when the Morse function f is perfect, i.e. the Betti numbers $b_k = \dim H^k_{dR}(M) = m_k$ for all $k = 0, \dots, 2n$.

As was our goal for the second question, our cone Morse inequalities (1.10)–(1.11) can certainly vary with the symplectic structure ω. Specifically, the b_k^ω's on the left-hand side and the ν_k's on the right-hand side both are defined with dependence on ω. This is in contrast to the m_k's which are fixed by the choice of the Morse function f on M.

To describe the rest of the paper, in Sect. 3, we will demonstrate the cone Morse inequalities in several examples. And the paper concludes in Sect. 4 where we will describe how the results proved for the $p = 0$ primitive case can be straightforwardly extended to the $p > 0$ case.

Finally, we mention two generalizations of the results presented in this work. First, the de Rham cone complex and its corresponding Morse theory defined here with respect to the symplectic structure ω can be studied in a more general context. In a follow-up paper [5], we describe a general cone complex and its cone Morse theory on any oriented manifold M, with respect to any degree ℓ form $\psi \in \Omega^\ell(M)$ that is d-closed. Second, it is possible to apply the Witten deformation method directly to the de Rham cone complex of ω. Doing so leads to interesting new phenomena, such as novel harmonic solutions localized at critical points of the Morse function. We shall present such a study in a future work [6].

2 Relation between the cone complex and the cone Morse complex

In this section, we derive the sharp Morse inequality bounds described in Theorem 1.4 by first proving the isomorphism of $H^k(\text{Cone}(\omega)) \cong H^k(\text{Cone}(c(\omega)))$ (Theorem 1.3).

2.1 Quasi-isomorphism between Cone(ω) and Cone($c(\omega)$)

Let us begin by briefly reviewing the relationship between de Rham cohomology, $H^k_{dR}(M)$, and Morse cohomology, $H^k_{C(f)}(M)$. Recall that there is a map $\mathcal{P}$ that links the de Rham complex with the Morse complex [3].

Definition 2.1 Define the map $\mathcal{P} : \Omega^k(M) \to C^k(M, f)$ by

$$\mathcal{P}\phi = \sum_{p_k \in Crit(f)} \left(\int_{\overline{U}_{p_k}} \phi \right) p_k$$

where $\phi \in \Omega^k(M)$ and U_p is the unstable submanifold consisting of gradient flow lines moving away from p.

Importantly, $\mathcal{P}$ is a chain map and induces an isomorphism on cohomology.

Theorem 2.2 [3, Theorem 2.9] *The map* $\mathcal{P} : \Omega^k(M) \to C^k(M, f)$ *is a chain map, i.e.*

$$\partial\, \mathcal{P} = \mathcal{P}\, d\,, \tag{2.1}$$

and moreover,

$$[\mathcal{P}] : H^k_{dR}(M) \to H^k_{C(f)}(M) \text{ is an isomorphism.} \tag{2.2}$$

The analytical Witten deformation proof of this theorem can be described by the following diagram:

$$\begin{tikzcd}
(\Omega^\bullet(M), d) \arrow[d, "\mathcal{P}"'] & (\Omega^\bullet(M), d_t) \arrow[l, "e^{tf}"'] \arrow[dl, "\mathcal{P}_t"] & ((F_t^{[0,1]})_\bullet, d_t) \arrow[l, "\iota_t^{[0,1]}"'] \arrow[dll, "\mathcal{P}_t\big|_{F_t^{[0,1]}}"] \\
(C^\bullet(M, f), \partial) & &
\end{tikzcd} \tag{2.3}$$

where $(\Omega^\bullet(M), d_t)$ is the Witten-deformed de Rham complex with $d_t = e^{-tf} \circ d \circ e^{tf}$ and $((F_t^{[0,1]})_\bullet, d_t)$ is the cochain complex consisting of eigenforms of $\Delta_t = d_t d_t^* + d_t^* d_t$ in $\Omega^\bullet(M)$ with eigenvalues in $[0, 1]$ (see, for example, [17, Section 6.4]). In the diagram, $\iota_t^{[0,1]}$ is the inclusion map, and also, the $\mathcal{P}$ map induces the map

$$\mathcal{P}_t = \mathcal{P}\, e^{tf} : (\Omega^k(M), d_t) \to (C^k(M, f), \partial)\,. \tag{2.4}$$

The proof of Theorem 2.2 involves showing that for t large enough, $\mathcal{P}_t\big|_{F_t^{[0,1]}} : ((F_t^{[0,1]})_\bullet, d_t) \to (C^\bullet(M, f), \partial)$ is a cochain isomorphism [17, Theorem 6.9]. This implies that the vertical map $\mathcal{P}$ gives an isomorphism on cohomology since the cohomologies of $\Omega(M)$ and $F_t^{[0,1]}$ are always identical, regardless of the value of t.

In considering $\mathrm{Cone}(c(\omega))$, let us first recall the definition of the map $c(\omega) : C^k(M, f) \to C^{k+2}(M, f)$ when acting on a critical point $p \in Crit(f)$ with index $n_f(p) = k$:

$$c(\omega)\, p = \sum_{q \in Crit(f)} \left(\int_{\overline{\mathcal{M}(q,p)}} \omega \right) q\,, \tag{2.5}$$

where the sum is over critical points q with index $n_f(q) = n_f(p) + 2$ and $\mathcal{M}(q, p)$ is the submanifold of flow lines from q to p. It was shown in [1, Section 3.5] and [15, Lemma 4] that

$$[\mathcal{P}][\omega] = [c(\omega)][\mathcal{P}]\,, \tag{2.6}$$

that is, they are cohomologous as maps from $H^k_{dR}(M)$ to $H^{k+2}_{C(f)}(M)$.

In the following, we assume that t is sufficiently large such that $\mathcal{P}_t = \mathcal{P}e^{tf}$ is an isomorphism between $(F_t^{[0,1]})_k$ and $C^k(M, f)$. By (2.6), we have that

$$[\mathcal{P}_t][\omega] = [c(\omega)][\mathcal{P}_t]$$

on cohomology. This motivates us to introduce an $\omega\wedge$ type map on $F_t^{[0,1]}$.

Definition 2.3 For t sufficiently large, we define the map $\tilde{\omega}_t : (F_t^{[0,1]})_k \to (F_t^{[0,1]})_{k+2}$ by

$$\tilde{\omega}_t = \mathcal{P}_t^{-1} c(\omega) \mathcal{P}_t\,.$$

Since $d_t = e^{-tf}\, d\, e^{tf}$, we can extend $\mathcal{P}\, d = \partial\, \mathcal{P}$ of (2.1) to

$$\mathcal{P}_t\, d_t = \partial\, \mathcal{P}_t\,. \tag{2.7}$$

Applying $\mathcal{P}_t^{-1}$ on both the left and the right of (2.7) gives $d_t\mathcal{P}_t^{-1} = \mathcal{P}_t^{-1}\partial$ acting on $C(M, f)$. Using the commutativity of $c(\omega)$ with ∂, i.e. $\partial c(\omega) = c(\omega)\partial$ in (1.8), it is straightforward to check that $\tilde{\omega}$ is a chain map:

$$\begin{aligned}\tilde{\omega}_t d_t &= \mathcal{P}_t^{-1}c(\omega)\mathcal{P}_t d_t\\ &= \mathcal{P}_t^{-1}c(\omega)\partial\mathcal{P}_t\\ &= \mathcal{P}_t^{-1}\partial c(\omega)\mathcal{P}_t\\ &= d_t\,\mathcal{P}_t^{-1}c(\omega)\mathcal{P}_t = d_t\tilde{\omega}_t.\end{aligned}$$

The induced map on the cohomology

$$[\tilde{\omega}_t] = [\mathcal{P}_t^{-1}][c(\omega)][\mathcal{P}_t] = [\mathcal{P}_t^{-1}][\mathcal{P}][\omega\wedge][\mathcal{P}]^{-1}[\mathcal{P}_t]\,, \tag{2.8}$$

is thus conjugate to the wedge product map $\omega\wedge$. With $\tilde{\omega}_t$, we can use it to define the following cone complex:

$$\mathrm{Cone}^k(\tilde{\omega}_t) = (F_t^{[0,1]})_k \oplus (F_t^{[0,1]})_{k-1}\,, \qquad \tilde{d}_{C,t} = \begin{pmatrix} d_t & \tilde{\omega}_t \\ 0 & -d_t \end{pmatrix}. \tag{2.9}$$

The cohomology of this cone Morse complex $(\mathrm{Cone}^\bullet(\tilde{\omega}_t), \tilde{d}_t)$ can be expressed in terms of the cokernels and kernels of the $\tilde{\omega}_t$ map in the following way:

$$\begin{aligned}H^k(\mathrm{Cone}(\tilde{\omega}_t))(M) \cong \mathrm{coker}&\left[[\tilde{\omega}_t] : H^{k-2}(F_t^{[0,1]}) \to H^k(F_t^{[0,1]})\right]\\ &\oplus \ker\left[[\tilde{\omega}_t] : H^{k-1}(F_t^{[0,1]}) \to H^{k+1}(F_t^{[0,1]})\right].\end{aligned} \tag{2.10}$$

This relation follows from the fact that the cone complex with elements $\mathrm{Cone}^k(\tilde{\omega}_t)$ sits in a short exact sequence of chain complexes

$$0 \longrightarrow ((F_t^{[0,1]})_k, \tilde{d}_t) \xrightarrow{\ \iota\ } (\mathrm{Cone}^k(\tilde{\omega}_t), \tilde{d}_{C,t}) \xrightarrow{\ \pi\ } ((F_t^{[0,1]})_{k-1}, -\tilde{d}_t) \longrightarrow 0 \tag{2.11}$$

where ι is the inclusion map and π is the projection onto the second component. The short exact sequence implies the following long exact sequence of cohomologies

$$\begin{tikzcd} \dots \arrow[r] & H^{k-2}(F_t^{[0,1]}) \arrow[r, "{[\tilde{\omega}_t]}"] & H^k(F_t^{[0,1]}) \arrow[r, "{[\iota]}"] & H^k(\mathrm{Cone}(\tilde{\omega}_t)) \arrow[dll, rounded corners, "{[\pi]}"'] \\ & H^{k-1}(F_t^{[0,1]}) \arrow[r, "{[\tilde{\omega}_t]}"] & H^{k+1}(F_t^{[0,1]}) \arrow[r] & \dots \end{tikzcd} \tag{2.12}$$

which implies (2.10).

We now point out two important properties of $H^k(\mathrm{Cone}(\tilde{\omega}_t))$ when t is sufficiently large such that $\mathcal{P}_t$ is an isomorphism.

(i) $H^k(\mathrm{Cone}(\tilde{\omega}_t)) \cong H^k(\mathrm{Cone}(c(\omega)))$. By construction, for t sufficiently large, the complex $(\mathrm{Cone}^\bullet(\tilde{\omega}_t), \tilde{d}_{C,t})$ is isomorphic to the cone Morse complex $(\mathrm{Cone}^\bullet(c(\omega)), \partial_C)$. Hence, their cohomologies must be isomorphic.

(ii) $H^k(\mathrm{Cone}(\tilde{\omega}_t)) \cong H^k(\mathrm{Cone}(\omega))$. Being both cone cohomologies, both $H^k(\mathrm{Cone}(\tilde{\omega}_t))$ and $H^k(\mathrm{Cone}(\omega))$ can be expressed in terms of the cokernels and kernels, of the $[\tilde{\omega}_t]$ map (1.3) and the $[\omega]$ map (2.10), respectively. Moreover, since $H^k(F_t^{[0.1]}) \cong H^k_{dR}(M)$ and also $[\tilde{\omega}_t]$ and $[\omega]$ have equivalent action on the cohomology level by (2.8), the two cohomologies are isomorphic.

Together, they imply the desired isomorphism that $H^k(\mathrm{Cone}(c(\omega))) \cong H^k(\mathrm{Cone}(\omega)) \cong PH^k(M,\omega)$ for $k = 0, 1, \ldots, 2n+1$. And this proves Theorem 1.3.

2.2 Cone Morse inequalities

Having proved $H(\mathrm{Cone}(\omega)) \cong H(\mathrm{Cone}(c(\omega)))$, we now proceed to derive Morse-type bounds for $b_k^\omega = \dim PH^k(M,\omega) = \dim H^k(\mathrm{Cone}(\omega))$.

To do so, we note that the cohomology of the cone Morse complex $(\mathrm{Cone}^\bullet(c(\omega)), \partial_C)$, like any cone cohomology, can be expressed in terms of cokernels and kernels of the $c(\omega)$ map:

$$\begin{aligned} H^k(\mathrm{Cone}(c(\omega)))(M) \cong \operatorname{coker}&\left[[c(\omega)] : H^{k-2}_{C(f)}(M) \to H^k_{C(f)}(M)\right] \\ &\oplus \ker\left[[c(\omega)] : H^{k-1}_{C(f)}(M) \to H^{k+1}_{C(f)}(M)\right] \end{aligned} \tag{2.13}$$

where $H_{C(f)}(M)$ is the cohomology of the standard Morse cochain complex $(C^\bullet(M,f), \partial)$. This relation can be derived similarly as that for $H^k(\mathrm{Cone}(\tilde{\omega}_t))$ in (2.10) by means of short exact sequence of chain complexes (2.11) resulting in a long exact sequence of cohomologies (2.12).

Since the dimensions of the cohomology of the Morse complex are given by the Betti numbers, i.e. $b_k = \dim H^k_{C(f)}(M)$, it follows from the isomorphism $H^k(\mathrm{Cone}(\omega)) \cong H^k(\mathrm{Cone}(c(\omega)))$ and (2.13) that

$$b_k^\omega = \dim H^k(\mathrm{Cone}(\omega)) = b_k - r_{k-2} + b_{k-1} - r_{k-1} \tag{2.14}$$

where

$$\begin{aligned} r_k &= \operatorname{rank}\Big([c(\omega)] : H^k_{C(f)}(M) \to H^{k+2}_{C(f)}(M)\Big) \\ &= \operatorname{rank}\Big([\omega] : H^k_{dR}(M) \to H^{k+2}_{dR}(M)\Big) \end{aligned} \tag{2.15}$$

is the rank of the $c(\omega)$ map on $H^k_{C(f)}(M)$, or equivalently, by (2.2) and (2.6), the rank of the ω map on $H^k_{dR}(M)$ as expressed in the second line of (2.15).

We recall the standard Morse inequalities bound for the Betti numbers in terms of the critical points of a Morse function f:

$$(weak\ Morse\ inequalities) \qquad b_k \le m_k\,, \qquad k = 0, 1, \ldots, 2n\,, \tag{2.16}$$

$$(strong\ Morse\ inequalities) \qquad \sum_{i=0}^{k} (-1)^i b_k \le \sum_{i=0}^{k} (-1)^i m_k\,, \quad k = 0, 1, \ldots, 2n\,, \tag{2.17}$$

where m_k is the number of index k critical points of f. But since the b_k^ω's can vary with the symplectic structure, we should expect that any sharp inequality bound of b_k^ω should have dependence on the symplectic structure as well. Hence, we introduce the rank of the map $c(\omega)$ on $C^k(M,f)$.

$$v_k = \operatorname{rank}\Big(c(\omega) : C^k(M,f) \to C^{k+2}(M,f)\Big)\,. \tag{2.18}$$

Notice that v_k differs from r_k with r_k being the rank of the $c(\omega)$ map acting on the cohomology, $H^k_{C(f)}(M)$, and not on the cochain, $C^k(M,f)$. However, since both the cochain and cohomology are generated by critical points, it is evident that

$$r_k \le v_k\,, \qquad k = 0, 1, \ldots, 2n-2, \tag{2.19}$$

$$r_k = v_k = 0\,, \quad k = 2n-1, 2n\,. \tag{2.20}$$

Below is another key property:

Proposition 2.4 *Let* (M^{2n}, ω, f, g) *be a closed symplectic manifold with the Morse function and the Riemannian metric satisfying the Morse-Smale transversality condition. Then, we have the following inequality:*

$$b_k - r_{k-1} \le m_k - v_{k-1}\,, \qquad k = 0, 1, \ldots, 2n+1\,. \tag{2.21}$$

Proof For $k = 2n+1$, each term in (2.21) vanishes and the inequality is trivial. For $k = 2n$, with $r_{2n-1} = v_{2n-1} = 0$ as noted in (2.20), the inequality is just the weak Morse inequality $b_{2n} \le m_{2n}$. In the remainder of the proof, we will only need to concern with the case $k = 0, \ldots, 2n-1$.

Let us note that $C^k(M, f)$ is a cochain vector space over $\mathbb{R}$ and is finitely-generated by the critical points of index k. Equipped with the differential ∂, we can decompose $C^k(M, f)$ as follows:

$$C^k = \partial A^{k-1} \oplus B^k \oplus A^k\,,$$

where $A^k = \frac{C^k}{\ker \partial \cap C^k}$ is the space of cochains modulo cocycles, ∂A^{k-1} is the coboundary space, and B^k is the space of cocycles modulo coboundaries (i.e. the cohomology). In this notation, the Betti number, $b_k = \dim B^k$. And if we let $a_k = \dim A^k$, then $\dim \partial A^{k-1} = a_{k-1}$, since ∂ is an injective map on A^{k-1}. Hence,

$$m_k = \dim C^k = \dim \partial A^{k-1} + \dim B^k + \dim A^k = a_{k-1} + b_k + a_k\,. \tag{2.22}$$

We consider the map $c(\omega) : C^{k-1} \to C^{k+1}$. Expressing the action of $c(\omega)$ on each component of $C^{k-1} = \partial A^{k-2} \oplus B^{k-1} \oplus A^{k-1}$, we observe that

$$\begin{aligned}
&\text{(i) } c(\omega)\, \partial A^{k-2} \subseteq \partial A^k \\
&\text{(ii) } c(\omega)\, B^{k-1} \subseteq \partial A^k \oplus B^{k+1} \\
&\text{(iii) } c(\omega)\, A^{k-1} \subseteq \partial A^k \oplus B^{k+1} \oplus A^{k+1} = C^{k+1}
\end{aligned}$$

having noted the property that $c(\omega)$ commutes with the differential ∂ as in (1.8), and hence, $c(\omega)$ maps coboundaries to coboundaries (i), and also cocycles to cocycles (ii). This can be expressed as a matrix operator

$$c(\omega) \begin{bmatrix} \partial A^{k-2} \\ B^{k-1} \\ A^{k-1} \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ O & R_{22} & R_{23} \\ O & O & R_{33} \end{bmatrix} \begin{bmatrix} \partial A^{k-2} \\ B^{k-1} \\ A^{k-1} \end{bmatrix} \subseteq \begin{bmatrix} \partial A^{k} \\ B^{k+1} \\ A^{k+1} \end{bmatrix}$$

Consider now v_{k-1} which is the rank of the above matrix. Note first that $r_{k-1} = \text{rank}([c(\omega)] : B^{k-1} \to B^{k+1})$, and therefore, r_{k-1} is the rank of R_{22}. Now the upper left block submatrix

$$\begin{bmatrix} R_{11} & R_{12} \\ O & R_{22} \end{bmatrix}$$

must have a rank that is greater than or equal to the rank of R_{22}. So let

$$\text{rank}\left(\begin{bmatrix} R_{11} & R_{12} \\ O & R_{22} \end{bmatrix}\right) = r_{k-1} + u_{k-1}\,, \quad \text{with } u_{k-1} \ge 0\,.$$

In particular, since there is a zero matrix in the lower left corner, only R_{11} and R_{12} can make u_{k-1} greater than zero. However, both R_{11}, R_{12} are in the first block-row, and thus can not contribute a rank greater than the size of the block-row, which is the dimension of ∂A^k. Thus, we have the bound $0 \leq u_{k-1} \leq a_k$. Now v_{k-1} is the rank of the whole matrix, which can not be less than $r_{k-1} + u_{k-1}$, so let

$$v_{k-1} = \operatorname{rank}\left(\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ O & R_{22} & R_{23} \\ O & O & R_{33} \end{bmatrix}\right) = r_{k-1} + u_{k-1} + t_{k-1}\,, \quad \text{with } t_{k-1} \geq 0\,. \tag{2.23}$$

Again, the zero matrices in the third row mean that only R_{13}, R_{23} and R_{33} can make t_{k-1} greater than zero. However, since these are in the third block-column, they cannot contribute a rank greater than the size of the block-column, which is the dimension of A^{k-1}, and we obtain the bound $0 \leq t_{k-1} \leq a_{k-1}$. Finally, combining (2.22) and (2.23), we obtain

$$\begin{aligned} m_k - v_{k-1} &= (a_{k-1} + b_k + a_k) - (r_{k-1} + u_{k-1} + t_{k-1}) \\ &= (b_k - r_{k-1}) + (a_k - u_{k-1}) + (a_{k-1} - t_{k-1}) \\ &\geq b_k - r_{k-1} \end{aligned}$$

since the last two terms of the second line are both nonnegative.

Proposition 2.4 above leads us to the desired cone Morse inequalities.

Theorem 2.5 *Let (M, ω, f, g) be a closed, symplectic manifold with the Morse function f and the Riemannian metric g satisfying the Morse-Smale transversality condition. Then, we have the following inequalities for the dimensions $b_k^\omega = \dim PH^k(M, \omega) = \dim H^k(\text{Cone})$:*
(A) Weak cone Morse inequalities:

$$b_k^\omega \leq m_k - v_{k-2} + m_{k-1} - v_{k-1}\,, \qquad k = 0, 1, \ldots, 2n+1\,; \tag{2.24}$$

(B) Strong cone Morse inequalities:

$$\sum_{i=0}^{k} (-1)^{k-i}\, b_i^\omega \leq m_k - v_{k-1}\,, \qquad k = 0, 1, \ldots, 2n+1\,. \tag{2.25}$$

Proof We first prove the strong inequality which follows directly from the isomorphism, $H^k(\text{Cone}(\omega)) \cong H^k(\text{Cone}(c(\omega)))$, and Proposition 2.4. Specifically, the isomorphism implies the expression in (2.14)

$$b_i^\omega = b_i - r_{i-2} + b_{i-1} - r_{i-1}\,.$$

This gives a telescoping sum

$$\begin{aligned} \sum_{i=0}^{k} (-1)^{k-i}\, b_i^\omega &= \sum_{i=0}^{k} (-1)^{k-i}\, (b_i - r_{i-2} + b_{i-1} - r_{i-1}) \\ &= b_k - r_{k-1} \leq m_k - v_{k-1} \end{aligned} \tag{2.26}$$

having applied the inequality of (2.21) which results in the desired strong cone Morse inequality.

As for the weak inequality, it can be derived directly from the strong cone Morse equalities, or equivalently, from (2.21) with (2.14). Let us write out the k-th and $(k-1)$-th inequalities of (2.21),

$$b_k - r_{k-1} \leq m_k - v_{k-1}\,, \qquad \text{and} \qquad b_{k-1} - r_{k-2} \leq m_{k-1} - v_{k-2}\,.$$

Adding these two inequalities together and using (2.14) gives the desired weak cone Morse inequality

$$b_k^{\omega} = b_k - r_{k-2} + b_{k-1} - r_{k-1} \le m_k - \upsilon_{k-2} + m_{k-1} - \upsilon_{k-1} \,. \tag{2.27}$$

Finally, let us consider the case when f is a perfect Morse function. By definition, a perfect Morse function implies $b_k = m_k$ for all values of k. This means that $\dim H^k_{C(f)} = \dim C^k(M, f)$, and in particular, the Morse differential ∂ acts by zero on all generators of $C^*(M, f)$. Clearly then, when f is perfect, we have both $b_k = m_k$ and also $r_k = \upsilon_k$. We can therefore conclude that the weak cone Morse inequalities as expressed in (2.27) and the strong Morse inequalities as in (2.26) would both become equalities when f is a perfect Morse function.

Altogether, the weak and strong cone Morse inequalities and that they become equalities when f is perfect are the statements of Theorem 1.4. We have thus completed the proof of Theorem 1.4.

3 Examples

In this section, we will consider on certain symplectic manifolds the cone Morse complex and check the cone Morse inequalities derived in the previous section:

$$(weak) \qquad\qquad b_k^{\omega} \le m_k - \upsilon_{k-2} + m_{k-1} - \upsilon_{k-1} \,, \tag{3.1}$$

$$(strong) \qquad \sum_{i=0}^{k} (-1)^{k-i} b_i^{\omega} \le m_k - \upsilon_{k-1} \,, \tag{3.2}$$

for $k = 0, 1, \dots, 2n+1$. In the first examples that we shall consider, the symplectic manifolds are Kähler. Due to the hard Lefschetz property, the wedge product map $[\omega^j] : H^{n-j}_{dR}(M) \to H^{n+j}_{dR}(M)$ for $j = 1, 2, \dots, n$, is an isomorphism. This implies in particular for $r_k = \operatorname{rank} [\omega]|_{H^k_{dR}(M)}$, that $r_k = \min(b_k, b_{k+2})$. It thus follows from the relation [10, 11]

$$H^k(\mathrm{Cone}(\omega)) \cong \operatorname{coker}\left[\omega : H^{k-2}_{dR}(M) \to H^k_{dR}(M)\right] \oplus \ker\left[\omega : H^{k-1}_{dR}(M) \to H^{k+1}_{dR}(M)\right] \tag{3.3}$$

that

$$b_k^{\omega} = \dim H^k(\mathrm{Cone}(\omega)) = \begin{cases} b_k - b_{k-2} & 0 \le k \le n \,, \\ b_{k-1} - b_{k+1} & n+1 \le k \le 2n+1 \,, \end{cases} \tag{3.4}$$

which are determined solely by the Betti numbers and do not vary with the symplectic structure. This is special to Kähler symplectic manifolds, as generally, the b_k^{ω}'s can vary with the class $[\omega] \in H^2_{dR}(M)$ (for explicit examples, see [12, 14]).

Remark 3.1 In the special case where the Morse function f and Riemannian metric g are chosen such that $c(\omega)^k : C^{n-k}(M, f) \to C^{n+k}(M, f)$ is bijective, mirroring the Lefschetz property but on the level of the Morse cochains, then $\upsilon_k = \operatorname{rank} c(\omega) = \min(m_k, m_{k+2})$. The weak cone Morse inequalities would then also be analogous to (3.4)

$$b_k^{\omega} \le \begin{cases} m_k - m_{k-2} & 0 \le k \le n, \\ m_{k-1} - m_{k+1} & n+1 \le k \le 2n+1. \end{cases} \tag{3.5}$$

While this does not hold generally, it always occurs when using a perfect Morse function manifolds on Kähler manifolds, which is the setting of our two Kähler examples below.

Example 3.2 Consider $(\mathbb{CP}^n, \omega_{FS} = \frac{i}{2}\partial\overline{\partial}\log|z_i|^2)$, the complex n-dimensional projective space equipped with the Fubini-Study metric as the Kähler structure. It is well-known to have a perfect Morse function that can be expressed as

$$f([z_0, \ldots, z_n]) = \frac{\sum \lambda_i |z_i|^2}{\sum |z_i|^2}$$

such that $\lambda_i \neq \lambda_j$ for $i \neq j$. If we consider $\lambda_0 < \lambda_1 < \ldots < \lambda_n$, then the critical points are $p_{2j} = [0 \ldots : 1 : \ldots 0]$ with 1 only in the j-th position and index $n_f(p_{2j}) = 2j$. Because the index of all the critical points are even, the Morse differential $\partial : C^k(M, f) \to C^{k+1}(M, f)$ necessarily vanishes for all $k = 0, \ldots, 2n$. Thus, f is no doubt a perfect Morse function and

$$m_k = b_k(\mathbb{CP}^n) = \begin{cases} 1 & 0 \leq k \leq 2n,\ k\ even, \\ 0 & otherwise. \end{cases} \tag{3.6}$$

Regarding the cone Morse differential $\partial_C = \begin{pmatrix} \partial & c(\omega_{FS}) \\ 0 & -\partial \end{pmatrix}$, it has a non-zero component coming from the $c(\omega_{FS})$ map. We note that

$$\mathcal{M}(p_{2j}, p_{2j+2}) = \{[0 : \ldots : z_j : z_{j+1} : \ldots : 0] : (z_j, z_{j+1}) \in \mathbb{C}^2 \backslash \{0\}\}$$

is isomorphic to $\mathbb{CP}^1$. Therefore,

$$c(\omega_{FS}) p_{2j} = \left(\int_{\mathbb{CP}^1} \omega_{FS} \right) p_{2j+2} = \pi\ p_{2j+2}\,,$$

and thus,

$$v_{2j} = \operatorname{rank} c(\omega_{FS})|_{C^{2j}(f)} = 1\,, \qquad j = 0, \ldots, n-1, \tag{3.7}$$

which is the same as $r_{2j} = \operatorname{rank} \omega|_{H^{2j}(\mathbb{CP}^n)}$. The cone Morse complex and cohomology can then be easily computed and we find

$$b_k^{\omega} = \begin{cases} 1 & k = 0, 2n+1\,, \\ 0 & otherwise\,, \end{cases} \tag{3.8}$$

which agrees exactly with the expectation from (3.4).

The cone Morse inequalities (3.1)–(3.2) can similarly be straightforwardly checked using (3.6)–(3.7), and they are, in fact, equalities, as would be expected for a perfect Morse function.

Example 3.3 Consider $(T^4 = \mathbb{R}^4/\mathbb{Z}^4, \omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4)$, the four-torus described using Euclidean coordinates, x_i with identification $x_i \sim x_i + 1$ for $i = 1, 2, 3, 4$. For this example, we will compute the Cone$(c(\omega))$ complex with respect to the flat metric, $g = \sum dx_i^2$, and the Morse function is taken to be

$$f = 2 - \frac{1}{2} \sum_{i=1}^{4} \cos(2\pi x_i)\,. \tag{3.9}$$

This Morse function has several desirable properties that are straightforward to prove:

(i) the non-degenerate critical points are located at $x_i = [0]$ or $x_i = [\frac{1}{2}]$ and have Morse index equal to the number of coordinates which are equal to $[\frac{1}{2}]$;

Table 1 Cohomology of Cone(ω) versus Cone($c(\omega)$) on $(T^4, \omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4)$

k	0	1	2	3	4	5
$H^k(\text{Cone}(\omega))$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}dx_1\\0\end{pmatrix}, \begin{pmatrix}dx_2\\0\end{pmatrix}$ $\begin{pmatrix}dx_3\\0\end{pmatrix}, \begin{pmatrix}dx_4\\0\end{pmatrix}$	$\begin{pmatrix}dx_{13}\\0\end{pmatrix}, \begin{pmatrix}dx_{14}\\0\end{pmatrix}, \begin{pmatrix}dx_{23}\\0\end{pmatrix},$ $\begin{pmatrix}dx_{24}\\0\end{pmatrix}, \begin{pmatrix}dx_{12}-dx_{34}\\0\end{pmatrix}$	$\begin{pmatrix}dx_{123}\\0\end{pmatrix}, \begin{pmatrix}dx_{124}\\0\end{pmatrix}, \begin{pmatrix}dx_{234}\\0\end{pmatrix},$ $\begin{pmatrix}dx_{234}\\0\end{pmatrix}, \begin{pmatrix}0\\dx_{12}-dx_{34}\end{pmatrix}$	$\begin{pmatrix}0\\dx_{123}\end{pmatrix}, \begin{pmatrix}0\\dx_{124}\end{pmatrix}$ $\begin{pmatrix}0\\dx_{134}\end{pmatrix}, \begin{pmatrix}0\\dx_{234}\end{pmatrix}$	$\begin{pmatrix}0\\dx_{1234}\end{pmatrix}$
$H^k(\text{Cone}(c(\omega)))$	$\begin{pmatrix}q_0\\0\end{pmatrix}$	$\begin{pmatrix}q_1\\0\end{pmatrix}, \begin{pmatrix}q_2\\0\end{pmatrix}$ $\begin{pmatrix}q_3\\0\end{pmatrix}, \begin{pmatrix}q_4\\0\end{pmatrix}$	$\begin{pmatrix}q_{13}\\0\end{pmatrix}, \begin{pmatrix}q_{14}\\0\end{pmatrix}, \begin{pmatrix}q_{23}\\0\end{pmatrix},$ $\begin{pmatrix}q_{24}\\0\end{pmatrix}, \begin{pmatrix}q_{12}-q_{34}\\0\end{pmatrix}$	$\begin{pmatrix}q_{123}\\0\end{pmatrix}, \begin{pmatrix}q_{124}\\0\end{pmatrix}, \begin{pmatrix}q_{234}\\0\end{pmatrix},$ $\begin{pmatrix}q_{234}\\0\end{pmatrix}, \begin{pmatrix}0\\q_{12}-q_{34}\end{pmatrix}$	$\begin{pmatrix}0\\q_{123}\end{pmatrix}, \begin{pmatrix}0\\q_{124}\end{pmatrix}$ $\begin{pmatrix}0\\q_{134}\end{pmatrix}, \begin{pmatrix}0\\q_{234}\end{pmatrix}$	$\begin{pmatrix}0\\q_{1234}\end{pmatrix}$

(ii) the number of critical points of index k, $m_k = b_k(T^4)$ for all k. Hence, f is perfect and the Morse differential ∂ acts by zero;

(iii) the pair (f, g) satisfies Smale transversality.

Because of (ii), the ∂_C map reduces to the $c(\omega)$ map. Hence, we are interested in pairs of critical points whose indices differ by two, e.g. q_{k+1} has two more $[\frac{1}{2}]$ coordinates than q_{k-1}. Also, note that $\mathcal{M}(q_{k+1}, q_{k-1})$ will be a two-dimensional face with two of the coordinates fixed and two coordinates spanning the entire coordinate interval $[0, 1]$ when we take the closure.

In Table 1, we give the cohomologies of $H(\text{Cone}(c(\omega)))$ and $H(\text{Cone}(\omega))$.

We use a multi-index notation of $I = \{i_1 \dots i_j\}$ in increasing order such that $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_j}$, q_0 denotes the index 0 point, and q_I denotes the point with $\frac{1}{2}$ in entry $i_1, \dots i_j$, i.e. $q_{13} = q_{\left[\frac{1}{2},0,\frac{1}{2},0\right]}$. The orientation of the submanifolds are chosen such that $\mathcal{P}dx_I = q_I$. (c.f. Definition 2.1.)

Notice that $c(\omega)q_I$ only picks out critical points that have two coordinates of q_I changed from $[0]$ to $[\frac{1}{2}]$ in either the 1–2 or 3–4 directions. Thus, we find that

$$c(\omega)q_0 = q_{12} + q_{34}\,, \qquad c(\omega)q_{12} = q_{1234}\,, \qquad c(\omega)q_{34} = q_{1234}\,,$$
$$c(\omega)q_1 = q_{134}\,, \qquad c(\omega)q_2 = q_{234}\,, \qquad c(\omega)q_3 = q_{123}\,, \qquad c(\omega)q_4 = q_{124}\,,$$

with all other critical points mapped to zero when acted upon by $c(\omega)$.

It is straightforward to see from above that $v_k = r_k$ and that cone Morse inequalities give the equalities $b_k^{\omega} = m_k - v_{k-2} - m_{k-1} - v_{k-1}$ for $0 \le k \le 5$. This is as expected with f in (3.9) being a perfect Morse function.

Next, we consider a non-Kähler symplectic manifold where the hard Lefschetz property does not hold.

Example 3.4 Let (M, ω) be the six-dimensional, closed, symplectic manifold constructed by Cho in [4] where the symplectic form ω is not hard Lefschetz type. Topologically, M can be described as a two-sphere bundle over a projective $K3$ surface and also has the following properties [4, Theorem 1.3]: (i) M is simply-connected; (ii) the odd degree cohomologies vanish, i.e. $H^1_{dR}(M) = H^3_{dR}(M) = H^5_{dR}(M) = 0$.

Consider the cohomology $PH(M, \omega) \cong H(\mathrm{Cone}(\omega))$. From (3.3), we find

$$\begin{aligned}
b_0^\omega &= b_7^\omega = 1\,,\\
b_1^\omega &= b_6^\omega = 0\,,\\
b_2^\omega &= b_5^\omega = b_2(X) - 1\,,\\
b_3^\omega &= \dim\left[\ker\left(\omega : H^2(X) \to H^4(X)\right)\right] > 0\,,\\
b_4^\omega &= \dim\left[\mathrm{coker}\left(\omega : H^2(X) \to H^4(X)\right)\right] > 0\,.
\end{aligned}$$

Note that $b_3^\omega = b_4^\omega > 0$ since (M, ω) is not hard Lefschetz, which implies that the map, $\omega : H^2_{dR}(M) \to H^4_{dR}(M)$, can not be an isomorphism.

For the cone Morse complex and inequalities, we can again choose to work with a perfect Morse function on M. That such exists is due to a a result of Smale [9, Theorem 6.3] which states that any simply-connected manifold of dimension greater than five that has no homology torsion has a perfect Morse function. (No homology torsion here can be seen from applying the Gysin sequence to M as a two-sphere bundle over $K3$.) Since M has trivial odd-degree cohomology, this implies that

$$\begin{aligned}
m_0 &= m_6 = 1\,,\\
m_1 &= m_3 = m_5 = 0\,,\\
m_2 &= m_4 = b_2(M)\,.
\end{aligned}$$

It is straightforward to check that the bounds (3.1)-(3.2) are satisfied. In particular, for the weak cone Morse bound of (3.1), the $k = 3, 4$ case corresponds to

$$\begin{aligned}
b_3^\omega &\le m_3 + m_2 - v_2 = m_2 - v_2\,,\\
b_4^\omega &\le m_4 + m_3 - v_2 = m_4 - v_2\,.
\end{aligned}$$

The above demonstrates the necessity of having both the m_k and the m_{k-1} term in the symplectic cone Morse inequalities.

Remark 3.5 We comment that there is a preprint [8] that presents some symplectic Morse-type inequalities which are different from those here and actually not valid generally. For instance, the inequality in [8, Corollary 3] can be expressed in our notation as $\dim F^pH^{n+p+1}(M, \omega) \le m_{n-p}$, which is not satisfied in the above Cho's non-Kähler six-dimensional example (M, ω) for a perfect Morse function. Specifically, it gives for $p = 0$ case the inequality relation, $b_4^\omega \le m_3 = 0$, which is inconsistent with $b_4^\omega > 0$ with ω being of non-hard Lefschetz type.

4 Discussion

In this paper, we have for simplicity focused on the $p = 0$ case of the TTY cohomologies, $F^pH(M, \omega) = H(\mathrm{Cone}(\omega^{p+1}))$. Let us comment in this final section the $p > 0$ case and lay out the results which generalize the $p = 0$ case. The cone Morse theory in the $p > 0$ case can be considered similar to the computations in this paper. In general, the TTY cohomologies for $p = 0, 1, \dots, n-1$ algebraically correspond to [10, 11]:

$$F^p H^k(M,\omega) \cong H^k(\mathrm{Cone}(\omega^{p+1})) \cong \mathrm{coker}\left([\omega^{p+1}] : H^{k-2p-2}_{dR}(M) \to H^k_{dR}(M)\right) \oplus \ker\left([\omega^{p+1}] : H^{k-2p-1}_{dR}(M) \to H^{k+1}_{dR}(M)\right) \quad (4.1)$$

with $k = 0, 1, \dots, 2n+2p+1$. The relevant cone complex for the general p case would have the elements and differential

$$\mathrm{Cone}^k(\omega^{p+1}) = \Omega^k(M) \oplus \theta\Omega^{k-2p-1}(M), \qquad d_C = \begin{pmatrix} d & \omega^{p+1} \\ 0 & -d \end{pmatrix}, \quad (4.2)$$

where θ is now a formal $(2p+1)$-form such that $d\theta = \omega^{p+1}$. And the corresponding cone Morse cochain complex would be

$$\mathrm{Cone}^k(c(\omega^{p+1})) = C^k(M,f) \oplus C^{k-2p-1}(M,f), \qquad \partial_C = \begin{pmatrix} \partial & c(\omega^{p+1}) \\ 0 & -\partial \end{pmatrix}, \quad (4.3)$$

with $c(\omega^{p+1}) : C^k(M,f) \to C^{k+2p+2}(M,f)$ given by

$$c(\omega^{p+1})\, q_k = \sum_{r_{k+2p+2}} \left(\int_{\overline{\mathcal{M}(r_{k+2p+2}, q_k)}} \omega^{p+1} \right) r_{k+2p+2}\,, \quad (4.4)$$

which integrates ω^{p+1} over the $2(p+1)$-dimensional submanifold $\overline{\mathcal{M}(r_{k+2p+2}, q_k)}$ of gradient flow lines from the index $k+2p+2$ critical point, r_{k+2p+2} , to q_k.

A generalization to the proof of Theorem 1.3 for the $p = 0$ case can be carried out which results in the isomorphism of the cohomologies of the cone complex (4.2) with that of the cone Morse complex (4.3). Hence, we can likewise obtain corresponding cone Morse inequalities for the $p > 0$ case. Explicitly, using the notation

$$s^p_k = \dim F^p H^k(M,\omega) = \dim H^k(\mathrm{Cone}(\omega^{p+1}))$$

to denote the dimension of the TTY cohomology, we can write down the following weak and strong cone Morse inequalities for all $p = 0, 1, \dots, n-1$:

$$s^p_k \le m_k - \nu_{k-2p-2} + m_{k-2p-1} - \nu_{k-2p-1}\,, \quad (4.5)$$

$$\sum_{i=0}^{k} (-1)^{k-i} s^p_i \le \left(\sum_{i=k-2p}^{k} (-1)^{k-i} m_i \right) - \nu_{k-2p-1}\,, \quad (4.6)$$

where $k = 0, 1, \dots, 2n+2p+1$ and

$$\nu_k = \mathrm{rank}\left(c(\omega^{p+1}) : C^k(M,f) \to C^{k+2p+2}(M,f) \right). \quad (4.7)$$

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References

1. Austin, D.M., Braam, P.J.: Morse-Bott Theory and Equivariant Cohomology, The Floer Memorial Volume, Progr. Math., vol. 133. Birkhäuser, Basel, pp. 123–183 (1995)
2. Bismut, J.-M., Lebeau, G.: Complex immersions and Quillen metrics. Inst. Hautes Étud. Sci. Publ. Math. **74**, 1–297 (1991)
3. Bismut, J.-M., Zhang, W.: An extension of a theorem by Cheeger and Müller, Astérisque 205, p.235 (1992) (**With an appendix by François Laudenbach**)
4. Cho, Y.: Hard Lefschetz property of symplectic structures on compact Kähler manifolds. Trans. Am. Math. Soc. **368**(11), 8223–8248 (2016)
5. Clausen, D., Tang, X., Tseng, L.-S.: Mapping cone and Morse theory. Camb. J. Math. (**in press**)
6. Clausen, D., Tang, X., Tseng, L.-S.: (**in preparation**)
7. Gibson, M., Tseng, L.-S., Vidussi, S.: Symplectic structures with non-isomorphic primitive cohomology on open 4-manifolds. Trans. Am. Math. Soc. **375**(12), 8399–8422 (2022)
8. Machon, T.: Some Morse-type inequalities for symplectic manifolds. arXiv:2109.13010v1 [math.SG]
9. Smale, S.: On the structure of manifolds. Am. J. Math. **84**, 387–399 (1962)
10. Tanaka, H.L., Tseng, L.-S.: Odd sphere bundles, symplectic manifolds, and their intersection theory. Camb. J. Math. **6**(3), 213–266 (2018)
11. Tsai, C.-J., Tseng, L.-S., Yau, S.-T.: Cohomology and Hodge theory on symplectic manifolds: III. J. Differ. Geom. **103**(1), 83–143 (2016)
12. Tseng, L.-S., Wang, L.: Symplectic boundary conditions and cohomology. J. Differ. Geom. **122**(2), 271–340 (2022)
13. Tseng, L.-S., Yau, S.-T.: Cohomology and Hodge theory on symplectic manifolds: I. J. Differ. Geom. **91**(3), 383–416 (2012)
14. Tseng, L.-S., Yau, S.-T.: Cohomology and Hodge theory on symplectic manifolds: II. J. Differ. Geom. **91**(3), 417–443 (2012)
15. Viterbo, C.: The Cup-Product on the Thom–Smale–Witten Complex, and Floer Cohomology, The Floer Memorial Volume, Progr. Math., vol. 133. Birkhäuser, Basel, pp. 609–625 (1995)
16. Witten, E.: Supersymmetry and Morse theory. J. Differ. Geom. **17**(4), 661–692 (1982)
17. Zhang, W.: Lectures on Chern–Weil Theory and Witten Deformations, Nankai Tracts in Mathematics, vol. 4. World Scientific Publishing Co., Inc, River Edge (2001)