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$\mathrm{SL}(2,\mathbb{Z})$ multiplets in $\mathcal{N}=4$ SYM theory

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Abstract: We discuss the action of $\mathrm{SL}(2,\mathbb{Z})$ on local operators in $D=4$, $\mathcal{N}=4$ SYM theory in the superconformal phase. The modular property of the operator's scaling dimension determines whether the operator transforms as a singlet, or covariantly, as part of a finite or infinite dimensional multiplet under the $\mathrm{SL}(2,\mathbb{Z})$ action. As an example, we argue that operators in the Konishi multiplet transform as part of a (p,q) $\mathrm{PSL}(2,\mathbb{Z})$ multiplet. We also comment on the non-perturbative local operators dual to the Konishi multiplet.

Keywords: Nonperturbative Effects, Duality in Gauge Field Theories, Conformal and W Symmetry, Supersymmetry and Duality.

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1. Introduction

The $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in four dimensions is widely believed to realize an SL$(2, \mathbb{Z})$ duality [1]. The duality group acts on the two parameters of the theory - the coupling, g, and the theta angle, θ. Writing the parameters as $\tau = \tau_1 + i\tau_2 \equiv \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$, the SL$(2, \mathbb{Z})$ action is that of the modular transformation

$$\tau \to \tau' = A(\tau) = \frac{a\tau + b}{c\tau + d}\,, \tag{1.1}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ SL$(2, \mathbb{Z})$.[1] By duality, theories with different τ's that are connected by an SL$(2, \mathbb{Z})$ transformation are physically equivalent. For $\theta = 0$ and $a = d = 0$, (1.1) reduces to $g \to 4\pi/g$, the weak-strong coupling duality of Montonen and Olive [2].

Discussions concerning SL$(2, \mathbb{Z})$ duality in $\mathcal{N} = 4$ SYM theory have mainly focused on the Coulomb phase of the theory, where the global SU(4) $\sim$ SO(6) R symmetry is spontaneously broken. In the Coulomb phase, duality has provided important insights for understanding the non-perturbative aspects of the theory. For example, it implies the invariance of the BPS mass spectrum under (1.1) (for reviews, see [3, 4]). Such invariance only occurs if the non-perturbative monopoles and dyonic states are taken into account. Indeed, by the BPS mass formulas, the W-bosons, monopoles, and dyons together are organized into (p, q) SL$(2, \mathbb{Z})$ multiplets. Dynamically, duality also implies that monopoles at strong coupling behave like W-bosons at weak coupling.

$\mathcal{N} = 4$ SYM theory has another important phase, the superconformal phase, where the theory is invariant under the superconformal group PSU$(2, 2|4)$, with SO$(4, 2) \times$ SU(4) as the bosonic subgroup. Here, the observables consist not of particles and solitons, but locally, operators with definite scaling dimensions organized into superconformal multiplets.[2] In

[1] The duality group is PSL$(2, \mathbb{Z})$, if identifying each matrix with its negative.

[2] Wilson loops, which are non-local observables, will not be discussed here. A discussion of Wilson loops and SL$(2, \mathbb{Z})$ duality can be found in [5].

this paper, we explore the action the SL$(2,\mathbb{Z})$ duality on the local observables in the superconformal phase. Whether an operator is mapped into itself or to a non-perturbative operator under an SL$(2,\mathbb{Z})$ transformation is determined by the invariance of its scaling dimension, as a function of τ, under the modular transformation. In general, operators can transform as an SL$(2,\mathbb{Z})$ singlet, or as part of a finite or infinite dimensional SL$(2,\mathbb{Z})$ multiplet.

As paradigms, we analyze two superconformal multiplets that have appeared prominently in the study of $D = 4$ quantum conformal algebra [6] and also AdS/CFT correspondence [7]. They are the 1/2-BPS supercurrent multiplet and the non-BPS Konishi multiplet. We show that operators in the supercurrent multiplet map into themselves up to a multiplicative factor similar to that conjectured by Intriligator [8] . However, using the perturbative and non-perturbative calculations for the scaling dimension of the Konishi operator in [9, 10, 11], we argue that the Konishi multiplet transforms covariantly under the SL$(2,\mathbb{Z})$ transformation. In particular, the Konishi multiplet is the $(1,0)$ element of a (p,q) PSL$(2,\mathbb{Z})$ multiplet of non-BPS superconformal multiplets in the $\mathcal{N} = 4$ SYM theory.[3]

In section 2, we briefly review the superconformal representations of $\mathcal{N} = 4$ SYM theory and set up our notation. In section 3, we discuss the implications of SL$(2,\mathbb{Z})$ duality on the spectrum of operators and examine in detail the transformation properties of the supercurrent and Konishi multiplets. We close in section 4 with some remarks on modular functions and non-perturbative duals of the Konishi multiplet.

2. Superconformal representations of the $\mathcal{N} = 4$ SYM theory

The $\mathcal{N} = 4$ SYM lagrangian is constructed from the component fields of the $\mathcal{N} = 4$ gauge multiplet transforming in the adjoint representation of the gauge group G. For simplicity, we will treat only the case $G = \mathrm{SU}(N)$. The fields consist of scalars, ϕ^I, with $I = 1, \ldots, 6$ in the $\mathbf{6}$ of SU(4) (R symmetry group), complex Weyl spinors, $\psi_{A\alpha}$ with $A = 1, \ldots, 4$ in the $\bar{\mathbf{4}}$ of SU(4), and a gauge field A_μ. The fields are normalized such that the action has the form

$$\mathcal{S} = \int d^4x \,\mathrm{Tr}\left\{-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{\theta g^2}{16\pi^2}F_{\mu\nu} * F^{\mu\nu} - D^\mu\phi^I D_\mu\phi^I + \cdots\right\}. \tag{2.1}$$

In the superconformal phase, with $\langle\phi^I\rangle = 0$, the quantum theory is described by operators that transform under scale transformations with definite scaling dimensions, Δ. Specifically, the operators are eigenfunctions of the dilation operator, D, with eigenvalue, $-i\Delta$. Besides its scaling dimension, each operator is also labelled by its Lorentz and SU(4) representations as required from the decomposition of the global bosonic symmetry $\mathrm{SO}(4,2) \times \mathrm{SU}(4) \supset \mathrm{SO}(1,1) \times \mathrm{SO}(3,1) \times \mathrm{SU}(4)$.

[3] In the AdS/CFT correspondence, the Konishi operator is expected to be associated with stringy states. From this perspective, comments on the covariant transformation of the Konishi operator under SL$(2,\mathbb{Z})$ were made in [9, 11].

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The operators are naturally organized into representations of the superconformal algebra. Such a representation module is constructed starting with a superconformal primary, the lowest weight (scaling dimension) operator in the module, and then acting on it with the 16 supersymmetry operators, Q^A_α and $\bar{Q}^A_{\dot\alpha}$, and momentum operators, P_μ. Acting by Q or $\bar{Q}$ increases Δ by 1/2 and generates conformal primaries while P_μ increases Δ by 1 and generates conformal descendants. The superconformal primary with the smallest scaling dimension is the identity operator and corresponds to the trivial one-dimensional representation with scaling dimension $\Delta = 0$. In the free theory with zero coupling, there are two superconformal primaries with $\Delta = 2$. One is the superconformal primary $O_{20'} = \mathrm{tr}\left(\phi^I\phi^J - \frac{\delta^{IJ}}{6}\sum_{L=1}^{6}\phi^L\phi^L\right)$, the lowest weight operator of the supercurrent multiplet. This multiplet is a 1/2 BPS short multiplet with the representation generated by only 8 supersymmetry generators. Being BPS, the scaling dimension of the supercurrent multiplet is "protected" or remains unchanged for all values of τ. Operators in this multiplet include the SU(4) R-current, $J^{IJ}_{\alpha\dot\alpha} = \delta\bar\delta O_{20'}$, the supercurrents, $S^A_{\alpha\beta\dot\alpha} = \delta^2\bar\delta O_{20'}$ and $\bar{S}_{A\alpha\dot\alpha\dot\beta} = \delta\bar\delta^2 O_{20'}$, and the energy-momentum tensor, $T_{\alpha\dot\alpha\beta\dot\beta} = \delta^2\bar\delta^2 O_{20'}$. Here, we follow the notation in [8] where, for example, $\delta\bar\delta O_{20'} = \{Q, [\bar{Q}, O_{20'}]\}$. The other dimension two superconformal primary is the Konishi operator, $K_1 = \sum_{K=1}^{6}\mathrm{tr}\ \phi^K\phi^K$, the lowest weight operator of the non-BPS Konishi multiplet. The free theory chiral current is a member of this multiplet. The scaling dimension of this long multiplet is not protected and the operator has a nonzero anomalous dimension when the theory is interacting.

3. SL(2, $\mathbb{Z}$) invariance of the superconformal $\mathcal{N} = 4$ SYM theory

In the Coulomb phase of the theory, SL(2, $\mathbb{Z}$) duality implies the invariance of the BPS mass spectrum under modular transformation of τ. However, in the superconformal phase, operators are labelled by their scaling dimensions and their Lorentz and SU(4) representations. Since the values of the Casimirs of Lorentz and SU(4) representations are discrete and not continuous, an operator's Lorentz and SU(4) representations can not vary with τ or under SL(2, $\mathbb{Z}$) transformation. As for scaling dimensions, we can consider the spectrum of scaling dimensions for all operators in the theory for each value of τ. Duality then implies that the scaling dimension spectrum is invariant under the transformation of (1.1).

The invariance of the scaling dimension spectrum constrains the transformation properties of operators under SL(2, $\mathbb{Z}$). Consider the theory at a specific value of τ. For a conformal primary operator $\mathcal{O}_\tau$ with scaling dimension $\Delta_{\mathcal{O}}(\tau_1, \tau_2)$, the two-point correlation function is fully determined by conformal invariance to be

$$\langle \mathcal{O}_\tau(x_1)\mathcal{O}_\tau(x_2)\rangle_\tau \sim \frac{1}{|x_1 - x_2|^{2\Delta_{\mathcal{O}}(\tau_1,\tau_2)}} \tag{3.1}$$

where we have ignored any constant factor that can be absorbed in the normalization of $\mathcal{O}_\tau$. In general, both $\mathcal{O}_\tau$ and $\Delta_{\mathcal{O}}(\tau_1, \tau_2)$ may have non-holomorphic dependence on τ. Now under an SL(2, $\mathbb{Z}$) transformation with $\tau \to \tau'$, duality implies the existence of a primary operator $\mathcal{O}'_{\tau'}$ in the theory at τ' that has the scaling dimension $\Delta_{\mathcal{O}'}(\tau_1', \tau_2') = \Delta_{\mathcal{O}}(\tau_1, \tau_2)$. Explicitly,

$$\langle \mathcal{O}'_{\tau'}(x_1)\mathcal{O}'_{\tau'}(x_2)\rangle_{\tau'} \;\sim\; \langle \mathcal{O}_{\tau}(x_1)\mathcal{O}_{\tau}(x_2)\rangle_{\tau} \;\sim \frac{1}{|x_1 - x_2|^{2\Delta_{\mathcal{O}}(\tau_1,\tau_2)}}\,. \qquad (3.2)$$

where $\mathcal{O}'_{\tau'}$ by $\mathrm{SL}(2,\mathbb{Z})$ invariance must have the same Lorentz and $\mathrm{SU}(4)$ representations as $\mathcal{O}_{\tau}$. Note that (3.2) must hold true for any values of τ and τ' related by an $\mathrm{SL}(2,\mathbb{Z})$ transformation.

Now if the scaling dimension satisfies the modular invariance condition $\Delta_{\mathcal{O}}(\tau_1', \tau_2') = \Delta_{\mathcal{O}}(\tau_1, \tau_2)$, then we simply have $\mathcal{O}'_{\tau'} \sim \mathcal{O}_{\tau}$.[4] Therefore, if the operator's scaling dimension is a modular function, (i.e. a function invariant under the modular transformation of (1.1)[5]), $\mathrm{SL}(2,\mathbb{Z})$ transforms the operator into itself, up to a possible multiplicative factor. This is the case for all BPS operators which have constant scaling dimensions. However, if the scaling dimension is not a modular function, then $\mathrm{SL}(2,\mathbb{Z})$ transformation will act non-trivially on the operator. The operator must necessarily transform covariantly as part of a multiplet under $\mathrm{SL}(2,\mathbb{Z})$.

Although $\mathrm{SL}(2,\mathbb{Z})$ is an infinite dimensional discrete group, the $\mathrm{SL}(2,\mathbb{Z})$ multiplet, in general, need not be infinite dimensional. It is possible that the scaling dimension is invariant under a subgroup, $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$. If Γ has finite index in $\mathrm{SL}(2,\mathbb{Z})$, then the $\mathrm{SL}(2,\mathbb{Z})$ multiplet will be finite dimensional. In fact, $\mathrm{SL}(2,\mathbb{Z})$ has infinitely many finite index subgroups (see [12, 13] and references therein). Well-known examples are the principal congruence subgroup of level N, $\Gamma(N)$, defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1 \bmod N,\, b \equiv c \equiv 0 \bmod N \right\} \qquad (3.3)$$

with index

$$[\mathrm{SL}(2,\mathbb{Z}) : \Gamma(N)] = N^3 \prod_{n|N} \left(1 - n^{-2}\right) \qquad (3.4)$$

where the product is over positive integers $n > 1$ that divide N. Nevertheless, if the index is not finite or if the scaling dimension is not invariant under any element of $\mathrm{SL}(2,\mathbb{Z})$, then the multiplet will be infinite dimensional.

It is worthwhile to point out a simple toy model exhibiting similar characteristics of conformal operators transforming under duality. This is the two dimensional gaussian model ($c = 1$ closed bosonic string theory) on a circle with lagrangian density $\mathcal{L} \sim \partial X \bar{\partial} X$. Here, the discrete duality group is the $\mathbb{Z}_2$ of T-duality, inverting the radius $R \to 1/R$. Operators with conformal dimension invariant under the $\mathbb{Z}_2$ action map to themselves up to a negative sign under T-duality. For example, $\mathcal{L} \to -\mathcal{L}$ because $X_R \to -X_R$ under

[4]We assume that there is no degeneracy of operators having identical global symmetry representations and scaling dimensions for all τ. Degeneracies of non-BPS operators that arise at $g = 0$ are typically broken by operator mixing at nonzero coupling.

[5]In the mathematical literature, the term modular function sometimes refers only to a meromorphic function of τ that are invariant under the modular group. Here, we call any holomorphic or non-holomorphic function $f(\tau_1, \tau_2)$ modular invariant if simply $f(\tau_1, \tau_2) = f(\tau_1', \tau_2')$.

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T-duality. Operators with conformal dimensions not invariant under radial inversion are transformed into other states. This results in the duality mapping between momentum and winding modes.

Below, we analyze the $\mathrm{SL}(2,\mathbb{Z})$ transformation property of the supercurrent and Konishi multiplets in detail to gain more insights on the action of $\mathrm{SL}(2,\mathbb{Z})$ on superconformal multiplets. Note that the scaling dimensions of all operators in a multiplet are determined by the scaling dimension of the superconformal primary. Therefore, the study of the scaling dimension of the primary will determine the $\mathrm{SL}(2,\mathbb{Z})$ multiplet structure for all operators in the superconformal multiplet.

3.1 $\mathrm{SL}(2,\mathbb{Z})$ action on the supercurrent multiplet

Since $\Delta_{O_{20'}}$ is a constant, operators in the supercurrent multiplet map into themselves up to a multiplicative factor under $\mathrm{SL}(2,\mathbb{Z})$ transformation. The multiplicative factor differs for different elements of the multiplet and in general can depend on τ. For some of the operators in the multiplet, the transformation factors have physical significance and can be simply deduced.

Consider first the R-current, $J^{IJ}_{\alpha\dot{\alpha}} = \delta\bar{\delta}O_{20'}$, and the energy momentum tensor, $T_{\alpha\dot{\alpha}\beta\dot{\beta}} = \delta^2\bar{\delta}^2 O_{20'}$. They are associated with the $\mathrm{SO}(1,1)\times\mathrm{SO}(3,1)\times\mathrm{SU}(4)$ symmetry charges of the theory. Since these charges are invariant under $\mathrm{SL}(2,\mathbb{Z})$ transformation, the multiplicative factor for both must be trivial and they transform invariantly.

Also of importance is the dimension four operator $\Phi = \delta^4 O_{20'}$ and its complex conjugate $\bar{\Phi} = \bar{\delta}^4 O_{20'}$. They are the exactly marginal operators invariant under all 16 supersymmetry generators and identified with the on-shell lagrangian density, $\mathcal{L} \sim Im[\frac{\tau}{\tau_2}\Phi]$.[6] As in the simple toy gaussian model on a circle, where $\mathcal{L}$ is negative of itself under T-duality, the $\mathcal{N}=4$ SYM lagrangian density also picks up a non-trivial factor under $\mathrm{SL}(2,\mathbb{Z})$ duality. This factor can be obtained as follows.

As marginal perturbation, $\Phi+\bar{\Phi}$ changes the coupling g of the theory while $\frac{1}{i}(\Phi-\bar{\Phi})$ changes θ. Let us consider a theory with parameter τ perturbed by

$$\begin{aligned}\delta\mathcal{L} &= \frac{\delta\tau_2}{\tau_2}(\Phi+\bar{\Phi}) + \frac{\delta\tau_1}{i\tau_2}(\Phi-\bar{\Phi}) \\ &= \frac{1}{i\tau_2}[\delta\tau\Phi - \delta\bar{\tau}\bar{\Phi}]\,,\end{aligned} \tag{3.5}$$

where $\delta\tau = \delta\tau_1 + i\delta\tau_2$ and its complex conjugates are constants parameterizing the perturbation. Under the marginal perturbation, $\tau \to \tau + \delta\tau$. Now apply the $\mathrm{SL}(2,\mathbb{Z})$ duality to the theory with the perturbation included. The dual theory at $\tau' + \delta\tau'$ is the theory at τ' perturbed by a dual perturbation

$$\delta\mathcal{L}' = \frac{1}{i\tau_2{}'}\left[\delta\tau'\Phi' - \delta\bar{\tau}'\bar{\Phi}'\right]\,, \tag{3.6}$$

where $\delta\tau' = \frac{\delta\tau}{(c\tau+d)^2}$. But since $\delta\tau$ is a constant and does not transform under $\mathrm{SL}(2,\mathbb{Z})$, Φ and $\bar{\Phi}$ must pick up a factor under $\mathrm{SL}(2,\mathbb{Z})$ transformation. From duality, the

[6]Explicitly, $\Phi \sim \mathrm{Tr}(F^2 + i * FF)$, from applying on-shell supersymmetry transformation relations.

transformation is required to be

$$\Phi \to \frac{1}{(c\tau+d)^2}\Phi \qquad \text{and} \qquad \bar{\Phi} \to \frac{1}{(c\bar{\tau}+d)^2}\bar{\Phi}\,. \tag{3.7}$$

Thus, Φ and $\bar{\Phi}$ transforms with modular weight $(-2,0)$ and $(0,-2)$, respectively, under modular transformation.[7]

Though the above conformal primary operators are in the same superconformal multiplet, they transform differently under duality. This implies that the action of $\mathrm{SL}(2,\mathbb{Z})$ and that of the supersymmetry generators, δ, $\bar{\delta}$ do not commute. Intriligator, in [8], has conjectured that all BPS operators transform under $\mathrm{SL}(2,\mathbb{Z})$ duality with a particular modular weight given by the $\mathrm{U}(1)_Y$ charge of the operator. The $\mathrm{U}(1)_Y$ is an outer automorphism of the $\mathcal{N}=4$ superconformal algebra that only acts on the fermionic generators. The conjecture is motivated by the AdS/CFT correspondence where the $\mathrm{U}(1)_Y$ is identified with the compact U(1) of the $\mathrm{SL}(2,\mathbb{R})$ symmetry in the type-IIB supergravity action. However, $\mathrm{U}(1)_Y$ is broken for non-zero coupling and its applicability for $\mathrm{SL}(2,\mathbb{Z})$ duality still needs to be clarified.

3.2 $\mathrm{SL}(2,\mathbb{Z})$ action on the Konishi multiplet

Being a long multiplet at non-zero coupling, the scaling dimension of the Konishi multiplet is not constant with respect to τ. Explicit calculations have been carried out to determine both the perturbative and non-perturbative contributions to the anomalous dimension of the Konishi operator, $\gamma_{K_1} = \Delta_{K_1} - 2$, for non-zero g and θ. From perturbative calculations in [6, 9, 10], it is known up to order g^4 that

$$\begin{aligned}\gamma_{K_1}(\tau) &= \frac{3N}{4\pi^2}g^2 - \frac{3N^2}{16\pi^4}g^4 + \cdots \\ &= \frac{3N}{\pi}\left(\frac{1}{\tau_2} - \frac{N}{\pi}\frac{1}{\tau_2{}^2} + \cdots\right),\end{aligned} \tag{3.8}$$

where again $\tau = \tau_1 + i\tau_2 \equiv \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$. As for the dependence on θ, note that θ only appears in the lagrangian coupled to the surface term $*FF$. For correlation functions, θ dependence is known only to arise from instanton sectors. Moreover, it was found in [9, 14, 11] that non-perturbative instanton effects do not contribute to γ_{K_1}. This is technically due to the inability of the two-point function of K_1 to provide the necessary fermion zero modes to match those of the instanton background (see [11] and also [15] for details). Thus, assuming only instanton effects may give a θ dependence to the scaling dimension, we conclude that γ_{K_1} is independent of θ.[8]

One can ask whether $\Delta_{K_1} = 2 + \gamma_{K_1}$ with no τ_1 dependence can possibly be a modular function. Indeed, one can prove that any modular function with no τ_1 dependence must be a constant.

[7] An operator $O(\tau,\bar{\tau})$ with modular weight $(w,\bar{w})$ transforms under the modular transformation as $O(\tau,\bar{\tau}) \to (c\tau+d)^w(c\bar{\tau}+d)^{\bar{w}}O(\tau,\bar{\tau})$

[8] We assume that no other non-perturbative effect contributes to the θ dependence of Δ_{K_1}.

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Theorem. *Let $f(\tau)$ with $\tau = \tau_1 + i\tau_2$ be a function on the upper half plane, i.e. $\tau_2 > 0$. If $f(\tau)$ is a modular invariant function and is also independent of τ_1 , then $f(\tau)$ is a constant function.*

Proof. With no dependence on τ_1, f is a function of only one variable $f(\tau_2)$. Now modular transformation of $\tau \to \tau' = \frac{a\tau+b}{c\tau+d}$ implies $\tau_2 \to \tau_2{}' = \frac{\tau_2}{|c\tau+d|^2}$. Therefore, $f(\tau_2)$ being a modular invariant function must satisfy

$$f(\tau_2) = f\left(\frac{\tau_2}{(c\tau_1+d)^2 + c^2\tau_2{}^2}\right) \tag{3.9}$$

for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ and for any τ_1 on the r.h.s. of (3.9). We will show that for any A with $c \neq 0$, (3.9) requires $f(\tau_2)$ is a constant.

First, choose $\tau_1 = -\frac{d+x}{c}$ and $\tau_2 = \frac{1}{|c|}$, (3.9) becomes

$$f\left(\tau_2 = \frac{1}{|c|}\right) = f\left(\tau_2 = \frac{1}{|c|(x^2+1)}\right) \tag{3.10}$$

for any real x. For $0 \leq x < \infty$, (3.10) implies $f(\tau_2) = f(\tau_2 = 1/|c|)$ for all $\tau_2 < 1/|c|$. Now, setting $\tau_1 = -d/c$ and $\tau_2 = \frac{1}{|cx|}$ in (3.9) , we obtain $f(\tau_2 = \frac{1}{|cx|}) = f(\tau_2 = \frac{|x|}{|c|})$. Taking $1 \leq x < \infty$, we conclude that $f(\tau_2) = f(\tau_2 = 1/|c|)$ for all τ_2.

By the above theorem, $\Delta_{K_1}(\tau_2)$ can not be a modular function. This implies that K_1 does not transform as a singlet under the $\mathrm{SL}(2,\mathbb{Z})$ duality action. For example, from the S transformation, $\tau \to -1/\tau$, there must exist a non-perturbative operator, $K_1{}'$ that has scaling dimension $\Delta_{K_1{}'} = 2$ as $g^2 \to \infty$. And because Δ_K is not invariant under S, $K_1{}'$ can not be proportional to K_1. More generally, from the proof of the above theorem, we know that $\Delta_{K_1}(\tau_2)$ is not invariant under any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ with $c \neq 0$. Hence, K_1 must be an element in an infinite dimensional multiplet of $\mathrm{SL}(2,\mathbb{Z})$ which we will call $\tilde{K}$. Since the $\mathrm{SL}(2,\mathbb{Z})$ transformation of τ_2 depends only on the two relatively prime integers (c,d) , elements in $\tilde{K}$ can be labelled by a pair of integers, (p,q), with p and q relatively prime. The (1,0) and (0,1) elements are respectively K_1 and $K_1{}'$. This representation is similar to that of the BPS (p,q)-string in type-II string theory. However, for non-BPS $\mathrm{SL}(2,\mathbb{Z})$ multiplets, the values of p and q do not correspond to any quantized U(1) charges. That is in $\tilde{K}$, both $(1,0)$ and $(-1,0)$ elements should be identified with the Konishi operator. Thus, the (p,q) representation is more accurately that of $\mathrm{PSL}(2,\mathbb{Z})$. This allows the imposition of the constraint that p be strictly non-negative.

We can easily write down the scaling dimensions of elements in $\tilde{K}$ in the small g^2 expansion. For each (p,q) element in $\tilde{K}$, the scaling dimensions is given by

$$\Delta_{(p,q)}(\tau) = 2 + \frac{3N}{\pi}\left[\frac{|p+q\tau|^2}{\tau_2} - \frac{N}{\pi}\left(\frac{|p+q\tau|^2}{\tau_2}\right)^2 + \cdots\right], \tag{3.11}$$

where we have simply applied a modular transformation to (3.8) by replacing τ_2 with $\frac{\tau_2}{|p+q\tau|^2}$. We expect that the scaling dimensions of all elements in $\tilde{K}$ with the exception of K_1 goes to infinity as $g^2 \to 0$ ($\tau_2 \to \infty$). Thus, in the small coupling regime, $K_1{}'$ and other elements of $\tilde{K}$ are highly non-perturbative.

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The above statements for K_1 also applies for all other operators in the Konishi multiplet. The scaling dimensions of the (p, q) element for an operator in the multiplet is that of (3.11) after replacing the Konishi operator's canonical dimension with that of the operator of interest.

4. Discussion

We have demonstrated that local operators in $\mathcal{N} = 4$ SYM theory in the superconformal phase may transform non-trivially under $\mathrm{SL}(2, \mathbb{Z})$ duality. How an operator transform is determined by the modular property of its scaling dimension function. If the function is modular invariant, then it is a singlet under the transformation. Otherwise, it sits in a finite or infinite dimensional multiplet of $\mathrm{SL}(2, \mathbb{Z})$. A class of singlets under $\mathrm{SL}(2, \mathbb{Z})$ are operators that have constant scaling dimensions. It would be interesting to identify perturbative operators that have non-trivial modular functions for their scaling dimensions. In the theory of automorphic forms, modular functions that are eigenfunctions of the laplacian on the upper half plane have been classified.[9] Taking into account of unitarity constraints which set a lower bound on the scaling dimension, a class of candidate modular scaling dimension functions is the non-holomorphic cusp forms.[10] Although no explicit form of these functions exists, they do exhibit characteristics of τ_1 dependence similar to those arising from instanton effects.

Without instanton contributions, a non-constant scaling dimension can not be modular invariant. Even though the scaling dimension is invariant under T transformation, $\tau \to \tau + 1$, the lack of τ_1 dependence requires that the operator in question transform as an $(1, 0)$ element in an infinite dimensional (p, q) $\mathrm{PSL}(2, \mathbb{Z})$ multiplet. This is the case for the operators in the Konishi multiplet. As a corollary, any operator that transforms in a finite dimensional $\mathrm{SL}(2, \mathbb{Z})$ multiplet must have a non-trivial τ_1 dependence. At present, no operator is known to transform in a finite $\mathrm{SL}(2, \mathbb{Z})$ multiplet. Nevertheless, it certainly would be interesting for such operators to appear or to prove that they are forbidden in the $\mathcal{N} = 4$ SYM theory.

As for the (p, q) multiplet, it consists almost exclusively of non-perturbative local operators whose fundamental roles arise at the large coupling regimes. (The exception is the $(1, 0)$ element.) This is evident from taking the $\mathrm{SL}(2, \mathbb{Z})$ dual of perturbative operator product expansions (OPEs) involving the Konishi operator. Consider the OPE of two $O_{20'}$'s. Schematically, it is given perturbatively by

$$O_{20'}(x_1) O_{20'}(x_2) \to \frac{c}{(x_{12})^4} + \frac{O_{20'}}{(x_{12})^2} + \frac{K_1}{(x_{12})^{2-\gamma_{K_1}(\tau_2)}} + \cdots \tag{4.1}$$

[9] The non-euclidean laplacian is $L = {\tau_2}^2 \left(\frac{\partial^2}{\partial {\tau_1}^2} + \frac{\partial^2}{\partial {\tau_2}^2} \right)$. The eigenfunctions are known to be of three types: constant, holomorphic, and non-holomorphic. We point out that although the eigenfunctions are by construction modular invariant, modular invariant functions are generally not eigenfunctions of the laplacian. For references on modular functions, see [13, 16, 17].

[10] In particular, the holomorphic modular functions are not bounded from below and the non-holomorphic Eisenstein series are not finite as $g \to 0$.

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where c is proportional to the central charge, and we have ignored all SU(4) indices and other proportionality constants. Under $\mathrm{SL}(2,\mathbb{Z})$ duality, $O_{20'}$ is invariant while K_1 transforms into an element in $\tilde{K}$. Thus, for example, at the large g coupling limit with $\theta = 0$, the OPE's of two $O_{20'}$'s contains the $(0,1)$ operator, ${K_1}'$. We point out that the structure constant of two 1/2 BPS short operators and a long operator, c_{SSL}, in general depend on τ. Thus, even with the aid of $\mathrm{SL}(2,\mathbb{Z})$ duality, understanding the interactions of ${K_1}'$ at perturbative coupling will require some knowledge of the dynamics of K_1 at strong coupling.

Obtaining a physical understanding of the non-perturbative (p,q) operators at finite small coupling is challenging. Because these operators are non-BPS, the (p,q) labels are just labels and do not pertain to any symmetry charges. It may be possible that a better understanding may be obtained from a more geometric perspective of $\mathrm{SL}(2,\mathbb{Z})$ duality, as in the toroidal compactification of the $D = 6$, $\mathcal{N} = (2,0)$ superconformal theory down to the superconformal $\mathcal{N} = 4$ SYM theory [18]. Unlike the $\mathcal{N} = 4$ theory, the corresponding Konishi-like operator in the $\mathcal{N} = (2,0)$ theory is found in a discrete series unitary representation of the superconformal algebra [19]. One may hope that the subtleties of the toroidal compactification will reveal the origin of the (p,q) operators and provide other insights into $\mathrm{SL}(2,\mathbb{Z})$ duality in the superconformal phase. We leave these questions for future investigations.

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