Heterotic Geometry and Fluxes

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Abstract. We begin by discussing the question, "What is string geometry?" We then proceed to discuss six-dimensional compactification geometry in heterotic string theory with fluxes. A class of smooth non-Kähler compact solutions is presented and relations between Kähler and non-Kähler solutions are pointed out.

1. String Geometry

Geometry plays an important role in many aspects of string theory. For spacetime compactifications, a six- (or seven-) dimensional internal compact geometry is often used to wrap the extra dimensions of string theory in order to connect with our low-energy four-dimensional world. The characteristics of the internal compact geometry then determine the physical observables we see in four dimensions. For the two-dimensional worldsheet (or higher-dimensional worldvolume), its dynamics is determined by a path integral over all possible embeddings in spacetime. The saddle points of the integration correspond to classical objects that typically have geometrically interesting embedding conditions.

An interesting question is what is the geometry string theory "sees"? Or more precisely, what geometries are identical or different from the perspective of string theory? That this differs from that of a point particle theory can already be seen in the closed bosonic string theory compactified in one dimension. Let us consider the moduli space of such compactifications. This is equivalent to the moduli space of $c = 1$ two-dimensional CFTs worked out by Dijkgraaf *et al.* [**DVV**] and Ginsparg [**G**]. This moduli space is interesting and relatively simple. Let us briefly describe it.

There are three types of $c = 1$ compactifications: the circle S^1 ($x \sim x + 2\pi R$), the interval $I = S^1/\mathbb{Z}_2$ (where $\mathbb{Z}_2 : x \to -x$), and three non-geometric ones. Both the circle and the interval have a radius parameter,

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R_S and R_I, respectively. As is well-known, string theory sometimes cannot distinguish what we would consider as different spaces. For example, T-duality equates R_S and $1/R_S$ theories. And even more unexpectedly, the circle theory at $R_S = 2$ is also identical to the interval theory at $R_I = 1$. This latter identification in fact connects the moduli space of the circle compactifications with that of the interval compactifications. The three non-geometrical compactifications are also interesting. They can all be described as orbifolds of the circle theory at the self-dual radius $R_S = 1$. But unlike the geometrical compactifications, they are three isolated points on the $c = 1$ moduli space.

The one-dimensional moduli space of string compactifications was certainly a surprise when first discovered in the mid-1980s. One would like to classify all string compactifications and describe their connectedness (or disconnectedness) for higher-dimensional cases. Studying the moduli space of compactifications will certainly help us understand better how string theory relates to geometrical and also "non-geometrical" spaces. Of course we expect that the higher-dimensional cases are likely very challenging to classify and certainly much richer as well. Below, we shall describe a step in this direction in the more phenomenologically relevant case of six-dimensional compactifications in heterotic string theory.

2. Heterotic Fluxes with Supersymmetry

For many years, the focus on six-dimensional compactification was on Calabi-Yau (CY) geometries, mainly because a CY preserves minimal four-dimensional supersymmetry without turning on any additional background fields. However, a CY geometry typically has many scalar moduli which result in unwanted massless scalars in four dimensions. Thus, in the past several years, there has been much work in incorporating fluxes, i.e. non-zero p-form background fields, into compactification models. Adding fluxes while preserving supersymmetry typically requires the geometry to be non-Kähler and hence non-CY.

I shall describe flux compactification in heterotic theory and present in the next section a class of smooth geometrical solutions that preserve supersymmetry in four dimensions. Before proceeding, let me mention some of the advantages of studying flux compactification in heterotic theory as compared to type II theories:

(1) The heterotic background fields are $\{g_{mn}, H_{mnp}, \phi, F_{mn}\}$. The three-form flux field, H, is the only additional field not present in standard Calabi-Yau compactification. The first three fields are the NS-NS fields of the type II theories. There are no additional R-R fields, and thus, in heterotic theory, the supersymmetry conditions incorporating the fluxes are relatively "simpler" and the possible types of geometry are more constrained.

(2) The internal six-dimensional geometry can be smooth since no sources or branes need to be present. In type II theories, supergravity no-go theorems for compactification to four-dimensional Minkowski spacetime stipulate that fluxes can only be non-zero if the compact geometry has singularities (which can arise from the presence of branes). The heterotic theory bypasses the no-go theorem with the anomaly cancellation condition which modifies the Bianchi identity for the three-form H.
(3) Gauge fields are naturally present. This allows for the possibility of constructing models with interesting phenomenology and possibly reproducing the Standard Model.

2.1. Review of $N = 1$ Supersymmetry Constraints. In the string frame, preserving supersymmetry requires the ten-dimensional geometry to be the product space $M = M^{3,1} \times X_6$. $N = 1$ supersymmetry requires the existence of a nowhere vanishing spinor η on X_6 that satisfies the following equations:

$$\begin{aligned} \delta\psi_m &= \nabla_m \eta + \frac{1}{8} H_{mnp}\, \gamma^{np}\, \eta = 0 \ , \\ \delta\lambda &= \gamma^m \partial_m \phi\, \eta + \frac{1}{12} H_{mnp}\, \gamma^{mnp}\, \eta = 0 \ , \\ \delta\chi &= \gamma^{mn} F_{mn} \eta = 0 \ . \end{aligned}$$

With a non-vanishing spinor η, we can write down a complex structure $J_m{}^n = -i\eta^\dagger \gamma_m{}^n \eta$ such that $J^2 = -1$ and the Nijenhuis tensor $N_{mn}{}^p = 0$. Furthermore, the Hermitian $(1,1)$-form and the holomorphic $(3,0)$-form which together define the geometry of X_6 can also be simply expressed as fermion bilinears as follows.

$$\begin{aligned} J^{(1,1)}_{mn} &= -i\eta^\dagger \gamma_{mn} \eta \ , \\ \Omega^{(3,0)}_{mnp} &= e^{-2\phi} \bar{\eta}^\dagger \gamma_{mnp} \eta \ . \end{aligned}$$

We now list the supersymmetry constraints on the background fields.

2.1.1. *Geometry.* The constraints on the six-dimensional geometry can be expressed as differential equations on the nowhere vanishing pair (J, Ω). We compare the case of Calabi-Yau versus the non-Kähler case in the following.

Calabi-Yau	Non-Kähler
$dJ = 0$ (Kähler condition)	$d(\lVert\Omega\rVert * J) = 0$ (conformally balanced)
$d\Omega^{(3,0)} = 0$	$d\Omega^{(3,0)} = 0$
$H = 0$	$H = i(\bar\partial - \partial)J \neq 0$
$\phi = \phi_0 = \text{constant}$	$e^{-2(\phi-\phi_0)} = \lVert\Omega\rVert$

Note that since $H^{(1,2)} = i\bar{\partial}J$, if $H \neq 0$, then the metric $J_{a\bar{b}} = ig_{a\bar{b}}$ must be non-Kähler. However, not any non-Kähler geometry is allowed; the metric with a non-zero H-flux turned on must be conformally balanced. The balanced condition is defined to be

$$d(*J) = \frac{1}{2}d(J \wedge J) = 0 \ .$$

Note that Kähler metrics are also balanced since $dJ = 0$ implies $d(J \wedge J) = 0$. Indeed, the balanced condition can be thought of as a relaxation of the Kähler condition. Explicitly, consider the number of constraint equations for three complex dimensions. For the Kähler condition, $dJ = 0$ gives 9 complex constraint equations. For the balanced condition, $*d(*J) = 0$ gives only 3 complex constraint equations.

Heuristically, the additional "degrees of freedom" in a conformally balanced geometry can be thought of as being parametrized by (H, ϕ) – H parametrizes the non-Kählerity and ϕ the conformal factor or the norm of Ω. Conversely, one can just forget about the H-flux and ϕ and simply consider the compactification geometry as being defined by J and Ω which are required to satisfy the above constraints.

Perhaps an analogy to closed curves in $\mathbb{R}^2$ would be useful here. The Calabi-Yau geometry is rigid and is the most constrained and we can compare it to a circle in $\mathbb{R}^2$. If we add flux and relax to non-Kähler geometry in heterotic theory, this can be said to correspond to an ellipse or a more curvy closed curve in $\mathbb{R}^2$. The additional degrees of freedom in the geometry, say for example the eccentricity of the ellipse or the curvature of a more general curve, would correspond physically to what we call fluxes. Flux geometries are clearly much more numerous and, as we also know, nature does not always choose the most symmetrical configurations, *e.g.* the elliptical shapes of planetary motion.

2.1.2. *Gauge Bundle.* The background gauge field satisfies the Hermitian-Yang-Mills condition:

$$F_{(2,0)} = F_{(0,2)} = 0 \ , \qquad\qquad F_{mn}J^{mn} = 0 \ .$$

The first part is the condition that the gauge bundle is holomorphic. The second is a primitivity condition. Together, the Hermitian-Yang-Mills condition is known from the work J. Li and Yau [**LY**] to correspond to the gauge bundle being "stable."

2.1.3. *Anomaly Condition.* In the heterotic theory, this is the modified Bianchi identity

$$dH = 2i\partial\bar{\partial}J = \frac{\alpha'}{4}(\mathrm{tr}R \wedge R - \mathrm{tr}F \wedge F) \ .$$

This is an important four-form equation which further relates the gauge bundle with the geometry.

3. A Class of Smooth Compact Solutions

Now we present a compactification model that satisfy all of the above supersymmetry conditions. We shall proceed in two steps. We first compactify from ten to six dimensions and then compactify another two dimensions to four dimensions.

3.1. Four Compact Dimensions to 6D. For four real compact dimensions, the required geometry was studied by Strominger [**St**] in 1986. The balanced condition $d(*J) = dJ = 0$ in four dimensions is just the Kähler condition. So the required geometry is that of a conformal Calabi-Yau with

$$J = e^{2\phi} J_{K3} \ , \qquad \Omega^{(2,0)} = \Omega_{K3} \ .$$

For gauge fields, one can have $U(1)$ gauge fields taking the field strengths to be the primitive (1,1) forms on $K3$:

$$U(1): \qquad F^{(1,1)} \wedge J_{K3} = 0 \ , \qquad F^{(1,1)} \in H^{(1,1)}(K3, \mathbb{Z}) \ .$$

Alternatively, one can look for stable $SU(r)$ bundles on $K3$. These have been studied by Mukai [**M**] and are known to exist as long as

$$SU(r): \qquad c_2(F) = -\frac{p_1}{2} \geq r - \frac{1}{r} \ .$$

The anomaly condition being a four-form equation on a four-manifold becomes a single Laplacian equation with added sources. It can be solved as long as the source, proportional to $*(\mathrm{tr}\, R \wedge R - \mathrm{tr}\, F \wedge F)$ integrates to zero. This gives the condition that

$$C_2(F) = \frac{1}{16\pi^2} \int_{K3} \mathrm{tr}\, F \wedge F = C_2(R_{K3}) = 24 \ .$$

There are certainly many possible stable bundles on $K3$ that satisfy the 2nd Chern class conditon. Note that the anomaly condition clearly shows that a conformal T^4 geometry would not be possible since $C_2(R_{T^4}) = 0$.

3.2. Six Compact Dimensions to 4D. We now take the conformal $K3$ solution and compactify two additional directions to get to four dimensions. We can compactify straightforwardly on a T^2 with complex coordinate $z = x + iy$. The metric and holomorphic three-form take the form

$$J = e^{2\phi} J_{K3} + \frac{i}{2} dz \wedge d\bar{z} \ , \qquad \Omega^{(3,0)} = \Omega_{K3} \wedge dz \ .$$

With the gauge field strength specified as above, we have a simple six-dimensional compactification on a Kähler $K3 \times T^2$ manifold.

But we can also compactify the two dimensions in a more sophisticated manner [**FY, BBFTY**]. Instead of a trivial T^2 metric, we can apply a Kaluza-Klein (KK) compactification on the T^2. The global one-form on the T^2 is now

$$\theta = (dz + \alpha_1 dz_1 + \alpha_2 dz_2)$$

with (z_1, z_2) the local complex coordinates on the $K3$ surface and α_i the KK gauge field. If the KK field strength $\omega = d\alpha \neq 0$, then the manifold is no longer $K3 \times T^2$. In fact, it is non-Kähler and is mathematically a T^2 bundle over a $K3$ surface. We can take the metric and holomorphic three-form to be

$$J = e^{2\phi} J_{K3} + \frac{i}{2}\theta \wedge \bar{\theta} \ , \qquad \Omega^{(3,0)} = \Omega_{K3} \wedge \theta \ . \tag{3.1}$$

In order that the manifold be complex and balanced, we need to require that $\omega = d\alpha \in H^{(1,1)}(K3, \mathbb{Z})$ and that ω is primitive, *i.e.* $\omega \wedge J_{K3} = 0$.[1] For the gauge field strengths, we can take as before the Hermitian-Yang-Mills field strengths on stable bundles of the $K3$. The only remaining question is whether the anomaly condition can be satisfied. This turns out to be an important question, for if the anomaly condition is not present, it can be shown that the above non-Kähler ansatz (3.1) would have an infinite number of scalar moduli [**BTY1**]. But inserting the non-Kähler ansatz into the anomaly equation gives a highly non-linear second order differential equation for ϕ on $K3$. Demonstrating that there exists a solution for ϕ is highly non-trivial and this was proved by Fu and Yau [**FY**] when the following topological condition is satisfied:

$$C_2(F) + \int_{K3} \|\omega\|^2 = 24 \ .$$

The expectation that there exists a non-trivial non-Kähler solution in heterotic theory was first motivated via duality from M-theory on $K3 \times K3$ with G-flux. This is the work of Dasgupta *et al.* [**DRS**] (see also [**BD**]) where they wrote down the form of the Hermitian metric. However, whether the anomaly condition can be satisfied in the heterotic theory on a $K3$ surface was not known until the proof of Fu and Yau [**FY**]. Hence, this non-Kähler geometry has been called in the literature [**CL**] the FSY geometry (after Fu, Strominger, and Yau). It is also worth pointing out that there is now a gauged linear sigma model construction of this non-Kähler solution that is valid to all orders in α' [**AEL**].

4. Connectedness of Kähler and non-Kähler Solutions

With a large class of six-dimensional compact solutions, which includes both Kähler and non-Kähler manifolds, *i.e.* $K3 \times T^2$ and T^2 bundles over $K3$, it is interesting to ask whether any of the solutions can be related. It turns out that in fact some of the non-Kähler solutions can be continuously deformed to Kähler solutions with $U(1)$ gauge bundles [**BTY2**] (see also [**A, Se**]). The easiest way to see this is via duality to M-theory compactification on $K3 \times K3$ with G-flux. The heterotic geometry that is dual to the M-theory solution is the above six-dimensional geometry (3.1) times a circle. We can recover the above six-dimensional solution by decompactifying the

[1] More generally, ω can have a component in $H^{(2,0)}(K3, \mathbb{Z})$ (see [**BBFTY**]).

circle, *i.e.* taking the limit of the circle radius to infinity. It turns out that the trivial duality of exchanging the two $K3$'s in the M-theory solution on $K3 \times K3$ can correspond in the heterotic theory to exchanging the T^2 twist ω with the $U(1)$ gauge field strength F. So for example, the heterotic Kähler solution with $(\omega = 0, F \neq 0)$ can be dualized to a non-Kähler solution with $(\omega \neq 0, F = 0)$.

Notice that the information of the T^2 twisting ω is encoded in the metric. The Kähler/non-Kähler duality of exchanging ω and F thus suggests that we should treat the the metric g and the field strength F (flux) on an equal footing when considering heterotic flux compactification. This is an important difference between Calabi-Yau and non-Kähler compactifications. In the Calabi-Yau case, all information is encoded in the geometry or specifically the metric. But in general string compactifications, we shall describe the "string" geometry with both metric and fluxes.

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