AMS/IP Studies in Advanced Mathematics
Volume 51, 2012

Cohomologies and Elliptic Operators on Symplectic Manifolds

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Abstract. In joint work with S.-T. Yau, we construct new cohomologies of differential forms and elliptic operators on symplectic manifolds. Their construction can be described simply following a symplectic decomposition of the exterior derivative operator into two first-order differential operators, which are analogous to the Dolbeault operators in complex geometry. These first-order operators lead to new cohomologies which are finite-dimensional and associated elliptic operators that exhibit Hodge theoretical properties. The symplectic cohomologies give new invariants for non-Kähler symplectic manifolds.

This article is an expanded version of the talk given at ICCM 2010, Beijing, where new results from joint work with S.-T. Yau [**18, 19, 20**] were reported concerning cohomologies on closed symplectic manifolds.

1. Introduction

A symplectic manifold M is an even dimensional manifold ($d = 2n$) equipped with a differential two-form, ω, that is both non-degenerate (e.g. $\omega^n > 0$) and d-closed. By the well-known Darboux's theorem, symplectic manifolds are locally diffeomorphic to $\mathbb{R}^{2n}$. So interesting symplectic invariants must be global objects.

Some of the simplest global invariants on manifolds come from cohomologies defined on differential forms. Certainly differential cohomologies and associated elliptic operators have played an important role in the analysis of Riemannian and complex manifolds. So it is natural to ask what types of differential cohomologies and elliptic operators are there on symplectic manifolds?

Now a symplectic manifold is of course a smooth manifold, so one can always consider the de Rham cohomology based on the standard exterior differential operator d. All symplectic manifolds also have an almost complex structure, J, and sometimes the almost complex structure is also integrable. If so, this would make a symplectic manifold a complex manifold (or perhaps even Kähler). In this case, one can also consider the Dolbeault cohomology and possibly other more refined complex cohomologies defined using the Dolbeault operators $(\partial, \bar{\partial})$. But none of these cohomologies make use of the symplectic structure. So this leads us naturally to ask the following questions:

	Differential operator	Cohomology	Invariants
Smooth M	d	de Rham	Betti
Complex (M, J)	$(\partial, \bar{\partial})$	Dolbeault	Hodge
Symplectic (M, ω)	?	?	?

TABLE 1. Standard cohomologies for Riemannian and complex manifolds and their associated numerical invariants. How about the sympletic case?

1. Are there any intrinsically symplectic cohomologies and elliptic operators on (M, ω)?
2. If so, what invariants do they encode?

As we shall explain, it turns out that there are quite a few intrinsically symplectic differential cohomologies giving new symplectic invariants especially for non-Kähler symplectic manifolds. In fact, as was found in [**18, 19**], symplectic cohomologies can be defined rather simply. As suggested in Table 1, the key is to first write-down the linear symplectic differential operators. Once they are found, new symplectic cohomologies can be easily constructed for instance similar to those of de Rham and Dolbeault. So let us begin first by introducing two first-order symplectic operators: ∂_+ and ∂_- .

2. First-order symplectic differential operators

The first-order symplectic operators (∂_+, ∂_-) found in [**19**] can be constructed in analogy with the Dolbeault operators $(\partial, \bar{\partial})$ in complex geometry. Let us motivate their construction by pointing out that on a complex manifold, there are two well-known decompositions which lead to the Dolbeault operators:

1. Decomposition of forms: Differential forms decompose into (p, q) components, $\mathcal{A}^{p,q}$, where $0 \leq\ p, q\ \leq n$. The space of all differential forms can be arranged into a "diamond" shape. For instance, for dimension $d = 4$, we have the (p, q)-diamond

$$\begin{array}{ccccc} & & \mathcal{A}^{2,2} & & \\ & \mathcal{A}^{2,1} & & \mathcal{A}^{1,2} & \\ \mathcal{A}^{2,0} & & \mathcal{A}^{1,1} & & \mathcal{A}^{0,2} \\ & \mathcal{A}^{1,0} & & \mathcal{A}^{0,1} & \\ & & \mathcal{A}^{0,0} & & \end{array}$$

2. Decomposition of the exterior derivative: $d = \partial + \bar{\partial}$. This follows from the fact that d acting on $A^{p,q}$ has only two components,

$$d\mathcal{A}^{p,q} \ \rightarrow\ \mathcal{A}^{p+1,q} \oplus\ \mathcal{A}^{p+1,q}\ ,$$

which defines the Dolbeault operators ∂ and $\bar{\partial}$ as the projections of $d\mathcal{A}^{p,q}$ onto $\mathcal{A}^{p+1,q}$ and $\mathcal{A}^{p,q+1}$, respectively. (Specifically, $\bar{\partial}\,(\mathcal{A}^{p,q}) := (d\mathcal{A}^{p,q})^{p,q+1}$). And importantly, because the decomposition consists of only two terms, it follows from $d^2 = 0$ that the Dolbeault operators have the useful properties

$$(\partial)^2 = (\bar{\partial})^2 = 0\ , \qquad \partial\bar{\partial} = -\bar{\partial}\partial\ .$$

Now on a symplectic manifold (M^{2n}, ω), there is also a decomposition of differential forms and the exterior derivative. The decompositions in this case is intrinsically symplectic as they require only the presence of a symplectic structure, ω. The two decompositions will lead us to (∂_+, ∂_-).

2.1. Differential forms decomposition: Lefschetz decomposition. When there exists a non-degenerate two-form, for instance ω, differential forms can be decomposed. This well-known decomposition is often called the Lefschetz decomposition. This decomposition is purely algebraic in nature and does not require that ω be d-closed.

Let $\Omega^*(M)$ be the space of differential forms and $A \in \Omega^*(M)$. In the presence of a non-degenerate two-form, ω, there are three operations one can apply to $\Omega^*(M)$: (1) exterior multiply A with ω; (2) contract two indices of A with ω^{-1} (since non-degeneracy implies ω is invertible when considered as a $2n \times 2n$ anti-symmetric matrix); (3) count the degree of A by multiplying by a degree-dependent constant. Explicitly, the operations are

$$\begin{aligned} L: &\ A \to \omega \wedge A \\ \Lambda: &\ A \to \frac{1}{2}(\omega^{-1})^{ij}\, i_{\partial_{x^i}} i_{\partial_{x^j}} A \\ H: &\ A \to (n-k)\, A \qquad for\ A \in \Omega^k(M) \end{aligned}$$

where $n = d/2$ is the half-dimension. Now what is somewhat remarkable is that these three simple operations (L, Λ, H) together generate the $sl(2)$ Lie algebra:

$$[H, \Lambda] = 2\Lambda\,; \quad [H, L] = -2L\,; \quad [\Lambda, L] = H\,.$$

Since there is an $sl(2)$ action on $\Omega^*(M)$, this means that differential forms can naturally decompose into finite-dimensional irreducible representation modules of $sl(2)$. In fact, Lefschetz decomposition is just an $sl(2)$ decomposition on forms in the presence of a non-degenerate two-form.

Let us construct the irreducible modules of $sl(2)$. Typically, this starts from the highest weight vector, or in our case the highest weight differential form. These are called primitive forms, whose space we shall label by $\mathcal{P}^*(M)$.

Definition 2.1 (**Primitive Differential Forms**). B_s is a *primitive* differential s-form (i.e. $B_s \in \mathcal{P}^s(M)$) if $\Lambda\, B_s = 0$, or equivalently, $L^{n+1-s}\, B_s = 0$, where $0 \leq s \leq n$.

In other words, a primitive form $B \in \mathcal{P}^*(M)$ is one that vanishes when two of its tensor indices are contracted by $(\omega)^{-1}$. Hence, zero-forms and one-forms are trivially always primitive, as they do not have even two indices to contract. Furthermore, there are no primitive forms of degree $(n+1)$ or greater, since contraction by two indices would always be non-zero. (If there were, then the condition $L^{n+1-s}B_s = 0$ which is empty for $s = n+1$ would not be equivalent to $\Lambda B_s = 0$.) From a highest weight B_s, the entire irreducible module are generated by acting by powers of L's to B_s. Hence, an irreducible $sl(2)$ module consists of the elements

$$\{B_s\,,\ \omega \wedge B_s\,,\ \omega^2 \wedge B_s\,,\ \ldots\,,\ \omega^{n-s} \wedge B_s\}$$

noting that by definition, $L^{n+1-s}B_s = \omega^{n+1-s} \wedge B_s = 0$. Let us note that each element consists of ω raised to a power r exterior multiplied with a primitive s-form.

So we can label each element of the $sl(2)$ module by the pair (r, s):

$$\mathcal{L}^{r,s}(M) = \left\{ A \in \Omega^{2r+s}(M) \,\middle|\, A = \omega^r \wedge B_s \right\}$$

In a rough sense, $\mathcal{L}^{r,s}$ are the symplectic analogs of $\mathcal{A}^{p,q}$ of complex geometry. (The analogy here is not precise since each $\mathcal{A}^{p,q}$ is an irreducible module of $U(n)$ while each $\mathcal{L}^{r,s}$ is only an element of an $sl(2)$ module.)

As an example, let us give the Lefschetz decomposition of differential forms for a four-manifold, i.e. $d = 4$ $(n = 2)$.

$$\begin{aligned}
A_0 &= B_0 \\
A_1 &= \qquad\quad B_1 \\
A_2 &= \omega \wedge B_0' \qquad + B_2 \\
A_3 &= \qquad \omega \wedge B_1' \\
A_4 &= \omega^2 \wedge B_0''
\end{aligned}$$

If we replace each term above by the corresponding $\mathcal{L}^{r,s} = \{\omega^r \wedge B_s\}$ and then rotate the entire decomposition counterclockwise by 90°, we have the diagram

$$\begin{array}{ccccc}
 & & \mathcal{L}^{0,2} & & \\
 & \mathcal{L}^{0,1} & & \mathcal{L}^{1,1} & \\
\mathcal{L}^{0,0} & & \mathcal{L}^{1,0} & & \mathcal{L}^{2,0}
\end{array}$$

which is what we call the symplectic (r, s)-"pyramid" (in analogy with the (p, q)-diamond).

2.2. Symplectic decomposition of the exterior derivative. In the complex case, $d = \partial + \bar{\partial}$ follows from the fact that d acting on $\mathcal{A}^{p,q}$ results in only two components. Thus, let us consider the action of d on $\mathcal{L}^{r,s}$:

$$d\mathcal{L}^{r,s} = d\left(\omega^r \wedge B_s\right) = \omega^r \wedge dB_s \ .$$

We see that the derivative acts solely on the primitive form. In a sense, this suggests that much if not all of the data of differential forms on symplectic manifolds are encoded within primitive forms.

Now what about the derivative acting on primitive s-form? In general, for any arbitrary differential s-form, Lefschetz decomposition implies that

$$dA_s = B^0_{s+1} + \omega \wedge B^1_{s-1} + \omega^2 \wedge B^2_{s-3} + \omega^3 \wedge B^3_{s-5} + \ldots \ . \tag{2.1}$$

where the upper index on the primitive forms on the right labels the associated power of ω. But in fact, if A_s is actually a primitive form, there can only be at most two non-zero terms on the right. Importantly, we have

$$dB_s = B^0_{s+1} + \omega \wedge B^1_{s-1} \tag{2.2}$$

This formula is known in the literature (see for example [**10**]) and also can be easily shown by exterior multiplying both sides of (2.1) by L^{n+1-s} and applying the primitive condition $L^{n+1-s}B_s = 0$ to each term. With (2.2), we therefore find

$$\begin{aligned}
d\mathcal{L}^{r,s} &= d\left(\omega^r \wedge B_s\right) = \omega^r \wedge dB_s \\
&= \omega^r \wedge B^0_{s+1} + \omega^{r+1} \wedge B^1_{s-1}
\end{aligned}$$

and thus,

$$d\mathcal{L}^{r,s} \ \rightarrow \ \mathcal{L}^{r,s+1} \ \oplus \ \omega \wedge \mathcal{L}^{r,s-1}$$

Importantly, we have arrived at the observation that $d\mathcal{L}^{r,s}$ has only two components! Projecting onto each component then, we can express the exterior derivative as

$$d = \partial_+ + \omega \wedge \partial_- \tag{2.3}$$

where the first-order differential operators (∂_+, ∂_-) are defined by the derivative mapping

$$\partial_\pm : \mathcal{L}^{r,s} \longrightarrow \mathcal{L}^{r,s\pm 1}$$
$$\partial_\pm : \mathcal{P}^s \longrightarrow \mathcal{P}^{s\pm 1} \qquad for\ r = 0$$

Notice that ∂_+ and ∂_-, respectively, raises and decreases the degree of the forms by one. Moreover, we see that (∂_+, ∂_-) are operators that maps primitive forms to primitive forms (in the case of $r = 0$).

With the exterior derivative symplectically decomposed to a sum of two operators only, we have from $d^2 = 0$, the desirable properties:

(1) $(\partial_+)^2 = (\partial_-)^2 = 0\,;$

(2) $L \wedge (\partial_+\partial_-) = L \wedge (-\partial_-\partial_+)\,;$

(3) $[\partial_+, L] = [L\,\partial_-, L] = 0\,.$

For (2),(3), when acting on $\mathcal{L}^{r,s}$ with $r + s < n$, they reduce to the relations

$$\partial_+\partial_- = -\partial_-\partial_+ \qquad and \qquad [\partial_\pm, L] = 0.$$

Hence, effectively (∂_+, ∂_-) are the symplectic analogues of $(\partial, \bar{\partial})$.

2.3. The symplectic adjoint operator. Let us also point out that besides (∂_+, ∂_-), there exists another useful first-order differential operator on symplectic manifolds, denoted by d^Λ. It can be defined simply as

$$d^\Lambda = d\,\Lambda - \Lambda\,d\ .$$

This operator acts like an adjoint operator, with $d^\Lambda : \Omega^k \to \Omega^{k-1}$, and in fact has an interpretation as the symplectic adjoint. It was actually first introduced by Ehresmann and Libermann [**5, 12**] in the late 1940s in their attempt to introduce a symplectic Hodge theory. In the late 1980s, the notion of symplectic Hodge theory was revived by Brylinski [**3**]. However, it is clear from the work of [**14**] and [**22**] that a Hodge theory defined using d and the adjoint d^Λ is rather problematic (see [**18**] for a discussion).

For our purpose, it is useful to point out another interpretation of the d^Λ operator. It is analogous to the operator $d^c = i(\bar{\partial} - \partial)$ of complex geometry from the Hitchin's generalized geometry point of view [**7, 4**]. Indeed, just like the d^c operator, d^Λ has the properties

$$(d^\Lambda)^2 = 0\ , \qquad dd^\Lambda = -d^\Lambda d\ ,$$

and can also be expressed as a linear combination of ∂_+ and ∂_- [**19**].

In summary, we gather the natural first-order differential operators on a symplectic manifold in Table 2.

	Symplectic (M, ω)	Complex (M, J)
Differential forms	$\mathcal{L}^{r,s}$	$\mathcal{A}^{p,q}$
Exterior derivative	$d = \partial_+ + \omega \wedge \partial_-$	$d = \partial + \bar{\partial}$
First-order operators	(∂_+, ∂_-)	$(\partial, \bar{\partial})$
	(d, d^Λ)	(d, d^c)

TABLE 2. The decomposition of forms and exterior derivative and first-order differential operators on a symplectic manifold, and also their complex analogues.

3. Constructing finite symplectic cohomologies

With first-order symplectic differential operators at hand, we can now easily construct a number of symplectic cohomologies. To begin, by analogy with the complex Dolbeault cohomology,

$$H^{p,q}_{\bar{\partial}}(M) = \frac{\ker \bar{\partial} \cap \mathcal{A}^{p,q}(M)}{\mathrm{im}\ \bar{\partial} \cap \mathcal{A}^{p,q}(M)} ,$$

it is natural to consider the following primitive cohomology for either ∂_+ or ∂_-:

$$PH^s_{\partial_\pm}(M) = \frac{\ker \partial_\pm \cap \mathcal{P}^s(M)}{\mathrm{im}\ \partial_\pm \cap \mathcal{P}^s(M)} \qquad \text{for } \ 0 \leq s < n \ .$$

Here, we have focused on the primitive subspace as we have seen that the derivative operator effectively acts only non-trivially on the primitive component. Nevertheless, we can easily consider the cohomology on the space of all differential forms and this is presented in detail in [**19**]. Let us also point out that the cohomologies $PH^s_{\partial_\pm}(M)$ are not well-defined for degree $s = n$. For since there are no primitive $(n{+}1)$-forms, all primitive n-forms are in the kernel of ∂_+, and likewise, (im $\partial_- \cap \mathcal{P}^s$) is an empty set.

Having written down $PH^s_{\partial_\pm}(M)$, the pressing question is whether they are finite-dimensional? In [**19**], this is demonstrated by constructing a symplectic elliptic complex. For if a differential complex is elliptic, then the cohomologies associated with the complex must be finite-dimenesional (see for example [**21**]). The elliptic complexes associated with de Rham and Dolbeault cohomologies are well-known. But can we construct a symplectic elliptic complex for $PH^s_{\partial_\pm}(M)$?

Since both ∂_+ and ∂_- square to zero, we can certainly write down the following two differential complexes:

$$0 \longrightarrow \mathcal{P}^0 \xrightarrow{\partial_+} \mathcal{P}^1 \xrightarrow{\partial_+} \ldots \xrightarrow{\partial_+} \mathcal{P}^{n-1} \xrightarrow{\partial_+} \mathcal{P}^n$$

$$0 \xleftarrow{\partial_-} \mathcal{P}^0 \xleftarrow{\partial_-} \mathcal{P}^1 \xleftarrow{\partial_-} \ldots \xleftarrow{\partial_-} \mathcal{P}^{n-1} \xleftarrow{\partial_-} \mathcal{P}^n$$

Checking for ellipticity (that is, the associated symbol sequence is exact), one quickly finds that ellipticity breaks down at $\mathcal{P}^n$ for both differential complexes above. Indeed, the two differential complexes just abruptly end at $\mathcal{P}^n$ since there are no degree $(n+1)$ primitive forms. So perhaps one should consider connecting the two since the first with ∂_+ have increasing degrees while the second with ∂_- have decreasing degrees. A simple way to connect the two complexes is just to identify the two $\mathcal{P}^n$'s which gives a complex which in the middle takes the form

$$\dots \xrightarrow{\partial_+} \mathcal{P}^{n-1} \xrightarrow{\partial_+} \mathcal{P}^n \xrightarrow{\partial_-} \mathcal{P}^{n-1} \xrightarrow{\partial_-} \dots$$

But unfortunately, this breaks the requirement of a differential complex since $\partial_-\partial_+$ (just like $\partial\bar{\partial}$) is not identically zero.

Thus one needs a more ingenious way of connecting the two complexes. Indeed a solution can be found if one is willing to make a leap and allow for second-order differential operators. In particular, notice that $\partial_+\partial_-$ preserves degree, in particular, $\partial_+\partial_- : \mathcal{P}^n \to \mathcal{P}^n$. Making use of $\partial_+\partial_-$ leads to the following proposition:

Proposition 3.1. *The following differential complex is elliptic.*

$$\begin{array}{ccccccccccc}
0 & \xrightarrow{\partial_+} & \mathcal{P}^0 & \xrightarrow{\partial_+} & \mathcal{P}^1 & \xrightarrow{\partial_+} & \dots & \xrightarrow{\partial_+} & \mathcal{P}^{n-1} & \xrightarrow{\partial_+} & \mathcal{P}^n \\
 & & & & & & & & & & \downarrow{\scriptstyle \partial_+\partial_-} \\
0 & \xleftarrow{\partial_-} & \mathcal{P}^0 & \xleftarrow{\partial_-} & \mathcal{P}^1 & \xleftarrow{\partial_-} & \dots & \xleftarrow{\partial_-} & \mathcal{P}^{n-1} & \xleftarrow{\partial_-} & \mathcal{P}^n
\end{array}$$

With an elliptic complex, it now follows that we have the desired finite-ness.

Corollary 3.2. $\dim PH^s_{\partial_\pm}(M) < \infty$, *for* $0 \le s < n$.

But as a bonus, we have actually found two additional finite-dimensional cohomologies by the inclusion of the $\partial_+\partial_-$ operator:

Corollary 3.3. *The two cohomologies*

$$PH^n_{\partial_+\partial_-}(M) = \frac{\ker \partial_+\partial_- \cap \mathcal{P}^n(M)}{\text{im } \partial_+ \cap \mathcal{P}^n(M)}$$

$$PH^n_{\partial_+ + \partial_-}(M) = \frac{\ker \partial_- \cap \mathcal{P}^n(M)}{\text{im } \partial_+\partial_- \cap \mathcal{P}^n(M)}$$

are also finite dimensional.

That two middle-degree cohomologies appeared in the elliptic complex is very satisfying since it would not be esthetically appealing if we were not able to define any cohomology for middle-degree primitive forms. It turns out that we can actually define $PH^s_{\partial_+\partial_-}(M)$ and $PH^s_{\partial_+ + \partial_-}(M)$ for all degree s, $0 \le s \le n$, and they are all finite-dimensional [**18**]. We list the finite-dimensional cohomologies and their complex analogues in Table 3

4. Properties of symplectic cohomologies

Having found new finite-dimensional symplectic cohomologies, let us now explore some of their properties on compact manifolds. For simplicity, we will here mainly mention those of $PH^s_{\partial_\pm}(M)$ as those of $PH^s_{\partial_+\partial_-}(M)$ and $PH^s_{\partial_+ + \partial_-}(M)$ are rather similar.

Symplectic (M,ω)	Complex (M,J)
$PH^s_{\partial_\pm} = \dfrac{\ker \partial_\pm \cap \mathcal{P}^s}{\text{im } \partial_\pm \cap \mathcal{P}^s}\,, \qquad s<n$	$H^{p,q}_{\bar\partial} = \dfrac{\ker \bar\partial \cap \mathcal{A}^{p,q}}{\text{im } \bar\partial \cap \mathcal{A}^{p,q}}$
$PH^s_{\partial_+\partial_-} = \dfrac{\ker \partial_+\partial_- \cap \mathcal{P}^s}{(\text{im } \partial_+ + \text{im } \partial_-) \cap \mathcal{P}^s}\,, \; s\leq n$	$H^{p,q}_{\partial\bar\partial} = \dfrac{\ker \partial\bar\partial \cap \mathcal{A}^{p,q}}{(\text{im } \partial + \text{im } \bar\partial) \cap \mathcal{A}^{p,q}}$
$PH^s_{\partial_++\partial_-} = \dfrac{\ker d \cap \mathcal{P}^s}{\text{im } \partial_+\partial_- \cap \mathcal{P}^s}\,, \qquad s\leq n$	$H^{p,q}_{\partial+\bar\partial} = \dfrac{\ker d \cap \mathcal{A}^{p,q}}{\text{im } \partial\bar\partial \cap \mathcal{A}^{p,q}}$

Table 3. Finite-dimensional primitive symplectic cohomologies and their complex analogs. Besides the Dolbeault cohomology, the two other complex cohomologies were introduced in the mid-1960s by Aeppli [**1**] and Bott-Chern [**2**].

4.1. Associated elliptic Laplacians. Just like for de Rham and Dolbeault cohomologies, we can write down the Laplacians associated with the symplectic cohomologies. Now on any symplectic manifold, there is a compatible triple of symplectic form, almost complex structure and Riemannian metric - (ω, J, g). We will make use of this compatible metric, g, to define the standard inner product

$$(A, A') = \int_M A \wedge * A' = \int_M g(A, A')\, d\text{vol}\,, \qquad A, A' \in \Omega^k(M)\ .$$

The adjoints $(\partial_+^*, \partial_-^*)$ are then defined with respect to this inner product. We thus have the second-order $\partial_\pm$-Laplacians

$$\Delta_{\partial_\pm} = \partial_\pm(\partial_\pm)^* + (\partial_\pm)^*\partial_\pm\ .$$

Now since $\Delta_{\partial_\pm}$ are associated with the elliptic complex in Proposition 3.1, they must be elliptic operators. Thus, by standard elliptic theory, $PH^s_{\partial_\pm}(M)$ exhibits Hodge theoretic properties. In particular, we will say that a primitive form $B_s \in \mathcal{P}^s(M)$ is $\partial_\pm$-harmonic if it satisfies

$$\partial_\pm B_s = 0\ , \qquad (\partial_\pm)^* B_s = 0\ .$$

Hodge theory then implies there exists a unique harmonic representative in each cohomology class of $PH^s_{\partial_\pm}(M)$.

Let us just add that the Laplacians for $PH^s_{\partial_+\partial_-}(M)$ and $PH^s_{\partial_++\partial_-}(M)$ can also be easily written down. They are however fourth-order differential operators. Nevertheless, harmonic forms can be similarly defined and Hodge theory also do apply [**18**].

4.2. Isomorphisms between cohomologies. We have argued that (∂_+, ∂_-) are analogous to $(\partial, \bar\partial)$. But then we should expect that $PH^s_{\partial_+}(M)$ is isomorphic to $PH^s_{\partial_-}(M)$, since Dolbeault cohomology defined using either ∂ or $\bar\partial$ are isomorphic by complex conjugation. Indeed, we can show that $PH^s_{\partial_+}(M) \cong PH^s_{\partial_-}(M)$ and interestingly, we will need to make use of a compatible almost complex structure.

Let us define the operator

$$\mathcal{J} = \sum_{p,q} (\sqrt{-1}\,)^{p-q}\ \Pi^{p,q}$$

where $\Pi^{p,q} : \Omega^* \to \mathcal{A}^{p,q}$. Here, $\mathcal{A}^{p,q}$ is defined with respect to a compatible almost complex structure. In complex geometry, $\mathcal{J}$ is used to define d^c, i.e. $d^c = -i(\partial - \bar{\partial}) = \mathcal{J}^{-1} d \mathcal{J}$. Interestingly, it can be shown, with respect to a compatible (ω, J, g), that

$$\begin{aligned} \mathcal{J}\,\partial_+\,\mathcal{J}^{-1} &= c_1\,\partial_-^*, \\ \mathcal{J}\,\partial_+^*\,\mathcal{J}^{-1} &= c_2\,\partial_-\ , \end{aligned}$$

where c_1, c_2 are non-zero constants [**19**]. This immediately implies that the space of ∂_+-harmonic forms are isomorphic to that of ∂_--harmonic forms. Hence, by Hodge theory, we have shown the isomorphism of $PH^s_{\partial_+}(M)$ with $PH^s_{\partial_-}(M)$. Similarly, by comparing the harmonic forms of $PH^s_{\partial_+\partial_-}(M)$ and $PH^s_{\partial_++\partial_-}(M)$, one can also show that these two cohomologies are also isomorphic [**19**].

4.3. Comparing $PH^s_{\partial_+}(M)$ with $H_d(M)$. Let us now ask whether the new symplectic cohomologies encode more information for a compact symplectic manifold then that already given by the de Rham cohomology, $H_d(M)$? Let us focus for simplicity on $PH^s_{\partial_+}(M)$. To start, recall that all zero-forms and one-forms are primitive, so it would not be surprising if for $s = 0, 1$, $PH^s_{\partial_+}(M)$ is related to the de Rham cohomology. In fact we have the following relations [**19**]:

Proposition 4.1. On a compact symplectic manifold, (M, ω)

(1) $PH^s_{\partial_+}(M) = H^s_d(M)$ *for* $s = 0, 1$;

(2) *For* $s > 1$,

$$PH^s_{\partial_+}(M) \cong H^s_d(M) \cap \mathcal{P}^s(M) = \frac{\ker d \cap \mathcal{P}^s}{d\Omega^{s-1} \cap \mathcal{P}^s(M)}$$

if the hard Lefschetz property holds, or equivalently the $\partial_+\partial_-$*-lemma holds.*

Hence, for $PH^s_{\partial_+}(M)$, $s = 0, 1$ does not tell us anything new. We need at least $s \geq 2$, which means $n \geq 3$ or dimension $d \geq 6$. This is because $PH^s_{\partial_+}(M)$ is not defined for $s = n$. Therefore, in order to have new interesting $d = 4$ invariants, we need to turn to the cohomologies $PH_{\partial_++\partial_-}(M)$ and $PH_{\partial_+\partial_-}(M)$ which are well-defined for $s = n$.

The second part of Proposition 4.1 states that for $s \geq 2$, $PH^s_{\partial_+}(M)$ is just a subset of the de Rham cohomology if the hard Lefschetz property holds. The hard Lefschetz property, a characteristic of Kähler geometry, is said to hold if the following map is an isomorphism for all $k \leq n$:

$$\begin{aligned} \varphi : H^k_d(M) &\longrightarrow\ H^{2n-k}_d(M) \\ A_k &\longrightarrow\ [w]^{n-k} \wedge A_k \end{aligned}$$

By the work of Merkulov [**15**] and Guillemin [**8**], the hard Lefschetz property is equivalent to the existence of $\partial_+\partial_-$-lemma on M. This $\partial_+\partial_-$-lemma is just the symplectic analogue of the $\partial\bar{\partial}$-lemma in complex geometry. All in all, we see that the cohomology $PH_{\partial_+}(M)$ may give new invariants for those non-Kähler symplectic manifolds that do not satisfy the hard Lefschetz property. This statement in fact applies also to the other symplectic cohomologies.

5. Example: Kodaira-Thurston four-fold KT^4

To get a taste of the new invariants that the symplectic cohomologies encode, let us analyze a specific non-Kähler symplectic manifold. The simplest one to consider is the Kodaira-Thurston four-fold [**11, 16**], which we will denote by KT^4. It can be described as a torus bundle over a torus,

$$\begin{array}{ccc} S^1_{x_1} \times S^1_{x_4} & \longrightarrow & KT^4 \\ & & \downarrow \\ & & T^2_{\{x_2,x_3\}} \end{array}$$

with global cotangent one-forms

$$e_1 = dx_1 \ , e_2 = dx_2 \ , e_3 = dx_3 \ , e_4 = dx_4 + x_2 dx_3 \ .$$

We can take as the symplectic form

$$\omega = e_1 \wedge e_2 + e_3 \wedge e_4$$

and clearly $d\omega = 0$ since $de_4 = e_2 \wedge e_3$.

Calculating the cohomology $PH^2_{\partial_+ + \partial_-}$, we find as basis

$$PH^2_{\partial_+ + \partial_-}(KT^4) = \left\{ H^2_d \cap P^2(KT^4),\ e_2 \wedge e_3 \right\}$$

consisting of the primitive elements of de Rham H^2_d and additionally $e_2 \wedge e_3$, which is d-exact but not $\partial_+\partial_-$-exact. To get a feel of what the element $e_2 \wedge e_3$ represents, let us consider the two-dimensional submanifold L located at a point on the base and wraps the fiber T^2 which spans the coordinates $\{x_1, x_4\}$. Clearly, L is a Lagrangian submanifold since $\omega\big|_L = 0$. But moreover, we can introduce a decomposable $(2,0)$-form

$$\Omega^{2,0} = (e_1 + \sqrt{-1}\, e_2) \wedge (e_3 + \sqrt{-1}\, e_4) \ ,$$

which gives us an almost complex structure on KT^4. But then, we find

$$\mathrm{Re}\,\Omega^{2,0}\big|_L = 0 \,, \qquad \mathrm{Im}\,\Omega^{2,0}\big|_L = vol(L) \,.$$

so in fact L is a special Lagrangian submanifold.

Now we can write the Poincaré dual current of L (situated at $x_2 = x_3 = 0$ on the base) as

$$\rho_L = \delta(x_2, x_3)\, e_2 \wedge e_3 \ .$$

Notice first that ρ_L is a primitive current. In fact, it can be shown that for a symplectic manifold, the Poincaré dual current of a submanifold is primitive if and only if the submanifold is lagrangian or more generally co-isotropic [**18**]. Now, ρ_L also involves $e_2 \wedge e_3$ and is trivial in $H^2_d(KT^4)$ but non-trivial in $PH^2_{\partial_+ + \partial_-}(KT^4)$. So what we find is that a special Lagrangian current, though trivial in de Rham cohomology can interestingly be non-trivial in the$PH^n_{\partial_+ + \partial_-}$ cohomology. And more generally, the primitive symplectic cohomologies can provide interesting data for the co-isotropic submanifolds of non-Kähler symplectic manifolds.

6. Application: string theory

As an application, we will show that the primitive cohomology $PH^s_{\partial_+ + \partial_-}$ arises when considering the deformation space of a system of differential equations in type IIA string theory [**20**]. The differential system of interest describes the geometry of a six-dimensional symplectic manifold (X^6, ω, Ω, g) where Ω here is a globally defined (3,0)-form, and hence the manifold has an $SU(3)$ structure. From physical requirements of preserving supersymmetry [**6, 17**], (ω, Ω) not only satisfy the algebraic $SU(3)$ condition

$$\omega \wedge \Omega = 0 \ , \qquad \sqrt{-1}\,\Omega \wedge \bar{\Omega} = e^{2f}\,\frac{\omega^3}{3!} \ ,$$

but also the differential conditions

$$\begin{aligned} d\,\omega &= 0\,, \\ d\,\mathrm{Re}\,\Omega &= 0\,, \\ (\partial_+\partial_-)^*\,\mathrm{Re}\,\Omega &= e^{-2f} * \rho_{SL} \ . \end{aligned}$$

where in above f is a function (or more precisely a distribution) and ρ_{SL} is the Poincaré dual current of special Lagrangian submanifolds. We are interested in parametrizing the local moduli space of solutions. In general, this is a very difficult problem, but it turns out that a subspace of the local deformation space is related to $PH^3_{\partial_+ + \partial_-}(X^6)$.

To motivate the appearance of the symplectic cohomology, consider the Maxwell equations on (M^4, g). They constrain the curvature two-form, F_2, of a principal $U(1)$ bundle to satisfy

$$\begin{aligned} dF_2 &= 0\,, \\ d^*F_2 &= *\rho_e\,, \end{aligned}$$

where ρ_e is the Poincaré dual of some electrical charge configuration. Now, it is easy to see that Maxwell's equations are related to the de Rham cohomology. For if we consider the deformation space varying, $F_2 \to F_2 + \delta F_2$, while keeping the source fixed, $\delta\rho = 0$, then δF_2 satisfies the de Rham harmonic condition $d(\delta F_2) = d^*(\delta F_2) = 0$. Hence, $\delta F_2 \in \mathcal{H}^2_d(M)$, the degree two de Rham harmonic space.

Now for the $SU(3)$ differential system of type IIA string theory, we can consider the subspace of deformations varying only $\Omega \to \Omega + \delta\Omega$ while keeping fixed (ω, ρ_{SL}, f). By the algebraic $SU(3)$ condition above, $\delta f = 0$ implies $(\delta\Omega)^{3,0} = 0$. With these conditions, we find that $\mathrm{Re}\,(\delta\Omega)^{2,1}$ satisfies

$$d\,\mathrm{Re}\,(\delta\Omega)^{2,1} = 0 \ , \qquad (\partial_+\partial_-)^*\mathrm{Re}\,(\delta\Omega)^{2,1} = 0 \ ,$$

which is the harmonic condition for $PH_{\partial_+ + \partial_-}$ cohomology. Thus, we find that a subspace of the local moduli space of solution is given by

$$\mathrm{Re}\,\delta\Omega \in P\mathcal{H}^3_{\partial_+ + \partial_-} \cap \mathrm{Re}\,\mathcal{A}^{2,1}(X^6) \ .$$

7. Concluding remarks

The primitive symplectic cohomologies discussed here were built from the linear differential operators (∂_+, ∂_-). Being first-order operators, they are the building blocks to construct higher-order elliptic symplectic differential operators. Though new, (∂_+, ∂_-) are not so foreign, as they act for the most part just like their complex counterparts, $(\partial, \bar{\partial})$.

We have also seen that symplectic cohomologies can give new invariants for non-Kähler symplectic manifolds. Now the Kähler condition can be heuristically thought of as a "symmetry" constraint for symplectic manifolds. And whenever a symmetry constraint is relaxed, the solution space is often enlarged allowing for more diverse solutions with interesting new characteristics. In a similar vein, we expect that non-Kähler symplectic manifolds generically will have many new properties that do not appear in Kähler symplectic manifolds. Today, there are now abundant examples of non-Kähler symplectic manifolds. But as a class of manifolds, they are still rather mysterious especially for those of dimensions six and higher. It is the hope that the linear symplectic differential operators and the new cohomologies introduced here will provide some of the analytical tools needed to uncover their properties.

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