Cohomologies on Symplectic Manifolds

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Abstract. We discuss some recently discovered cohomologies of differential forms on symplectic manifolds. We show that these new cohomologies encode the data of the kernels and cokernels of the Lefschetz maps and are also associated with some novel symplectic elliptic complexes.

This article is an expanded version of the talk given at ICCM 2013, Taipei, where we reported on some new results of the cohomologies of differential forms on symplectic manifolds based on a joint work with C.-J. Tsai and S.-T. Yau [**6**].

1. Introduction

A symplectic manifold (M, ω) is an even dimensional manifold $(d = 2n)$ equipped with a differential two-form, ω , that is both non-degenerate (i.e. $\omega^n > 0$) and d-closed. By the celebrated Darboux's theorem, symplectic manifolds are locally diffeomorphic to $\mathbb{R}^{2n}$. Thus, to characterize and distinguish symplectic manifolds, we must study invariants that are global. In general, it is a challenging task to find useful global invariants that are also calculable.

In this article, we describe some new symplectic invariants arising from cohomologies of differential forms, $\Omega(M)$. Of course, the most well-known cohomology of forms on smooth manifolds is the de Rham cohomology:

$$H^k_d(M) = \frac{\ker d \cap \Omega^k(M)}{\text{im } d \cap \Omega^k(M)} \ .$$

It provides some of the most basic invariants on manifolds and the dimensions of this cohomology are the well-known Betti numbers. However, the invariants associated with the de Rham cohomology are topological and they do not at all care about the presence of the symplectic form, ω. Hence, we can ask what other cohomologies of differential forms are there on (M, ω)? Perhaps a bit of a surprise, the answer as we know now is that there are quite a few!

To motivate their construction and gain insights into their properties, it is worthwhile to return to the de Rham cohomology and try to introduce the symplectic form into the picture. To do so, we recall that one of the advantages of the de Rham cohomology versus its dual homology (e.g. singular homology) is that the de Rham cohomology $H^*_d(M)$ has a ring structure with the product given by the exterior wedge product. Now on a symplectic manifold, there is a most distinguished set of elements of $H^*_d(M)$ consisting of ω and its higher powers $\{\omega^2, \ldots, \omega^n\}$. Thus, it is natural to focus on the product involving these distinguished elements, ω^r, for $r = 1, \ldots, n$, with other elements of the de Rham cohomology, i.e. $\omega^r \otimes H^*_d(M)$. Such a product by ω^r can be considered as a map, taking an element of $H^k_d(M)$ into an element of $H^{k+2r}_d(M)$. This action is often referred to as the Lefschetz map (of degree r):

$$\begin{aligned} L^r : H^k_d(M) \ &\rightarrow \ H^{k+2r}_d(M) \ , \\ [A_k] \qquad &\rightarrow \ [\omega^r \wedge A_k] \ , \end{aligned}$$

where $[A_k] \in H^k_d(M)$.

Clearly, Lefschetz maps are linear and only depend on the cohomology class of $[\omega^r] \in H^{2r}_d(M)$. But of particular interest, Lefschetz maps are in general neither injective nor surjective. For instance, if (M^{2n}, ω) is closed and connected, then the degree one Lefschetz map, $L : H^0_d(M) \rightarrow H^2_d(M)$, by simple dimension counting can not be onto as long as the second Betti number $b_2 > 1$. Similarly, the map, $L : H^{2n-2}_d(M) \rightarrow H^{2n}_d(M)$, can not be one-to-one if $b_2 = b_{2n-2} > 1$. But non-trivially, even for the map $L : H^{n-1}_d \rightarrow H^{n+1}_d$ where the dimensions do match up via Poincaré duality, i.e. $b_{n-1} = b_{n+1}$, injectivity and surjectivity can also fail in general if the manifold M is symplectic but non-Kähler. Hence, a most basic question one can ask for any symplectic manifold in general is what are the kernels and cokernels of the Lefschetz maps?

It turns out that the data of the kernels and cokernels of the Lefschetz maps are encoded precisely by the symplectic cohomologies of differential forms. For Lefschetz maps of degree $r = 1$, the cohomologies are the primitive cohomologies, $PH^*(M)$, introduced in joint works with S.-T. Yau in [**4, 5**]. And for Lefschetz maps of $r > 1$, they are what are called the filtered cohomologies $F^pH^*(M)$, recently introduced in a joint work with C.-J. Tsai and S.-T. Yau [**6**].

Below, we shall make clear how the data of the Lefschetz maps are encoded within cohomologies which are intrinsically symplectic. To do so, we will begin first by review the decomposition of differential forms $\Omega(M)$ in the presence of a non-degenerate two-form, ω, known as the Lefschetz decomposition. Then we shall proceed to discuss how Lefschetz maps act on $\Omega(M)$. In fact, we will be able to write short exact sequences which encode the Lefschetz action. And finally, these short exact of forms will result in a long exact sequence of cohomology which naturally introduces the symplectic cohomologies of differential forms. For simplicity, we will primarily focus in this article on the degree $r = 1$ Lefschetz map case. The method we introduce here is however general and can be easily extended to arbitrary degree r as we shall note in the concluding remarks.

2. Lefschetz decomposition on $\Omega^*(M)$

The presence of a non-degenerate two-form, ω, leads to an $\mathfrak{sl}(2)$ action on $\Omega^k(M)$. In fact, on (M^{2n}, ω), we can write down three natural operators on $\Omega^k(M)$:

$$\begin{aligned} L: \ & A \to \omega \wedge A \ , \\ \Lambda: \ & A \to \frac{1}{2}(\omega^{-1})^{ij}\, i_{\partial_{x^i}} i_{\partial_{x^j}} A \ , \\ H: \ & A \to (n-k)\, A \qquad for \ A \in \Omega^k(M) \ . \end{aligned}$$

where L is simply exterior multiplying A by ω, Λ is the contraction by the Poisson bivector field $\frac{1}{2}(\omega^{-1})^{ij}\, i_{\partial_{x^i}} i_{\partial_{x^j}}$, and H counts the degree of A by multiplying it by the normalized constant $(n-k)$. The three operators (L, Λ, H) generate the $\mathfrak{sl}(2)$ Lie algebra:

$$[H, \Lambda] = 2\Lambda\,; \quad [H, L] = -2L\,; \quad [\Lambda, L] = H\,.$$

Hence, $\Omega(M)$ decomposes into irreducible representations of $\mathfrak{sl}(2)$. Such a decomposition is typically called the Lefschetz decomposition.

As is standard, the description of the irreducible representation modules of $\mathfrak{sl}(2)$ starts with the highest weight vector, or here the highest weight differential form called the primitive form.

Definition 2.1 (**Primitive Differential Forms**). B_s is a *primitive* differential s-form (i.e. $B_s \in \mathcal{P}^s(M)$) if

$$\Lambda\, B_s = 0 \quad \Longleftrightarrow \quad L^{n-s+1}\, B_s = 0 \ ,$$

where $0 \leq s \leq n\,$.

From a highest weight primitive form B_s, the entire irreducible module is generated by acting by powers of L's to B_s. Hence, an irreducible $sl(2)$ module consists of the elements

$$\left\{ B_s\,,\ \omega \wedge B_s\,,\ \omega^2 \wedge B_s\,,\ \dots\,,\ \omega^{n-s} \wedge B_s \right\}$$

noting that by definition, $L^{n+1-s} B_s = \omega^{n+1-s} \wedge B_s = 0\,$. Let us note that each element consists of ω raised to a power r exterior multiplied with a primitive s-form. So each element of the $sl(2)$ module can be labelled by the pair of indices (r, s):

$$\mathcal{L}^{r,s}(M) = \left\{ A \in \Omega^{2r+s}(M) \,\middle|\, A = \omega^r \wedge B_s \right\}$$

The decomposed elements can be arranged into we called an (r, s) pyramid diagram, as exhibited for the dimension $d = 6$ case in Figure 1.

3. Lefschetz maps on $\Omega^*(M)$

Let us now consider the action of Lefschetz maps on differential forms. Notice that the degree one Lefschetz map, $L = \omega \wedge$, is one of the three $\mathfrak{sl}(2)$ generators that acts naturally on $\Omega(M)$. It is thus not surprising that we can neatly re-package the information of the Lefschetz decomposition in terms of a series of short exact

Ω^0	Ω^1	Ω^2	Ω^3	Ω^4	Ω^5	Ω^6
			$\mathcal{P}^3$			
		$\mathcal{P}^2$		$\omega \wedge \mathcal{P}^2$		
	$\mathcal{P}^1$		$\omega \wedge \mathcal{P}^1$		$\omega^2 \wedge \mathcal{P}^1$	
$\mathcal{P}^0$		$\omega \wedge \mathcal{P}^0$		$\omega^2 \wedge \mathcal{P}^0$		$\omega^3 \wedge \mathcal{P}^0$

FIGURE 1. The $\mathfrak{sl}(2)$ or Lefschetz decomposition of differential forms in dimension $d = 6$. The degree of the forms starts from zero (on the left) to $2n = 6$ (on the right).

sequences of differential forms involving Lefschetz maps. For instance, for $d = 6$ and $r = 1$, we can write the following short exact sequences based on Fig. 1.

$$0 \longrightarrow \mathcal{P}^2 \xrightarrow{*_r} \Omega^4 \xrightarrow{\omega\wedge} \Omega^6 \longrightarrow 0$$

$$0 \longrightarrow \mathcal{P}^3 \xrightarrow{*_r} \Omega^3 \xrightarrow{\omega\wedge} \Omega^5 \longrightarrow 0$$

$$0 \longrightarrow \Omega^2 \xrightarrow{\omega\wedge} \Omega^4 \longrightarrow 0$$

$$0 \longrightarrow \Omega^1 \xrightarrow{\omega\wedge} \Omega^3 \xrightarrow{\Pi^0} \mathcal{P}^3 \longrightarrow 0$$

$$0 \longrightarrow \Omega^0 \xrightarrow{\omega\wedge} \Omega^2 \xrightarrow{\Pi^0} \mathcal{P}^2 \longrightarrow 0$$

where $*_r$ is defined to be a reflection operator with respect to the central axis of the pyramid diagram of Fig. 1, that is, $*_r : \mathcal{L}^{r,s} \to \mathcal{L}^{n-r,s}$, or more specifically,

$$*_r : \omega^r \wedge B_s \to \omega^{n-r} \wedge B_s \ ,$$

and moreover, Π^0 is the projection operator,

$$\Pi^0 : \Omega^k(M) \to \mathcal{P}^k(M) \ .$$

To relate Lefschetz maps on forms to that on cohomologies, we can try to make the above exact sequences into a diagram of chain complexes. Notice first that the two middle columns would give the de Rham complex if we insert the exterior derivative, d, to relate the adjacent rows. What we need are derivative operators that maps between the primitive forms on the two outer columns. On the right column, the needed linear operator is $\partial_+ : \mathcal{L}^{r,s}(M) \to \mathcal{L}^{r,s+1}(M)$ which raises the degree of a primitive form by one (in the $s = 0$ case) and satisfies the required commutation condition,

$$\partial_+ \, \Pi^0 = \Pi^0 \, d \ .$$

On the left column, the linear derivative operator is $\partial_- : \mathcal{L}^{r,s}(M) \to \mathcal{L}^{r,s-1}(M)$ which lowers the degree of a primitive form by one (again in the case of $s = 0$) and satisfies the relation,

$$*_r \, \partial_- = d \, *_r \ .$$

In general, (∂_+, ∂_-) gives an intrinsically symplectic decomposition of the exterior derivative operator [**5**]

$$d = \partial_+ + \omega \wedge \partial_-$$

that holds true for any differential form.

With the three differential operators $(d, \partial_+, \partial_-)$, we can write down the following commutative diagram:

(3.1)
$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{P}^0 & \xrightarrow{*_r} & \Omega^6 & \longrightarrow & 0 & & & & \\
 & & {\scriptstyle \partial_-}\uparrow & & {\scriptstyle d}\uparrow & & & & & & \\
0 & \longrightarrow & \mathcal{P}^1 & \xrightarrow{*_r} & \Omega^5 & \longrightarrow & 0 & & & & \\
 & & {\scriptstyle \partial_-}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & & & \\
0 & \longrightarrow & \mathcal{P}^2 & \xrightarrow{*_r} & \Omega^4 & \xrightarrow{\omega\wedge} & \Omega^6 & \longrightarrow & 0 & & \\
 & & {\scriptstyle \partial_-}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & & & \\
0 & \longrightarrow & \mathcal{P}^3 & \xrightarrow{*_r} & \Omega^3 & \xrightarrow{\omega\wedge} & \Omega^5 & \longrightarrow & 0 & & \\
 & & & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & & & \\
 & & 0 & \longrightarrow & \Omega^2 & \xrightarrow{\omega\wedge} & \Omega^4 & \longrightarrow & 0 & & \\
 & & & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & & & \\
 & & 0 & \longrightarrow & \Omega^1 & \xrightarrow{\omega\wedge} & \Omega^3 & \xrightarrow{\Pi^0} & \mathcal{P}^3 & \longrightarrow & 0 \\
 & & & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle \partial_+}\uparrow & & \\
 & & 0 & \longrightarrow & \Omega^0 & \xrightarrow{\omega\wedge} & \Omega^2 & \xrightarrow{\Pi^0} & \mathcal{P}^2 & \longrightarrow & 0 \\
 & & & & & & {\scriptstyle d}\uparrow & & {\scriptstyle \partial_+}\uparrow & & \\
 & & & & 0 & \longrightarrow & \Omega^1 & \xrightarrow{\Pi^0} & \mathcal{P}^1 & \longrightarrow & 0 \\
 & & & & & & {\scriptstyle d}\uparrow & & {\scriptstyle \partial_+}\uparrow & & \\
 & & & & 0 & \longrightarrow & \Omega^0 & \xrightarrow{\Pi^0} & \mathcal{P}^0 & \longrightarrow & 0
\end{array}$$

Let us briefly remark on the commutative diagram for the case of degree r Lefschetz maps for $r \neq 1$. In this case, the special forms that arise are no longer the primitive forms, $\mathcal{P}^*(M)$, but instead are the $(r-1)$-*filtered* forms, $F^{r-1}\Omega^*(M)$ [**6**]. To define filtered forms, consider the Lefschetz decomposition of a differential k-form:

$$A_k = B_k + \omega \wedge B_{k-2} + \omega^2 \wedge B_{k-4} + \omega^3 \wedge B_{k-6} + \ldots \ ,$$

where the B's are the primitive forms. We have already introduced the projection operator Π^0 which projects to the primitive component. But we can also introduce a projection operator Π^p that keeps up to the ω^p-th term of the Lefschetz decomposition:

$$\begin{aligned}
\Pi^0 A_k &= B_k \ , \\
\Pi^1 A_k &= B_k + \omega \wedge B_{k-2} \ , \\
&\vdots \\
\Pi^p A_k &= B_k + \omega \wedge B_{k-2} + \omega^2 \wedge B_{k-4} + \ldots + \omega^p \wedge B_{k-2p} \ , \\
&\vdots
\end{aligned}$$

For a filtration degree p, the projection operator $\Pi^p : \Omega^*(M) \to F^p\Omega^*(M)$ is defined to project to the space of p-filtered forms, $F^p\Omega^*(M)$. Notice in particular that the primitive forms are just the special case of zero-filtered forms, i.e. $\mathcal{P}^* = F^0\Omega^*$. And we call $F^p\Omega^*(M)$ the space of p-filtered forms, since we have the natural filtration

$$P^k = F^0\Omega^k \subset F^1\Omega^k \subset F^2\Omega^k \subset \ldots \subset F^n\Omega^k = \Omega^k \ .$$

For the commutative diagram involving Lefschetz maps, we have seen that the degree $r = 1$ case involves zeroth filtered forms (i.e. the primitive forms) as in (3.1). More generally, for the degree r Lefschetz maps, the commutative diagram will require $(r-1)$-filtered forms, $F^{r-1}\Omega^*(M)$ [**6**].

4. Lefschetz maps on $H^*(M)$

In general, a short exact sequence of chain complex implies a long exact sequence of cohomologies. In the previous section, we have seen that we can write down commutative diagrams consisting of short exact sequences involving Lefschetz maps (e.g. (3.1)). Although they are not standard short exact sequence of chain complexes, nevertheless, we can still derive long exact sequences of cohomologies involving the Lefschetz maps from them [**6**]

For instance, the commutative diagram (3.1) involving the degree $r = 1$ Lefschetz maps in dimension $d = 6$ results in the long exact sequence of cohomologies [**2, 6**]

$$
\begin{array}{ccccc}
 & \longrightarrow H^5_d(M) & \xrightarrow{L} & 0 & \\
 & \longrightarrow H^4_d(M) & \xrightarrow{L} H^6_d(M) & \longrightarrow PH^1_{\partial_-}(M) & \\
 & \longrightarrow H^3_d(M) & \xrightarrow{L} H^5_d(M) & \longrightarrow PH^2_{\partial_-}(M) & \\
 & \longrightarrow H^2_d(M) & \xrightarrow{L} H^4_d(M) & \longrightarrow PH^3_{d+d^\Lambda}(M) & \\
 & \longrightarrow H^1_d(M) & \xrightarrow{L} H^3_d(M) & \longrightarrow PH^3_{dd^\Lambda}(M) & \\
 & \longrightarrow H^0_d(M) & \xrightarrow{L} H^2_d(M) & \longrightarrow PH^2_{\partial_+}(M) & \\
 & 0 & \longrightarrow H^1_d(M) & \longrightarrow PH^1_{\partial_+}(M) &
\end{array} \tag{4.1}
$$

where the new cohomologies are defined standardly from the commutative diagram to be [**4, 5**]

$$PH^s_{\partial_\pm}(M) = \frac{\ker \partial_\pm \cap \mathcal{P}^s(M)}{\text{im } \partial_\pm \cap \mathcal{P}^s(M)} \ , \qquad 0 < s < n \tag{4.2}$$

$$PH^n_{dd^\Lambda}(M) = \frac{\ker \partial_+\partial_- \cap \mathcal{P}^n(M)}{(\text{im } \partial_+ + \text{im } \partial_-) \cap \mathcal{P}^n(M)} \ , \tag{4.3}$$

$$PH^n_{d+d^\Lambda}(M) = \frac{\ker d \cap \mathcal{P}^n(M)}{\text{im } \partial_+\partial_- \cap \mathcal{P}^n(M)} \ , \tag{4.4}$$

here defined generally on (M^{2n}, ω). Note that these primitive cohomologies are cohomologies defined purely on the space of primitive forms, $\mathcal{P}^s(M)$, for $s = 0, 1, \ldots, n$. Clearly, the long exact sequence above implies generally that the primitive cohomologies are isomorphic to the direct sum of kernels and cokernels of Lefschetz maps of degree one. For example, from the long exact sequence (4.1), we have for $d = 6$ that

$$\begin{aligned} PH^2_{\partial_+}(M) &\cong \text{coker}[L : H^0_d(M) \to H^2_d(M)] \oplus \ker[L : H^1_d(M) \to H^3_d(M)] \ , \\ PH^3_{dd^\Lambda}(M) &\cong \text{coker}[L : H^1_d(M) \to H^3_d(M)] \oplus \ker[L : H^2_d(M) \to H^4_d(M)] \ . \end{aligned}$$

So we see that in the Lefschetz maps of degree one case, there is a two-sided resolution that involves precisely the primitive cohomologies

$$PH^*(M) = \{PH^*_{\partial_+}(M), PH^n_{d+d^\Lambda}(M), PH^n_{dd^\Lambda}(M), PH^*_{\partial_-}(M)\} \ . \tag{4.5}$$

These primitive cohomologies turn out to arise naturally in an elliptic complex [**3, 4, 1**]

$$\begin{array}{ccccccccccccc} 0 & \xrightarrow{\partial_+} & \mathcal{P}^0 & \xrightarrow{\partial_+} & \mathcal{P}^1 & \xrightarrow{\partial_+} & \ldots & \xrightarrow{\partial_+} & \mathcal{P}^{n-1} & \xrightarrow{\partial_+} & \mathcal{P}^n \\ &&&&&&&&&& \Big\downarrow{\scriptstyle \partial_+\partial_-} \\ 0 & \xleftarrow{\partial_-} & \mathcal{P}^0 & \xleftarrow{\partial_-} & \mathcal{P}^1 & \xleftarrow{\partial_-} & \ldots & \xleftarrow{\partial_-} & \mathcal{P}^{n-1} & \xleftarrow{\partial_-} & \mathcal{P}^n \end{array} \tag{4.6}$$

Thus, we can encapsulate the resolution of the degree one Lefschetz map simply in a triangle diagram of cohomologies [**6**]:

$$\begin{array}{ccc} & PH^*(M) & \\ \swarrow & & \nwarrow \\ H^*_d(M) & \xrightarrow{\quad L \quad} & H^*_d(M) \end{array} \tag{4.7}$$

5. Concluding remarks

Concerning the resolution of Lefschetz maps, L^r, for $r > 1$, one can proceed similarly as in the degree one case (e.g. (3.1)) and build up a commutative diagram from a series of short exact sequences involving L^r maps and p-filtered forms $F^p\Omega^*(M)$, with $p = r - 1$. And just like for primitive forms, there are filtered cohomologies that are grouped together by the filtration degree p, with cohomologies denoted by

$$F^pH^*(M) = \{F^pH^*_+(M), F^pH^*_-(M)\} \tag{5.1}$$

in analogy with (4.5). The p-filtered cohomologies can be associated with a p-filtered elliptic complex, and hence, they are all finite-dimensional [**6**]. And so generally, the resolution of the Lefschetz maps of any degree r can be expressed simply by the exact triangle [**6**]

$$\begin{array}{ccc} & F^{r-1}H^*(M) & \\ \swarrow & & \nwarrow \\ H^*_d(M) & \xrightarrow{\quad L^r \quad} & H^*_d(M) \end{array} \tag{5.2}$$

Thus, we see that the data of the kernels and cokernels of the Lefschetz maps are indeed encoded in symplectic cohomologies of differential forms!

Lastly, it is noteworthy that the symplectic cohomologies defined *differentially* by first- or second-order differential operators acting on forms (see e.g. (4.2)-(4.4)) are isomorphic to *algebraic* properties of the Lefschetz maps. In fact, the symplectic cohomologies themselves have good algebraic properties. In particular, the group of p-filtered cohomologies $F^pH^*(M)$ in (5.1) actually forms a ring [**6**]. That is, we can define a product operation on the space of p-filtered forms, $F^p\Omega^*(M)$, and show that the required Leibniz rules are satisfied for such a product. (For details, see [**6**].)

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