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Non-Kähler Calabi-Yau Manifolds

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Abstract. String theory has had a profound influence on research in Calabi-Yau spaces over the past twenty-five years. We first briefly mention some of the work in Kähler Calabi-Yau manifolds that was influenced by the discovery of mirror symmetry in the late 1980s. We then discuss some of the mathematical motivations behind the recent work on non-Kähler Calabi-Yau manifolds, which arise in string compactifications with fluxes. After extending mirror symmetry to non-Kähler Calabi-Yau manifolds, we show how this leads to new cohomologies and invariants of non-Kähler symplectic manifolds.

1. Introduction

String theory and mathematics have had a very close interaction over the past thirty years. Indeed, the interaction has been extremely fruitful and produced many beautiful results. As a prime example, mathematical research on Calabi-Yau spaces over the past two decades has been strongly motivated by string theory, and in particular, mirror symmetry.

Mirror symmetry started from the simple observation by Dixon [**17**] and Lerche-Vafa-Warner [**42**] around 1989, of a possible geometric realization of flipping the sign of a representation of the superconformal algebra. Geometrically, it implied that Calabi-Yaus should come in pairs with the pair of Hodge numbers, $h^{1,1}$ and $h^{2,1}$, exchanged. Shortly following this observation, Greene-Plesser [**33**] gave an explicit construction of the mirror of the Fermat quintic using an orbifold construction. And soon after, Candelas-de la Ossa-Green-Parkes [**13**] discovered as a consequence of mirror symmetry a most surprising formula for counting rational curves on a general quintic.

The identification of the topological A- and B-models by Witten [**59**] further inspired a lot of rigorous mathematical work needed to justify various definitions and relations, such as the Gromov-Witten invariants, multiple cover formula, and other related topics. More works by Witten [**60**], Kontsevich [**40, 41**] and many others led to the proofs, independently by Givental [**29**] and Lian-Liu-Yau [**46**], of

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the Candelas *et al.* formula for the genus zero Gromov-Witten invariants in the mid-1990s. As for the genus one Gromov-Witten invariants, the string prediction of Bershadsky-Cecotti-Ooguri-Vafa (BCOV) [**9**] made in 1993 for the quintic was only proved by Zinger and Jun Li [**62, 44**] about five years ago.

Though we now know much about mirror symmetry, many important questions remain open and progress continues to be made. In fact, the higher genus $g \geq 2$ case is still mathematically not well-understood. In the celebrated work of BCOV [**10**], a holomorphic anomaly equation for higher genus partition functions, F_g, was written down. Yamaguchi-Yau [**61**] in 2004 were able to show that F_g for $g \geq 2$ are polynomials of just five generators: $(V_1, V_2, V_3, W_1, Y_1)$. When these generators are assigned degrees $(1, 2, 3, 1, 1)$ respectively, F_g becomes a quasi-homogeneous polynomial of degree $(3g-3)$. This result was used by Huang-Klemm-Quackenbush [**38**] to compute the partition function on the mirror quintic up to genus $g = 51$.

As noted by BCOV, the higher genus B-model partition function can come from the quantization of the Kodaira-Spencer gauge theory. Towards this aim, Costello and Si Li have recently made significant progress. They have found a prescription for quantizing the Kodaira-Spencer theory and have successfully carried it out in the elliptic curve case [**45**].

Separately, much of the work on mirror symmetry has been based on toric geometry. To go beyond the toric cases, one needs to study period integrals and the differential equations which govern them under complex structure deformations. In this regard, Lian, Song, and Yau [**47, 48**] have very recently been able to describe explicitly a Picard-Fuchs type differential system for Calabi-Yau complete intersections in a Fano variety or a homogeneous space.

And finally, from the geometric perspective, Strominger-Yau-Zaslow [**53**] gave a T-duality explanation of mirror symmetry. This viewpoint has been clarified in much detail in the work of Gross-Siebert [**34, 35, 36**] during the past decade.

As we can see from the influence of mirror symmetry, string theory has had a strong effect on the development of mathematics. But let us now turn to a more recent developing area of string-math collaboration. This is the study of non-Kähler manifolds with trivial canonical bundle. They are sometimes called non-Kähler Calabi-Yaus. For string theory, they play an important role as they appear in supersymmetric flux compactifications. However, we will begin first by describing why mathematicians were interested in them prior to string theory.

2. Non-Kähler Calabi-Yau

A large class of compact non-Kähler Calabi-Yau threefolds were already known in the mid-1980s by a construction of Clemens [**16**] and Friedman [**25**]. Their construction starts from a smooth Kähler Calabi-Yau threefold, Y.

Suppose Y contains a collection of mutually disjoint rational curves. These are curves that are isomorphic to $\mathbb{CP}^1$ and have normal bundles $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

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Following Clemens, we can contract these rational curves and obtain a *singular* Calabi-Yau threefold X_0 with ordinary double-point singularities. Friedman then gave a condition to deform X_0 into a smooth complex manifold X_t. What we have described is just the compact version of the local conifold transition which physicists are familiar with

$$Y \dashrightarrow X_0 \dashrightarrow X_t \ .$$

Here, X_t's canonical bundle is also trivial, so it is a Calabi-Yau too. But in general X_t is non-Kähler. To see this, we can certainly contract enough rational curves so that $H^2(Y)$ is killed and $b_2 = 0$. In this case, after smoothing, we end up with a complex non-Kähler complex manifold which is diffeomorphic to a k-connected sum of $S^3 \times S^3$, with $k \geq 2$.

In 1987, Reid [**50**] put forth an interesting proposal, often called Reid's fantasy. Reid wanted to make sense of the vast collection of diverse Calabi-Yau threefolds. He speculated that all (Kähler) Calabi-Yau threefolds that can be deformed to Moishezon manifolds fit into a single universal moduli space in which families of smooth Calabi-Yaus of different homotopy types are connected to one another by the Clemens-Friedman conifold transitions that we have just described.

Now if we want to test this proposal, understanding non-Kähler Calabi-Yau manifolds becomes essential. For example, a question one can ask is: what geometrical structures exist on these non-Kähler Calabi-Yau manifolds? If the metrics are no longer Kähler, do they have some other property?

2.1. Balanced Metrics. A good geometric structure to consider is the one studied by Michelsohn [**49**] in 1982. Recall that a hermitian metric, with an associated $(1,1)$-form ω, is Kähler if

$$d\omega = 0 \qquad\qquad (\text{Kähler}) \ .$$

For threefolds, Michelsohn analyzed the weaker *balanced* condition:

$$d(\omega \wedge \omega) = 2\,\omega \wedge d\omega = 0 \qquad (\text{balanced}) \ .$$

As should be clear, a Kähler metric is always balanced but a balanced metric need not be Kähler.

The balanced condition has good mathematical properties. It is preserved under proper holomorphic submersions and also under birational transformations as shown by Alessandrini-Bassanelli [**2**]. There are also simple non-Kähler compact balanced manifolds. For example:

- Calabi [**14**] showed that a non-trivial bundle of complex tori over a Riemann surface cannot be Kähler, but it does have a balanced metric [**32**].
- The natural metric on compact six-dimensional twistor spaces is balanced. As Hitchin showed, only those associated with S^4 and $\mathbb{CP}^2$ are Kähler [**37**]. One can get a non-Kähler Calabi-Yau by taking branched covers of twistor spaces. Sometimes, if the four-manifold is an orbifold, the singularities on the twistor space may be resolved to also give a non-Káhler Calabi-Yau.
- Three-dimensional Moishezon spaces are balanced.

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So how about the non-Kähler Calabi-Yaus from conifold transitions? Do they admit a balanced metric? In this regard, J. Fu, J. Li, and S.-T. Yau proved the following theorem.

Theorem 2.1 (Fu-Li-Yau [**26**]). *Let Y be a smooth Kähler Calabi-Yau threefold and let $Y \to X_0$ be a contraction of mutually disjoint rational curves. Suppose X_0 can be smoothed to a family of smooth complex manifolds X_t. Then for sufficiently small t, X_t admits smooth balanced metrics.*

This constructive theorem provides us with balanced metrics on a large class of complex threefolds. In particular, for the Clemens-Friedman construction, the theorem implies

Corollary (Fu-Li-Yau [**26**]). *There exists a balanced metric on $\#_k(S^3 \times S^3)$ for any $k \geq 2$.*

Knowing that a balanced metric is present is useful. But to really understand Reid's proposal for Calabi-Yau moduli space, it is important to define some *canonical* balanced metric which would satisfy an additional condition, like the Ricci-flatness condition for the Kähler Calabi-Yau case. So we would like to have a natural condition, and here string theory gives some suggestions. As Calabi-Yau has played an important role in strings, one may ask what would be the natural setting to study compact conifold transitions and non-Kähler Calabi-Yau in physics.

Physicists have been interested in non-Kähler manifolds for more than a decade now in the context of compactifications with fluxes and model building (see e.g. [**30, 19**]). In this scenario, if one desires compact spaces without singularities from branes, then one should consider working in heterotic string theory.

2.2. Strominger's System. In the heterotic theory, the conditions for preserving $N = 1$ supersymmetry with H-fluxes were written down by Strominger [**52**] in 1986. Strominger's system of equations specifies the geometry of a complex threefold X (with a holomorphic three-form Ω) and in addition a holomorphic vector bundle E over X. The Hermitian metric ω of the manifold X and the metric h of the bundle E satisfy the following system of differential equations:

(1) $d(\| \Omega \|_\omega \, \omega \wedge \omega) = 0\,;$

(2) $F_h^{2,0} = F_h^{0,2} = 0\,, \quad F_h \wedge \omega^2 = 0\,;$

(3) $i\partial\overline{\partial}\, \omega = \frac{\alpha'}{4} \left[\mathrm{tr}\big(R_\omega \wedge R_\omega\big) - \mathrm{tr}\big(F_h \wedge F_h\big)\right]\,.$

Notice that the first equation is equivalent to the existence of a (conformally) balanced metric. The second is the Hermitian-Yang-Mills equations which is equivalent to E being a stable bundle. The third equation is the anomaly equation. When X is Kähler and E is the tangent bundle T_X, the system is then solved with $h = \omega_{CY}$, the Kähler Calabi-Yau metric.

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Using a perturbation method, J. Li and S.-T. Yau [**43**] have constructed smooth solutions on a class of Kähler Calabi-Yau manifolds with *irreducible* solutions for vector bundles with gauge group $SU(4)$ and $SU(5)$. Andreas and Garcia-Fernandez [**5, 6**] have generalized our construction on Kähler Calabi-Yau manifolds for any stable bundle E that satisfies $c_2(X) = c_2(E)$. In recent years, our collaborators and other groups have also constructed solutions of the Strominger system on non-Kähler Calabi-Yaus [**28, 7, 8, 27, 21, 58**].

As is clear in heterotic string theory, understanding stable bundles on Calabi-Yau threefolds is important. In this regard, Donagi, Pantev, Bouchard and others have done nice work constructing stable bundles on Kähler Calabi-Yaus to obtain realistic heterotic models of nature [**18, 12**]. Also, Andreas and Curio [**3, 4**] have done analysis on the Chern classes of stable bundles on Calabi-Yau threefolds, verifying in a number of cases a proposal of Douglas-Reinbacher-Yau [**20**].

But returning to conifold transitions on compact Calabi-Yaus, it has been proposed by Yau to use Strominger's system to study Reid's proposal. Certainly the first condition that there exists a balanced metric can be satisfied. As we have already mentioned, Fu-Li-Yau [**26**] showed the existence of a balanced metric under conifold transitions. However, the second condition from the heterotic string adds a stable gauge bundle into the picture. So one needs to know about the stability of holomorphic bundles through a global conifold transition. In his recent PhD thesis, M.-T. Chuan [**15**] examined how to carry a stable vector bundle through a conifold transition, from a Kähler to a non-Kähler Calabi-Yau. Under the assumption that the initial stable holomorphic bundle is trivial in a neighborhood of the contracting rational curves, he proved that the resulting holomorphic bundle on the non-Kähler Calabi-Yau also has a Hermitian Yang-Mills metric, and hence is stable. This shows that two of the three conditions of Strominger's system, the existence of a balanced metric and a Hermitian-Yang-Mills metric on the bundle, can be satisfied. The last condition, the anomaly equation, which couples the two metrics, is perhaps the most demanding and difficult to analyze.

J. Fu and S.-T. Yau have analyzed carefully the anomaly equation when the manifold is a T^2 bundle over a $K3$ surface [**28**]. In this case, the anomaly equation reduces down to a Monge-Ampère type equation on the $K3$:

$$\triangle(e^u - \frac{\alpha'}{2} f e^{-u}) + 4\alpha' \frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + \mu = 0\,,$$

where f and μ are functions on the $K3$ satisfying $f \geq 0$ and $\int_{K3} \mu = 0$ and $u_{i\bar{j}}$ is the $\partial\overline{\partial}$ partial derivative matrix on the function u. It would be interesting to show that the anomaly equation can be satisfied throughout the non-Kähler Calabi-Yau moduli space.

2.3. Symplectic Conifold Transitions: Smith-Thomas-Yau. So far we have discussed conifold transitions between Calabi-Yaus that although non-Kähler still maintain a complex structure. The contraction of a rational curve $\mathbb{CP}^1$ (and the inverse operation of resolution) is naturally a complex operation. The smoothing of a conifold singularity by S^3 on the other hand is naturally symplectic. Friedman's

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condition is needed to ensure that a smoothed out Calabi-Yau contains a global complex structure.

But instead of preserving the complex structure, we can preserve the symplectic structure throughout the conifold transition. This would be the symplectic *mirror* of the Clemens-Friedman conifold transition. In this case, we would collapse disjoint Lagrangian three-spheres, and then replace them by symplectic two-spheres. Such a symplectic transition was proposed in a work of Smith-Thomas-Yau [**51**] in 2002.

Locally, of course, there is a natural symplectic form in resolving the singularity by a two-sphere. But there may be obstructions to patching the local symplectic forms to get a global one. Smith-Thomas-Yau wrote down the condition (analogous to Friedman's complex condition) that ensures a global symplectic structure. This symplectic structure however may not be compatible with the complex structure. So in general, the symplectic conifold transitions result in non-Kähler manifolds, but they all have $c_1 = 0$ and so they are called symplectic Calabi-Yaus. In fact, Smith-Thomas-Yau used conifold transitions to construct many real six-dimensional non-Kähler symplectic Calabi-Yaus.

In the symplectic conifold transition, if we can collapse all disjoint three-spheres, then such a process should result in a manifold diffeomorphic to a connected sum of $\mathbb{CP}^3$s. This mirrors the complex case, which after collapsing all disjoint rational curves gives a connected sum of $S^3 \times S^3$s. More recently, Fine-Panov [**23, 24**] have also constructed interesting simply-connected symplectic Calabi-Yaus with Betti number $b_3 = 0$, which means that they cannot be Kähler.

As mentioned above, a balanced structure can always be found in a complex conifold transition. So similarly, we can ask if there is any geometric structure present before and after a symplectic conifold transition? Here we will be looking for a condition on the globally $(3,0)$-form which in the general non-Kähler case is no longer d-closed. Again, we can turn to string theory for a suggestion. Is there a mirror dual of a complex balanced manifold in string theory that is symplectic and generally non-Kähler?

Such a symplectic mirror will not be found in heterotic string theory. All supersymmetric solutions satisfy the Strominger system in heterotic string. So the mirror dual of a complex balanced manifold with a bundle should be another complex balanced manifold with a bundle. But it turns out the answer can be found in type II string theories. As we will describe below, the equations for non-Kähler Calabi-Yaus in type II string also give us insights into the natural cohomologies on non-Kähler manifolds.

3. Type II Strings: Non-Kähler Calabi-Yau Mirrors

In type II string theory, supersymmetric compactifications preserving a $SU(3)$ structure have been studied by many authors in the last ten years. Since we are interested in non-Kähler geometries of compact manifolds, any supersymmetric solution will have orientifold sources. The type of sources helps determine the

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type of non-Kähler geometries. We shall describe the supersymmetric equations written in a form very similar to that of Graña-Minasian-Petrini-Tomasiello [**31**] and Tomasiello [**54**]. More details of our description here can be found in [**57**].

3.1. Complex Balanced Geometry in Type IIB. The supersymmetric equations that involve complex balanced threefolds are found in type IIB theory in the presence of orientifold 5-branes (and possibly also D5-branes). These branes are wrapped on holomorphic curves. In this case, the conditions on the Hermitian $(1,1)$-form ω and $(3,0)$-form Ω can be written as [**54**]

$$\begin{aligned} d\Omega &= 0 && \text{(complex integrability)} \\ d(\omega\wedge\omega) &= 0 && \text{(balanced)} \\ 2i\,\partial\overline{\partial}(e^{-2f}\omega) &= \rho_B && \text{(source)} \end{aligned}$$

where ρ_B is the sum of the currents Poincaré dual to the holomorphic curves that the five-brane sources wrap around, and f is a distribution that satisfies

$$i\,\Omega\wedge\overline{\Omega} = 8\,e^{2f}\,\frac{\omega^3}{3!}\ .$$

The balanced and the source equations together are noteworthy in that they share a resemblance with the Maxwell equations. With the Hodge star operator defined with respect to the compatible Hermitian metric, we can write $\omega = *(\omega^2/2)$. The equations can then be expressed (ignoring the conformal factor)

$$d(\omega^2/2) = 0\ ,$$
$$2i\,\partial\overline{\partial}*(\omega^2/2) = \rho_B\ .$$

Now this might have been somewhat expected as the five-brane sources are associated with the three-form field strength F_3 which is hidden in the source equation. These two equations however do tell us something more.

Let us recall the Maxwell case. Maxwell's equations in four dimensions are

$$d\,F_2 = 0\,,$$
$$d*\,F_2 = \rho_e\,,$$

where ρ_e is the Poincaré dual current of some electric charge configuration. Now, if we consider the deformation $F_2 \to F_2 + \delta F_2$ with the source fixed, that is, $\delta\rho_e = 0$, this leads to

$$d(\delta F_2) = d*(\delta F_2) = 0\ ,$$

which are the harmonic conditions for a degree two form in de Rham cohomology. Thus, the de Rham cohomology is naturally associated with Maxwell's equations.

For the type IIB complex balanced equations, we can also deform $\omega^2 \to \omega^2 + \delta\omega^2$. If this deformation is performed with the source current and the conformal factor fixed (i.e. $\delta\rho_A = \delta A = 0$), then we arrive at the conditions

$$d(\delta\omega^2) = \partial\overline{\partial}*(\delta\omega^2) = 0\ ,$$

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which turn out to be precisely the harmonicity conditions for a (2,2)-element of the Bott-Chern cohomology:

$$H^{p,q}_{BC} = \frac{\ker d \cap \mathcal{A}^{p,q}}{\text{im } \partial\overline{\partial} \cap \mathcal{A}^{p,q}} \ .$$

This cohomology was introduced by Bott-Chern [**11**] and Aeppli [**1**] in the mid-1960s.

The string equations thus strongly suggest that the Bott-Chern cohomology (or the dual Aeppli cohomology, see Table 1) is the natural one to use for studying complex balanced manifolds. Let us point out that when the manifold is Kähler, the $\partial\overline{\partial}$-lemma holds. In this case, the Bott-Chern and the Dolbeault cohomology are in fact isomorphic. So the Bott-Chern cohomology is really most useful in the non-Kähler setting, and especially when the $\partial\overline{\partial}$-lemma fails to hold.

3.2. Symplectic Mirror Dual Equations in Type IIA. The mirror dual to the complex balanced manifold is found in the type IIA string. Roughly, the type IIA equations can be obtained from the IIB equations by first replacing $\omega^2/2$ with $(\text{Re}\, e^{i\,\omega})$ and then exchanging $e^{i\,\omega}$ with Ω [**22**]:

$$d(\omega^2/2) = 0 \ \Leftrightarrow \ d(\text{Re}\, e^{i\,\omega}) = 0 \qquad \longleftrightarrow \qquad d\,\text{Re}\ \Omega = 0 \ .$$

Thus, $d\,\text{Re}\,\Omega = 0$ is the condition that is suggested by string theory for symplectic conifold transitions.

This condition turns out to be part of the type IIA supersymmetry conditions in the presence of orientifold (and D-) six-branes wrapping special Lagrangian submanifolds. The type IIA equations that are mirror to the type IIB complex balanced system can be written as follows:

$$\begin{aligned} d\omega &= 0\,, && \text{(symplectic)} \\ d\,\text{Re}\ \Omega &= 0\,, && \text{(almost complex)} \\ \partial_+\partial_- * (e^{-2f}\,\text{Re}\,\Omega) &= \rho_A\,, && \text{(source)} \end{aligned}$$

with

$$8\,\frac{\omega^3}{3!} = i\,e^{2f}\,\Omega\wedge\overline{\Omega} \ .$$

In the above system, ρ_A is the current Poincaré dual to the wrapped special Lagrangian submanifolds. The operators ∂_+ and ∂_- are linear symplectic operators that can be thought of as the symplectic analogues of the Dolbeault operators, ∂ and $\overline{\partial}$, and were recently introduced by us in [**56**]. If we naively compare the above symplectic system with the mirror complex one, a natural question arises: does the type IIA symplectic system suggest the existence of a symplectic cohomology of the form

$$\frac{\ker d}{\text{im } \partial_+\partial_-}$$

analogous to the complex Bott-Chern cohomology? Interestingly, as we found in [**55, 56**], such a cohomology is indeed natural and finite-dimensional on a compact symplectic manifold. Moreover, it provides new invariants for non-Kähler symplectic manifolds. But to discuss more about this symplectic cohomology and its

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relation to the IIA symplectic system, we will need certain properties of the symplectic differential operators, (∂_+, ∂_-). Since these linear operators are new, let us proceed now to give some more details.

4. Symplectic Differential Operators and Cohomologies

4.1. Linear Differential Symplectic Operators. Like their Dolbeault counterparts, (∂_+, ∂_-) can be naturally defined by an intrinsically symplectic decomposition of the exterior derivative. Recall that in the complex case, the differential forms are decomposed into (p,q) components $\mathcal{A}^{p,q}$. The exterior derivative d acting on each component gives two terms:

$$d: \mathcal{A}^{p,q} \;\to\; \mathcal{A}^{p+1,q} \oplus \; \mathcal{A}^{p,q+1} \,.$$

This then defines the Dolbeault operators ∂ and $\overline{\partial}$ as the projections of $d\mathcal{A}^{p,q}$ onto $\mathcal{A}^{p+1,q}$ and $\mathcal{A}^{p,q+1}$, respectively.

On a symplectic space (M,ω) of dimension $d = 2n$, we can do the analogous analysis. Indeed, there is also a decomposition of differential forms, specifically into representations of the $\mathfrak{sl}(2)$ lie algebra. This is well-known in the Kähler literature as the Lefschetz decomposition. Let us however emphasize that this decomposition requires only a non-degenerate two-form, which we do have here in ω. More explicitly, acting on a differential form $A \in \Omega^*(M)$, the $\mathfrak{sl}(2)$ generators take the form

$$\begin{aligned} L:&\;\; A \to \omega \wedge A \\ \Lambda:&\;\; A \to \frac{1}{2}(\omega^{-1})^{ij}\, i_{\partial_{x^i}} i_{\partial_{x^j}} A \\ H:&\;\; A \to (n-k)\, A \qquad \text{for } A \in \Omega^k(M) \end{aligned}$$

with commutation relations

$$[H,\Lambda] = 2\Lambda\,, \quad [H,L] = -2L\,, \quad [\Lambda, L] = H\,.$$

The $\mathfrak{sl}(2)$ irreducible modules are standardly constructed from the highest weight forms, which are commonly called primitive forms. Denoting the space of primitive forms by $\mathcal{P}^*(M)$, let us recall that a differential form is primitive, i.e. $B_s \in \mathcal{P}^s(M)$, if

$$\Lambda\, B_s = 0 \quad \text{or equivalently,} \quad L^{n+1-s}\, B_s = 0\,.$$

Hence, an irreducible $\mathfrak{sl}(2)$ module is the span of the elements

$$\left\{B_s\,,\; \omega\wedge B_s\,,\; \omega^2 \wedge B_s\,,\; \ldots\,,\; \omega^{n-s}\wedge B_s\right\}\,.$$

Since each element of this basis consists of ω raised to some power r exterior multiplied with a primitive s-form, it is natural to label basis elements of the $\mathfrak{sl}(2)$ module by the pair (r,s) and define

$$\mathcal{L}^{r,s}(M) = \left\{A \in \Omega^{2r+s}(M) \,\middle|\, A = \omega^r \wedge B_s \text{ and } \Lambda\, B_s = 0\right\}\,.$$

In a rough sense then, the space of forms $\mathcal{L}^{r,s}$ are the symplectic analogs of $\mathcal{A}^{p,q}$ of complex geometry.

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Continuing the analogy with the complex case, let us act on $\mathcal{L}^{r,s}$ by the exterior derivative d. Since $d\omega = 0$, we have

$$d\mathcal{L}^{r,s} = d\,(\omega^r \wedge B_s) = \omega^r \wedge dB_s \ .$$

Clearly, the derivative only acts on the primitive forms. This is suggestive that much if not all of the data of differential forms on symplectic manifolds are encoded within primitive forms. Now as for d acting on a primitive form, it can be shown (see for example [**39**]) that

$$dB_s = B^0_{s+1} + \omega \wedge B^1_{s-1} \ .$$

Combining the above two equations, we find that

$$d:\ \mathcal{L}^{r,s} \ \rightarrow\ \mathcal{L}^{r,s+1} \ \oplus\ \omega \wedge \mathcal{L}^{r,s-1}$$

which has only two components on the right hand side just as in the complex case. Therefore, projecting onto each component, we can express the exterior derivative as [**56**]

$$d \ = \ \partial_+ \ + \ \omega \wedge \partial_-$$

where the first-order differential operators (∂_+, ∂_-) are defined by the derivative mapping

$$\begin{aligned} &\partial_\pm : \mathcal{L}^{r,s} \longrightarrow \mathcal{L}^{r,s\pm 1} \\ &\partial_\pm : \mathcal{P}^{s} \ \longrightarrow\ \mathcal{P}^{s\pm 1} \qquad \text{for } r = 0 \end{aligned}$$

By the above definition, ∂_+ and ∂_-, respectively, raise and decrease the degree of the forms by one. Moreover, (∂_+, ∂_-) are operators that map primitive forms to primitive forms (in the case of $r = 0$). And similarly to their complex counterparts, it follows from $d^2 = 0$ and the Lefschetz decompostion that they square to zero, i.e.

$$(\partial_+)^2 = (\partial_-)^2 = 0 \ ,$$

and anticommute: $\omega \wedge (\partial_+\partial_-) = -\,\omega \wedge (\partial_-\partial_+)\,.$

4.2. Symplectic Cohomologies and a Type IIA System. With the linear symplectic operators, (∂_+, ∂_-) and their properties at hand, we can now write down an interesting primitive symplectic elliptic complex.

Proposition (Tseng-Yau [**56**])**.** *On a symplectic manifold of dimension $d = 2n$, the following differential complex is elliptic.*

$$\begin{tikzcd} 0 \arrow[r, "\partial_+"] & \mathcal{P}^0 \arrow[r, "\partial_+"] & \mathcal{P}^1 \arrow[r, "\partial_+"] & \cdots \arrow[r, "\partial_+"] & \mathcal{P}^{n-1} \arrow[r, "\partial_+"] & \mathcal{P}^n \arrow[d, "\partial_+\partial_-"] \\ 0 & \mathcal{P}^0 \arrow[l, "\partial_-"'] & \mathcal{P}^1 \arrow[l, "\partial_-"'] & \cdots \arrow[l, "\partial_-"'] & \mathcal{P}^{n-1} \arrow[l, "\partial_-"'] & \mathcal{P}^n \arrow[l, "\partial_-"'] \end{tikzcd}$$

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Symplectic (M, ω)		Complex (M, J)	
$PH^s_{\partial_\pm} = \dfrac{\ker \partial_\pm \cap \mathcal{P}^s}{\text{im } \partial_\pm \cap \mathcal{P}^s}$,	$s < n$	$H^{p,q}_{\overline{\partial}} = \dfrac{\ker \overline{\partial} \cap \mathcal{A}^{p,q}}{\text{im } \overline{\partial} \cap \mathcal{A}^{p,q}}$	(Dolbeault)
$PH^s_{\partial_+\partial_-} = \dfrac{\ker \partial_+\partial_- \cap \mathcal{P}^s}{(\text{im } \partial_+ + \text{im } \partial_-) \cap \mathcal{P}^s}$,	$s \le n$	$H^{p,q}_{\partial\overline{\partial}} = \dfrac{\ker \partial\overline{\partial} \cap \mathcal{A}^{p,q}}{(\text{im } \partial + \text{im } \overline{\partial}) \cap \mathcal{A}^{p,q}}$	(Aeppli)
$PH^s_{\partial_+ + \partial_-} = \dfrac{\ker d \cap \mathcal{P}^s}{\text{im } \partial_+\partial_- \cap \mathcal{P}^s}$,	$s \le n$	$H^{p,q}_{\partial + \overline{\partial}} = \dfrac{\ker d \cap \mathcal{A}^{p,q}}{\text{im } \partial\overline{\partial} \cap \mathcal{A}^{p,q}}$	(Bott-Chern)

TABLE 1. Finite-dimensional primitive symplectic cohomologies (Tseng-Yau [**55, 56**]) and their complex analogs. The complex cohomologies involving $\partial\overline{\partial}$ were introduced in the mid-1960s by Aeppli [**1**] and Bott-Chern [**11**].

Since the complex is elliptic, we can write down four different types of finite-dimensional primitive cohomologies associated with it.

$$\begin{aligned} PH^s_{\partial_\pm}(M) &= \frac{\ker \partial_\pm \cap \mathcal{P}^s(M)}{\text{im } \partial_\pm \cap \mathcal{P}^s(M)} \qquad \text{for } 0 \le s < n \ , \\ PH^n_{\partial_+\partial_-}(M) &= \frac{\ker \partial_+\partial_- \cap \mathcal{P}^n(M)}{\text{im } \partial_+ \cap \mathcal{P}^n(M)} \ , \\ PH^n_{\partial_+ + \partial_-}(M) &= \frac{\ker \partial_- \cap \mathcal{P}^n(M)}{\text{im } \partial_+\partial_- \cap \mathcal{P}^n(M)} \ . \end{aligned}$$

Furthermore, by considering extended elliptic complexes involving non-primitive forms, it is possible to also define $PH^s_{\partial_+\partial_-}(M)$ and $PH^s_{\partial_+ + \partial_-}(M)$ for all degree s, $0 \le s \le n$. On a compact symplectic manifold, these cohomologies are also finite-dimensional [**55**]. We list the finite-dimensional symplectic cohomologies and their complex analogues in Table 1.

In [**55, 56**], we analyzed some of the basic properties of the new symplectic cohomologies. Since they are associated with an elliptic complex, each has an associated elliptic Laplacian and thus have standard nice Hodge theoretical properties. Moreover, we have calculated the cohomologies explicitly for some non-Kähler symplectic nilmanifolds and found that the cohomologies indeed lead to new symplectic invariants. Perhaps not too surprisingly, the new invariants do not contain new information when the manifold is Kähler.

Finally, returning back to the type IIA symplectic system of equations in Section 3.2. the middle-degree cohomology

$$PH^n_{\partial_+ + \partial_-} = \frac{\ker d \cap \mathcal{P}^n}{\text{im } \partial_+\partial_- \cap \mathcal{P}^n}$$

with $n = 3$ turns out to be particularly relevant. Consider deforming the type IIA system by $\Omega \longrightarrow \text{Re}\,\Omega + \delta\text{Re}\,\Omega$ with $\delta\rho_A = 0$ and conformal factor remaining fixed,

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$\delta A = 0$. Then, $\delta \mathrm{Re}\,\Omega$ is required to satisfy (dropping the conformal factor)

$$d(\delta \mathrm{Re}\,\Omega) = 0\ , \qquad\qquad \partial_+\partial_- * (\delta \mathrm{Re}\,\Omega) = 0\ ,$$

which are the harmonicity conditions of the primitive $PH^n_{\partial_+ + \partial_-}$ cohomology. In fact, assuming $\delta\rho_A = \delta A = 0$, it can be shown [**57**] that a subspace of the linearized deformation of the type IIA symplectic system can be parametrized by the primitive cohomology with

$$\delta \mathrm{Re}\,\Omega \in P\mathcal{H}^3_{\partial_+ + \partial_-} \cap \mathrm{Re}\,\mathcal{A}^{2,1}\ .$$

5. Concluding Remarks

String theory has motivated much research in non-Kähler geometry in the past couple of years. We fully expect that investigations especially revolving around six-dimensional non-Kähler geometry will remain very active in the near future too. Of particular interest, six-dimensional non-Kähler geometries can have relations with four- and three-dimensional manifolds. As has been known for some time, one can construct many non-Kähler six-manifolds by the twistor construction. The twistor space of an anti-self dual four-manifolds has a complex structure, and the twistor space of a hyperbolic four-manifold has a symplectic structure. The S^3 bundle over a hyperbolic three-manifold is also complex. (Fine-Panov have given examples of the hyperbolic constructions [**24**].) There should also be interesting dualities relating complex and symplectic structures on non-Kähler six-manifolds. We have no doubt that string theory will continue to be a guiding influence in future work on non-Kähler geometry.

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