Mapping Cone Connections and their Yang-Mills Functional

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Abstract

For a given closed two-form, we introduce the cone Yang-Mills functional which is a Yang-Mills-type functional for a pair (A, B), a connection one-form A and a scalar B taking value in the adjoint representation of a Lie group. The functional arises naturally from dimensionally reducing the Yang-Mills functional over the fiber of a circle bundle with the two-form being the Euler class. We write down the Euler-Lagrange equations of the functional and present some of the properties of its critical solutions, especially in comparison with Yang-Mills solutions. We show that a special class of three-dimensional solutions satisfy a duality condition which generalizes the Bogomolny monopole equations. Moreover, we analyze the zero solutions of the cone Yang-Mills functional and give an algebraic classification characterizing principal bundles that carry such cone-flat solutions when the two-form is non-degenerate.

Contents

1 Introduction

Let (M^m, g) be a Riemannian manifold and P a principal G-bundle over M. For simplicity, we will assume that the Lie group G is compact and a subgroup of $SO(N)$.

In this paper, we study a Yang-Mills type functional that involves a pair (A, B), with $A \in \Omega^1(M, Ad\, P)$ being the connection one-form and $B \in \Omega^0(M, Ad\, P)$ a section of the associated adjoint bundle $Ad\, P$, and also a two-form $\zeta \in \Omega^2(M)$ that is d-closed. We shall call this functional the *cone Yang-Mills functional* and it is defined by

$$\begin{aligned} S_{cYM}(A,B) &= \|F_A + \zeta B\|^2 + \|d_A B\|^2 \\ &= c_G \int_M \mathrm{tr}\left[(F_A + \zeta B) \wedge *(F_A + \zeta B) \,+\, d_A B \wedge * d_A B\right], \end{aligned} \tag{1.1}$$

where $F_A = dA + A \wedge A$ is the curvature, $d_A B = dB + [A, B]$ is the covariant derivative of B, and in the second line, we have inserted the Killing form of the Lie group G which for $SO(N)$ and $SU(N)$ is given by the trace times a constant dependent on N, denoted by c_G. The critical points of the functional are solutions of the associated Euler-Lagrange equations

$$d_A^*(F_A + \zeta B) + [B, d_A B] = 0\,, \tag{1.2}$$

$$\zeta^*(F_A + \zeta B) + d_A^* d_A B = 0\,, \tag{1.3}$$

where $\zeta^* : \Omega^k(M) \to \Omega^{k-2}(M)$ is the adjoint of the wedging map, $\zeta\wedge$, and can be explicitly expressed as

$$\zeta^* = (-1)^{(m-k)k} * \zeta * \,. \tag{1.4}$$

Let us give both a physical and a mathematical motivation for considering the cone Yang-Mills functional.

For the physical motivation, the cone Yang-Mills functional can be considered as the dimensional reduction of the standard Yang-Mills functional over the S^1 fiber of a circle bundle,

$\pi : X \to M$. Unlike a product space, a circle bundle generically is one where the fiber circle is non-trivially twisted over M. Such arises for instance in Kaluza-Klein monopole solutions with a non-vanishing background two-form flux ζ. On any circle bundle X, there exists what is called a global angular one-form θ which can be locally expressed as $\theta = dz + a$ where z denotes the S^1 fiber coordinate and a is the $U(1)$ Kaluza-Klein gauge field. The two-form flux, $\zeta = d\theta = d(dz + a) = da$, is then the background field strength of the $U(1)$ gauge field and effectively measures the twisting of the S^1 fiber.

Now for the dimensional reduction of the Yang-Mills functional over the S^1 fiber of the circle bundle $\pi : X \to M$, we take the metric to be

$$g_X = \pi^* g_M + \theta \otimes \theta \,, \tag{1.5}$$

and further require that the local connection one-form (Yang-Mills gauge field) $\mathcal{A}$ on X be invariant under translation of the circle fiber. Such a connection form on X can be expressed as

$$\mathcal{A} = A + \theta\, B \,. \tag{1.6}$$

Here, both A and B takes values on $Ad\, P$, and B represents the component of the connection in the circle direction. Since $\mathcal{A}$ is invariant under S^1 translation, both A and B only have dependence on the coordinates of M. The curvature two-form of $\mathcal{A}$ (Yang-Mills field strength) on X then has the following expression:

$$\mathcal{F}_{\mathcal{A}} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = (F_A + \zeta B) + \theta(-d_A B) \,, \tag{1.7}$$

where we have used $d\theta = \zeta$. Comparing with the cone Yang-Mills functional in (1.1), we see that $S_{cYM} = \|\mathcal{F}_{\mathcal{A}}\|^2$, and therefore, it is just the Yang-Mills functional on X dimensionally reduced to M over the fiber circle. And we can also interpret the cone Yang-Mills Euler-Lagrange equations (1.2)-(1.3) as the dimensionally-reduced Yang-Mills equations over the S^1 fiber of X.

In physical terms, the cone Yang-Mills functional on M is a Euclidean or Wick-rotated action, situated in a curved background, with Riemannian metric g:

$$\begin{aligned} S_{cYM}(A, B) &= \int_M c_G \operatorname{tr}\left(|F_A + \zeta B|^2 + |d_A B|^2\right) dV \\ &= \int_M c_G \operatorname{tr}\left(|F_A|^2 + |d_A B|^2 + |\zeta|^2 B^2 + 2\,\zeta^* F_A B\right) dV \,, \end{aligned}$$

$$
\begin{tikzcd}
\dots \arrow[r, "d_C"] & \mathrm{Cone}^k(\zeta) \arrow[r, "d_C"] & \mathrm{Cone}^{k+1}(\zeta) \arrow[r, "d_C"] & \mathrm{Cone}^{k+2}(\zeta) \arrow[r, "d_C"] & \dots \\
\dots \arrow[r, "d"] & \Omega^k(M) \arrow[r, "d"] & \Omega^{k+1}(M) \arrow[r, "d"] & \Omega^{k+2}(M) \arrow[r, "d"] & \dots \\
\dots \arrow[r, "-d"] & \Omega^{k-1}(M) \arrow[r, "-d"] \arrow[ur, "\zeta\wedge"] & \Omega^k(M) \arrow[r, "-d"] \arrow[ur, "\zeta\wedge"] & \Omega^{k+1}(M) \arrow[r, "-d"] & \dots
\end{tikzcd}
$$

Figure 1: The mapping cone complex of the d-closed two-form ζ. The complex of elements $\mathrm{Cone}^*(\zeta) = \Omega^*(M) \oplus \Omega^{*-1}(M)$ arises from the ζ map between two de Rham complexes.

where we have used a simplified notation, e.g. $F_A \wedge *F_A = |F_A|^2 dV$. The action thus consists of a Yang-Mills part, an adjoint-valued scalar with a "mass" $|\zeta|$ (which may vary over M), and an interaction term between A and B. Heuristically, the Euler-Lagrange equations of motion (1.2)-(1.3) can be thought of as a mixing of the Yang-Mills equations with a Klein-Gordon-type scalar plus an interaction term.

The cone Yang-Mills functional also has a mathematical motivation. In fact, the functional arises naturally from its relation with a mapping cone complex which we will provide an explanation here. (For a reference on the general mapping cone complex, see for example [11] and also [3].)

The d-closed two-form $\zeta \in \Omega^2(M)$ can be thought of as an operator or a *map*, by wedge product between differential forms, i.e. $\zeta\wedge : \Omega^*(M) \to \Omega^{*+2}(M)$. If desired, the form ζ can represent a geometric structure of interest on M, such as a symplectic or hermitian structure, but generally, the form ζ can be any closed two-form on M. Letting ζ map between two de Rham chain complexes leads to a mapping cone complex (see Figure 1). The elements of the mapping cone complex consist of pairs of differential forms:

$$
\mathrm{Cone}^k(\zeta) := \Omega^k(M) \oplus \Omega^{k-1}(M)\,, \qquad k = 0, 1, \dots, m+1\,. \tag{1.8}
$$

The differential of the mapping cone complex $d_C : \mathrm{Cone}^k(\zeta) \to \mathrm{Cone}^{k+1}(\zeta)$ is given by

$$
\begin{aligned}
d_C\, \mathrm{Cone}^k(\zeta) &= d_C \left(\Omega^k \oplus \Omega^{k-1}\right) \\
&= \left(d\,\Omega^k + \zeta \wedge \Omega^{k-1}\right) \oplus -d\,\Omega^{k-1}\,.
\end{aligned}
$$

Since the grading of the two components of $\mathrm{Cone}^k(\zeta)$ in (1.8) are different, it is useful to introduce a formal one-form θ to supplement the second component so that we can express a cone form as a sum with both components having the same total degree k, i.e.

$$\mathrm{Cone}^k(\zeta) = \Omega^k \oplus\, \theta \wedge \Omega^{k-1}\,. \tag{1.9}$$

Additionally, it is useful for us to impose that the formal one-form θ satisfy $d\theta = \zeta$. For this will allow us to interpret the cone differential d_C simply as the exterior derivative:

$$\begin{aligned} d_C\, \mathrm{Cone}^k(\zeta) &= d\left(\Omega^k \oplus\, \theta\wedge\Omega^{k-1}\right) \\ &= \left(d\,\Omega^k + \zeta\wedge\Omega^{k-1}\right) \oplus\, \theta\wedge\left(-d\,\Omega^{k-1}\right)\,. \end{aligned}$$

It then becomes self-evident that $d_C\, d_C = 0$. Moreover, there is a natural product on the cone space $\mathrm{Cone}^*(\zeta)$ given by the usual wedge product

$$\begin{aligned} \mathrm{Cone}^j(\zeta)\times\mathrm{Cone}^k(\zeta) : &= \left(\Omega^j \oplus\, \theta\wedge\Omega^{j-1}\right)\wedge\left(\Omega^k \oplus\, \theta\wedge\Omega^{k-1}\right) \\ &= \left(\Omega^j\wedge\Omega^k\right) \oplus\, \theta\wedge\left(\Omega^{j-1}\wedge\Omega^k + (-1)^j\Omega^j\wedge\Omega^{k-1}\right)\,. \end{aligned}$$

This product satisfies the Leibniz rule with respect to d_C. In all, we see that we have a mapping cone algebra, $(\mathrm{Cone}^*(\zeta), d_C, \times)$, that satisfies the conditions of a differential graded algebra (DGA).

Now in the presence of the associated adjoint bundle, we should consider the twisted cone forms:

$$\mathrm{Cone}^k(\zeta)(M, Ad\,P) = \Omega^k(M, Ad\,P) \oplus\, \theta\wedge\Omega^{k-1}(M, Ad\,P)$$

which take values on the associated adjoint bundle $Ad\,P$. In this context, the differential d_C must also be twisted by a cone connection one-form $\mathcal{A} \in \mathrm{Cone}^1(\zeta)(M, Ad\,P)$

$$\ldots \xrightarrow{d_C+\mathcal{A}} \mathrm{Cone}^k(\zeta)(M, Ad\,P) \xrightarrow{d_C+\mathcal{A}} \mathrm{Cone}^{k+1}(\zeta)(M, Ad\,P) \xrightarrow{d_C+\mathcal{A}} \mathrm{Cone}^{k+2}(\zeta)(M, Ad\,P) \xrightarrow{d_C+\mathcal{A}} \ldots$$

where $\mathcal{A} = A + \theta\wedge B$, with $A \in \Omega^1(M, Ad\,P)$ and $B \in \Omega^0(M, Ad\,P)$. The cone curvature then takes the form

$$\mathcal{F}_{\mathcal{A}} = (d_C + \mathcal{A})^2 = d_C\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = (F_A + \zeta B) + \theta\wedge(-d_A B)\,, \tag{1.10}$$

where $F_A = dA + A\wedge A$. The above twisted complex is only a differential complex if $\mathcal{F}_{\mathcal{A}} = 0$. This requires that the cone curvature $\mathcal{F}_{\mathcal{A}}$ vanishes, which from (1.10) corresponds to (A, B)

satisfying what we shall call the cone-flat condition with respect to ζ, or simply just the cone-flat condition,

$$F_A + \zeta\, B = 0\,, \qquad d_A\, B = 0\,. \tag{1.11}$$

The cone Yang-Mills functional of (1.1) is then just the normed square of the cone curvature, $\|\mathcal{F}_{\mathcal{A}}\|^2$. It is worthwhile to emphasize that this mapping cone perspective only requires that the two-form $\zeta \in \Omega^2(M)$ be d-closed and nothing more. This is in contrast with the dimensional reduction perspective, where ζ mathematically represents the Euler class of the circle bundle $\pi : X \to M$ and hence would need to be an element of $H^2(M, \mathbb{Z})$.

In this paper, we take an important first step in understanding cone Yang-Mills solutions. We will show that a subset of the solutions involves Yang-Mills connections. For instance, in dimension two when M is compact and ζ is taken to be the volume form, the above cone-flat condition (1.11) implies exactly the two-dimensional Yang-Mills condition

$$d_A^*\, F_A = -*\, d_A *\, F_A = *\, d_A B = 0\,, \tag{1.12}$$

having noted that $*\,\zeta = 1$. Conversely, in two dimensions, if (A, B) is a cone-Yang-Mills solution with A also a Yang-Mills connection, then the pair (A, B) must be cone-flat.

In higher dimensions, certain special classes of Yang-Mills connections can be paired with a scalar B to obtain cone Yang-Mills solutions. In the trivial case where we set $\zeta = 0$, any Yang-Mills connection A together with a covariantly constant B is trivially a cone Yang-Mills solutions. When $\zeta \neq 0$, Yang-Mills connections A such that the curvature two-form satisfy $\zeta^* F_A = 0$ are cone Yang-Mills solutions with $B = 0$. When ζ is a harmonic form, cone-flat solutions are always composed of a Yang-Mills connection with an appropriate scalar section B.

On the other hand, it should be evident that the space of cone Yang-Mills solutions (A, B) is generally much richer and different from that of Yang-Mills solutions. As we will show in explicit examples, not all Yang-Mills connection A can be paired with a scalar B to form a cone Yang-Mills solutions. Conversely, there are also cone Yang-Mills solutions (A, B) where A is not Yang-Mills. Furthermore, given a cone Yang-Mills solutions (A, B), there may be other scalars B' such that (A, B') remain cone Yang-Mills.

Interestingly, when M is three-dimensional, we are able to write down a Bogomolny-type condition, that gives a sufficient condition on (A, B) to be a cone Yang-Mills solution. Such a condition can be motivated by considering $\mathcal{F}_{\mathcal{A}}$ as the Yang-Mills curvature on a four-dimensional circle bundle X and imposing the (anti-)self-dual condition. Then, dimensionally reducing on the fiber S^1 by expressing $\mathcal{F}_{\mathcal{A}}$ as in (1.10), the (anti-)self-dual condition implies the three-dimensional condition

$$F_A + \zeta B = \pm * d_A B\,. \tag{1.13}$$

Notice that if we set $\zeta = 0$, the above equation is just the Bogomolny monopole equation. Similar to the (anti-)self-dual Yang-Mills condition in four dimensions, (1.13) is a first-order condition whose solutions solve (1.2)-(1.3) in three dimensions. This duality condition will assist us in finding non-abelian, cone Yang-Mills solutions. Indeed, we shall give in Section 3.2 an explicit $SU(2)$ solution of (1.13) that comes from dimensionally reducing the four-dimensional Taub-NUT gravitational instanton solution.

Concerning the cone-flat condition, we are able to characterize principal bundles that carry a cone-flat connection, i.e. a pair (A, B) satisfying (1.11), when ζ is a non-degenerate two-form. This type of cone-flat connections can interestingly be classified similar to Atiyah-Bott's classification of bundles carrying Yang-Mills connections on Riemann surfaces [2]. Our classification of the cone-flat bundles however depends on the given ζ and the second homotopy group, $\pi_2(M)$.

Theorem 1.1. *Let M be a path connected manifold, $\zeta \in \Omega^2(M)$ be a non-zero, non-degenerate, closed two-form, and G be a Lie group. There exists a bijective correspondence between the following sets:*

$$\left\{\begin{matrix} \textit{isomorphism classes of cone-flat connections} \\ \textit{with respect to } \zeta \textit{ on } G\textit{-bundles over } M \end{matrix}\right\} \simeq \left\{\begin{matrix} \textit{conjugacy classes of} \\ \textit{homomorphisms } \rho : \Gamma \to G \end{matrix}\right\},$$

where Γ is an $\mathbb{R}/\overline{H}$ extension of $\pi_1(M)$ with $\overline{H} \subset \mathbb{R}$ being the closure of the group

$$H := \left\{ \int_{\mathcal{S}} \zeta \;\middle|\; \mathcal{S} \textit{ is a representative in } \pi_2(M) \right\}.$$

This paper is organized as follows. In Section 2, after a brief description of our notations/conventions, we proceed to consider the first-order variation of the cone Yang-Mills functional to obtain its Euler-Lagrange equations. We also show that modulo gauge equivalence, the cone Yang-Mills equations are elliptic and hence has a finite-dimensional solution space on a closed manifold. We also describe properties of cone Yang-Mills solutions under certain conditions for the two-form ζ and structure group G, and especially emphasizing its relationship to Yang-Mills connections. In Section 3, we work out the special case of abelian cone Yang-Mills solutions in dimension two. We also discuss the three-dimensional Bogomolny-type monopole condition (1.13) and give an explicit non-trivial $SU(2)$ cone Yang-Mills solutions on $M = \mathbb{R}^3 - \{0\}$, i.e. the Euclidean space with the origin removed. Finally, in Section 4, we consider bundles that can carry cone-flat solutions when ζ is a non-degenerate, closed two-form, and prove the classification of cone-flat bundles of Theorem 1.1.

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2 Properties of cone Yang-Mills solutions

2.1 Preliminaries

Let M be a smooth manifold and P be a principal bundle over M. We consider the associated adjoint bundle $Ad\,P$ equipped with an inner product that is invariant under the adjoint action. Hence, for arbitrary $x, y, z \in \Omega^*(M, Ad\,P)$,

$$\langle [x, y], z\rangle = \langle x, [y, z]\rangle. \tag{2.1}$$

We can extend the inner product $\langle -, -\rangle$ on $\Omega^*(M, Ad\,P)$ to $\langle -, -\rangle_{\mathcal{C}}$ on $\mathrm{Cone}^*(\zeta)(M, Ad\,P) = \Omega^*(M, Ad\,P) \oplus \theta\Omega^{*-1}(M, Ad\,P)$ for a d-closed two-form, $\zeta \in \Omega^2(M)$. We do so by setting

$$\langle \eta_1 + \theta\xi_1, \eta_2 + \theta\xi_2\rangle_{\mathcal{C}} = \langle \eta_1, \eta_2\rangle + \langle \xi_1, \xi_2\rangle. \tag{2.2}$$

Note that θ is a formal one-form with property $d\theta = \zeta$. When paired with another differential form, there is a wedge product, $\theta\wedge$, which for notational simplicity, as in (2.2), we will just assume without writing out explicitly.

In this paper, we take the inner product on $\Omega^*(M, Ad\,P)$ to be the standard one induced from the Riemannian metric g and the Killing form of the Lie group G. Specifically,

$$\langle \eta_1, \eta_2 \rangle = c_G \int_M \mathrm{tr}\,[\eta_1 \wedge *\eta_2]\ , \tag{2.3}$$

where the constant $c_G = N-2$ if $G = SO(N)$. With (2.2) and (2.3), we can also express $\langle -, - \rangle_{\mathcal{C}}$ as an integral by introducing the Hodge star operator for cone forms

$$\begin{aligned} *_{\mathcal{C}} : \mathrm{Cone}^k(\zeta)(M, Ad\,P) \ &\rightarrow \quad \mathrm{Cone}^{m+1-k}(\zeta)(M, Ad\,P) \\ \eta + \theta\xi \qquad &\mapsto \quad *\,\xi + \theta\,(-1)^{|\eta|} *\eta \end{aligned} \tag{2.4}$$

which allows us to write

$$\langle \eta_1 + \theta\xi_1, \eta_2 + \theta\xi_2 \rangle_{\mathcal{C}} = c_G \int_M \mathrm{tr}\, \frac{\partial}{\partial\theta} [(\eta_1 + \theta\xi_1) \wedge *_{\mathcal{C}}(\eta_2 + \theta\xi_2)] \tag{2.5}$$

where $\frac{\partial}{\partial\theta}(\theta\xi) = \xi$ for any $\xi \in \Omega^*(M, Ad\,P)$.

Having defined the inner product on $\mathrm{Cone}^*(\zeta)(M, Ad\,P)$, we write down the Yang-Mills functional associated with the cone connection. Given a connection $A \in \Omega^1(M, Ad\,P)$ and a section $B \in \Omega^0(M, Ad\,P)$, we write the cone connection as $\mathcal{A} = A + \theta B \in \mathrm{Cone}^1(\zeta)(M, Ad\,P)$. Locally, if we write the usual covariant derivative as $d_A = d + A$, then the cone covariant derivative is given by $D_{\mathcal{C}} = d + \mathcal{A} = d + (A + \theta B)$ and the cone curvature takes the form

$$\begin{aligned} \mathcal{F}_{\mathcal{A}} = D_{\mathcal{C}} D_{\mathcal{C}} &= d(A + \theta B) + (A + \theta B) \wedge (A + \theta B) \\ &= (F_A + \zeta B) - \theta\, d_A B\,, \end{aligned}$$

having used the relation $\zeta = d\theta$. The norm squared of the cone curvature is the cone Yang-Mills functional

$$\begin{aligned} S_{cYM}(A + \theta B) = \|\mathcal{F}_{\mathcal{A}}\|^2_{\mathcal{C}} &= \|(F_A + \zeta B) - \theta\, d_A B\|^2_{\mathcal{C}} \\ &= \|F_A + \zeta B\|^2 + \|d_A B\|^2\,. \end{aligned}$$

The zero points of the functional gives the cone-flat condition for the pair (A, B):

$$F_A + \zeta B = 0, \qquad d_A B = 0. \tag{2.6}$$

If satisfying (2.6), we will call the pair (A, B) cone-flat connections with respect to ζ, or often, just simply cone-flat connections.

Now to obtain the equations for the critical points of the cone Yang-Mills functional, we consider the first order variation, $(A, B) \to (A + t\eta, B + t\xi)$ where $\eta \in \Omega^1(M, Ad\,P)$ and $\xi \in \Omega^0(M, Ad\,P)$. For the standard curvature, we find

$$F_{A+t\eta} = F_A + t d_A\eta + t^2 \eta \wedge \eta \,,$$

where locally, $d_A\eta = d\eta + A \wedge \eta + \eta \wedge A =: d\eta + [A, \eta]$. Furthermore, we have

$$\begin{aligned}
&S_{cYM}(A + t\eta, B + t\xi)\\
&\qquad = \|F_{A+t\eta} + \zeta(B + t\xi)\|^2 + \|d_{A+t\eta}(B + t\xi)\|^2\\
&\qquad = \|F_A + \zeta B\|^2 + \|d_A B\|^2 + 2t\left(\langle F_A + \zeta B, d_A\eta + \zeta\xi\rangle + \langle d_A B, [\eta, B] + d_A\xi\rangle\right) + o(t)\\
&\qquad = \|\mathcal{F}_{\mathcal{A}}\|^2_{\mathcal{C}} + 2t\big(\langle d_A^*(F_A + \zeta B) + [B, d_A B], \eta\rangle + \langle \zeta^*(F_A + \zeta B) + d_A^* d_A B, \xi\rangle\big) + o(t)
\end{aligned}$$

where for $\dim M = m$,

$$d_A^* = (-1)^{mk+m+1} * d_A * \,, \tag{2.7}$$

and $\zeta^* \colon \Omega^k(M) \to \Omega^{k-2}(M)$ is the adjoint of the $\zeta\wedge$ map and defined to be

$$\zeta^* = (-1)^{(m-k)k} * \zeta * \,. \tag{2.8}$$

Hence, the stationary solutions of the cone Yang-Mills functional satisfy

$$d_A^*(F_A + \zeta B) + [B, d_A B] = 0 \,, \tag{2.9}$$

$$\zeta^*(F_A + \zeta B) + d_A^* d_A B = 0 \,. \tag{2.10}$$

We will call a pair (A, B) satisfying the above cone Yang-Mills equations (2.9)-(2.10) cone Yang-Mills connections.

2.2 Elliptic property of the cone Yang-Mills solutions

In this subsection, we shall prove the following theorem.

Theorem 2.1. *On a closed manifold M, the space of cone Yang-Mills connections modulo gauge equivalence is finite-dimensional.*

Our proof will follow the same line of arguments as Atiyah-Bott's proof of the analogous statement [2, Sec. 4] for the Yang-Mills functional.

Proof. We compute the linearized variation of the cone Yang-Mills equations. Let $(A,B) \to (A+t\eta, B+t\xi)$ where $\eta \in \Omega^1(M, Ad\,P)$ and $\xi \in \Omega^0(M, Ad\,P)$. The linearized variation of the first cone Yang-Mills equation (2.9) is of the form

$$\begin{aligned} &d^*_{A+t\eta}\big(F_{A+t\eta} + \zeta(B+t\xi)\big) + [B+t\xi, d_{A+t\eta}(B+t\xi)] \\ &\quad = d^*_A(F_A+\zeta B) + [B, d_A B] + t\Big(d^*_A d_A\eta - [B,[B,\eta]] + d^*_A(\zeta\xi) + [B, d_A\xi] \\ &\qquad + [\xi, d_A B] + (-1)^{m+1} * [\eta, *(F_A+\zeta B)]\Big) + o(t)\,, \end{aligned} \tag{2.11}$$

and for the second equation (2.10), we find

$$\begin{aligned} &\zeta^* F_{A+t\eta} + \zeta^*\Big(\zeta(B+t\xi)\Big) + d^*_{A+t\eta} d_{A+t\eta}(B+t\xi) \\ &\quad = \zeta^* F_A + \zeta^*(\zeta B) + d^*_A d_A B + t\Big(d^*_A d_A \xi + \zeta^*(\zeta\xi) + \zeta^* d_A\eta - d^*_A[B,\eta] - *[\eta, *d_A B]\Big) + o(t)\,. \end{aligned} \tag{2.12}$$

With (2.11)-(2.12), we see that $A + \theta B + t(\eta+\theta\xi) + o(t)$ describes a curve of the critical points of the cone Yang-Mills functional if and only if

$$\Big\{d^*_A d_A\eta - [B,[B,\eta]] + d^*_A(\zeta\xi) + [B, d_A\xi]\Big\} + [\xi, d_A B] + (-1)^{m+1} * [\eta, *(F_A+\zeta B)] = 0 \tag{2.13}$$

$$\Big\{d^*_A d_A\xi + \zeta^*(\zeta\xi) + \zeta^* d_A\eta - d^*_A[B,\eta]\Big\} - *[\eta, *d_A B] = 0 \tag{2.14}$$

We can write (2.13)-(2.14) more simply in terms of $D_{\mathcal{C}}$, the cone covariant derivative. Locally, if $d_A = d + A$, then $D_{\mathcal{C}} = d + (A+\theta B)$ and

$$D_{\mathcal{C}}(\eta+\theta\xi) = d_A\eta + \zeta\wedge\xi + \theta([B,\eta] - d_A\xi)\,,$$

which we can express in matrix form as

$$D_{\mathcal{C}}\begin{pmatrix}\eta\\ \xi\end{pmatrix} = \begin{pmatrix} d_A & \zeta\wedge \\ [B,-] & -d_A\end{pmatrix}\begin{pmatrix}\eta\\ \xi\end{pmatrix}\,, \tag{2.15}$$

and its adjoint with respect to the metric in (2.5) by

$$D^*_{\mathcal{C}}\begin{pmatrix}\eta\\ \xi\end{pmatrix} = \begin{pmatrix} d^*_A & -[B,-] \\ \zeta^* & -d^*_A\end{pmatrix}\begin{pmatrix}\eta\\ \xi\end{pmatrix}\,. \tag{2.16}$$

Here, we have noted that the adjoint $[B,-]^* = -[B,-]$, since for any $\gamma, \gamma' \in \Omega^k(M, Ad\,P)$, we have

$$\langle [B,-]^*(\gamma), \gamma'\rangle = \langle \gamma, [B,\gamma']\rangle = \langle [\gamma, B], \gamma'\rangle = -\langle [B,\gamma], \gamma'\rangle\,,$$

having used the adjoint-invariance property of the inner product (2.1). Now, the composition

$$D_{\mathcal{C}}^* D_{\mathcal{C}} \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} d_A^* d_A - [B,[B,-]] & d_A^*(\zeta \wedge -) + [B, d_A -] \\ \zeta^* d_A - d_A^*[B,-] & d_A^* d_A + \zeta^*(\zeta \wedge -) \end{pmatrix} \begin{pmatrix} \eta \\ \xi \end{pmatrix}$$

which reproduces exactly the terms within the curly brackets $\{\ \ldots\}$ in (2.13)-(2.14). The remaining terms can be expressed as

$$\begin{aligned}(-1)^m *_{\mathcal{C}} [\,\eta + \theta\xi\,,\, *_{\mathcal{C}}\, \mathcal{F}_{\mathcal{C}}\,] &= (-1)^m *_{\mathcal{C}} [\,\eta + \theta\xi\,,\, *_{\mathcal{C}}(F_A + \zeta\, B - \theta\, d_A B)] \\ &= (-1)^m *_{\mathcal{C}} [\eta + \theta\xi\,, \theta * (F_A + \zeta\, B) - * d_A B] \\ &= (-1)^{m+1} *_{\mathcal{C}} \{[\eta, * d_A B] + \theta\,([\xi, * d_A B] + [\eta, *(F_A + \zeta\, B)])\} \\ &= [\xi, d_A B] + (-1)^{m+1} * [\eta, *(F_A + \zeta\, B)] - \theta\, * [\eta, * d_A B]\end{aligned}$$

The last equation follows from the relation $(-1)^{m+1} * [\xi, * d_A B] = [\xi, d_A B]$. This can be seen by writing $d_A B = \sum \mu_i \otimes \alpha_i$ and $\xi = \sum \xi_i \otimes \alpha_i$ where $\mu_i \in \Omega^1(M)$, $\xi_i \in \Omega^0(M)$ and $\{\alpha_i\}$ is a basis of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Then

$$\begin{aligned}(-1)^{m+1} * [\xi, * d_A B] &= (-1)^{m+1} \sum *[\xi_i \otimes \alpha_i, (*\mu_j) \otimes \alpha_j] \\ &= (-1)^{m+1} \sum * * (\xi_i \mu_j) \otimes [\alpha_i, \alpha_j] \\ &= \sum \xi_i \mu_j \otimes [\alpha_i, \alpha_j] \\ &= [\xi, d_A B]\,.\end{aligned}$$

In all, the linearized variation condition (2.13)-(2.14) can be expressed concisely as

$$D_{\mathcal{C}}^* D_{\mathcal{C}}(\eta + \theta\xi) + (-1)^m *_{\mathcal{C}} [\,\eta + \theta\xi\,,\, *_{\mathcal{C}}\, \mathcal{F}_{\mathcal{C}}\,] = 0\,. \tag{2.17}$$

Now, under a gauge transformation

$$A + \theta B \to g(A + \theta B)g^{-1} + g d g^{-1} = A + \theta B - t(D_C\, \alpha) + o(t)$$

having substituted on the right-hand-side $g = e^{t\alpha}$ for $\alpha \in \mathfrak{g}$. To quotient out a linear variation that is a gauge transformation, we impose that the deformation satisfy the gauge-fixing condition

$$D_{\mathcal{C}}^*(\eta + \theta\xi) = 0\,. \tag{2.18}$$

Combining (2.17) with (2.18), the linearized deformation $(\eta + \theta\xi)$ is characterized by solving the differential system

$$(D_{\mathcal{C}}^* D_{\mathcal{C}} + D_{\mathcal{C}} D_{\mathcal{C}}^*)(\eta + \theta\xi) + (-1)^m *_{\mathcal{C}} [\, \eta + \theta\xi \,, *_{\mathcal{C}} \mathcal{F}_{\mathcal{C}} \,] = 0 \,. \tag{2.19}$$

This is an elliptic system since the cone Laplacian $\Delta_{\mathcal{C}} = D_{\mathcal{C}}^* D_{\mathcal{C}} + D_{\mathcal{C}} D_{\mathcal{C}}^*$ is elliptic. (With (2.15)-(2.16), it is easily seen that the contribution to the principal symbol of $\Delta_{\mathcal{C}}$ comes only from the standard Laplacian $d^*d + dd^*$ on the diagonal components.) Hence, this implies that the tangent space of cone Yang-Mills connections modulo gauge equivalence is finite-dimensional. $\square$

2.3 Comparison with Yang-Mills solutions

Since the cone Yang-Mills functional is closely related to the Yang-Mills functional, it is natural to ask about the relationship between their critical points (i.e. solutions). We shall study this issue starting first with some special cases.

(i) Cone Yang-Mills solutions with $B = 0$.

Consider first the case of cone Yang-Mills solutions with $B = 0$. In this setting, the cone Yang-Mills equations simplify to

$$d_A^* F_A = 0 \,, \qquad \zeta^* F_A = 0 \,. \tag{2.20}$$

Hence, cone Yang-Mills solutions with $B = 0$ are a subset of Yang-Mills solutions satisfying additionally the second condition of (2.20).

In particular, when M is a Kähler manifold and $\zeta = \omega$ is the Kähler metric, a hermitian Yang-Mills connection that satisfies the conditions

$$F_A^{2,0} = F_A^{0,2} = 0 \,, \qquad \omega^{n-1} \wedge F_A = 0 \,,$$

is also a cone Yang-Mills solution satisfying (2.20) with $B = 0$. This is because $\omega^* F_A = 0$ is equivalent to $\omega^{n-1} \wedge F_A = 0$ for a Kähler metric.

(ii) Cone Yang-Mills solutions with $\zeta = 0$.

Another special case is that of setting $\zeta = 0$. Notice first for the zero point of the cone Yang-Mills functional satisfying the cone-flat condition (2.6), the pair (A, B) is cone-flat if and only if A is a flat connection (i.e. $F_A = 0$) and $d_A B = 0$. In general, we have the following:

Lemma 2.2. *Let M be a closed manifold and let $\zeta = 0$. Then (A, B) is a cone Yang-Mills solution if and only if A is a Yang-Mills connection and $d_A B = 0$.*

Proof. When $\zeta = 0$, ζ^* is a zero map. The second equation of cone Yang-Mills condition (2.10) becomes just $d_A^* d_A B = 0$, which implies $d_A B = 0$ on a compact manifold. The first equation (2.9) then simplifies to $d_A^* F_A = 0$. Hence, A must be a Yang-Mills connection with B covariantly constant. □

(iii) Cone Yang-Mills solutions when ζ is a harmonic form.

Instead of vanishing, suppose ζ is a harmonic two-form, i.e. $d\zeta = d^*\zeta = 0$. We can obtain a similar statement to Lemma 2.2 if we require additionally that the cone Yang-Mills solution is cone-flat.

Lemma 2.3. *Suppose (A, B) is a cone-flat solution, i.e. a zero point of the cone Yang-Mills functional. If ζ is a harmonic form, then A is a Yang-Mills connection.*

Proof. By assumption, we have $F_A = -\zeta B$ and $d_A B = 0$. So

$$d_A * F_A = -d_A(* \zeta B) = -(d * \zeta)B - (-1)^{m-2}(*\zeta) d_A B = 0\,,$$

implying that A is a Yang-Mills connection. □

Remark 2.4. When (M^{2n}, ω) is a symplectic manifold and $\zeta = \omega$, then a connection satisfying the curvature condition $F_A = -\omega B$ such that $d_A B = 0$ is called a symplectically-flat connection as introduced in [10]. Symplectically-flat connections are Yang-Mills connections with respect to a compatible metric. This agrees with Lemma 2.3 above since with respect to a compatible metric, $* \omega = \omega^{n-1}/(n-1)!$ which implies $d^*\omega = 0$, i.e. ω is a harmonic form.

(iv) Cone Yang-Mills solutions with connection one-form not Yang-Mills.

Thus far, we have described special cone Yang-Mills solution pairs (A, B) where the connection part A is Yang-Mills. Such is not the generic case. Below, we shall give a simple cone-flat solution (A, B) where A is not Yang-Mills. In order not to contradict Lemma 2.3, the ζ in the example below is not harmonic.

Example 2.5. Let Σ be a Riemann surface and $A' \in \Omega^1(\Sigma, Ad\,P)$ be a non-flat Yang-Mills connection on Σ. Let ζ be the volume form of Σ normalized such that the total volume of Σ is one. Define M to be the three-dimensional circle bundle $\pi : M \to \Sigma$ with Euler class given by ζ. Since A' is Yang-Mills, we can write its curvature as $F_{A'} = \zeta\Phi$ such that $d_{A'}\Phi = 0$. For simplicity, we will also use ζ, A' and Φ to denote their pullbacks on M.

Let $\theta \in \Omega^1(M)$ be the global angular one-form of the circle bundle M, i.e. $d\theta = \zeta$, and also let $c \in \mathbb{R}$. For $A = A' + c\,\theta\Phi$ and $B = -(1+c)\Phi$ on the pullback bundle $\pi^*(Ad\,P)$, we find

$$F_A = F_{A'} + c\,\zeta\Phi - c\,\theta d_{A'}\Phi = (1+c)\zeta\Phi = -\zeta B\,,$$

and

$$d_A B = -(1+c)d_{A'}\Phi - c(1+c)\theta[\Phi,\Phi] = 0\,,$$

that is, (A, B) is a cone-flat solution. However, the connection form A is not Yang-Mills in general with respect to the volume form on the circle bundle M, $dvol_M = \zeta \wedge \theta$, since

$$d_A^* F_A = *\, d_A * F_A = - *\, d_A(\theta B) = - * \zeta B = (1+c)\theta\Phi$$

which is non-zero unless $c = -1$. (We have assumed A' is not a flat connection, and therefore, $F_{A'} = \zeta\Phi \neq 0$.) Hence, generally, for any $c \neq -1$, A is not a Yang-Mills connection.

(v) Yang-Mills connections that can not be a part of a cone Yang-Mills solution.

For a cone Yang-Mills solution, (A, B), we have seen that A need not be a Yang-Mills connection. In the reverse direction, we can ask if given a Yang-Mills connection A, will there always exist a B such that (A, B) is a cone Yang-Mills solution? The answer is no, as is shown in the example below.

Example 2.6. Let $M = T^4 = \mathbb{R}^4/2\pi\mathbb{Z}^4$ be the 4-torus described by identifications $x_i \sim x_i + 2\pi n_i$ for $i = 1, 2, 3, 4$ and $n_i \in \mathbb{Z}$. Let $\zeta = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. We will take the Riemannian metric to be

$$g = (dx_1)^2 + (dx_2)^2 + \frac{1}{f}(dx_3)^2 + f\,(dx_4)^2$$

where $f = \dfrac{3 + 2\sin 2x_2 \cos x_3}{1 - \frac{1}{2}\sin 2x_2 \cos x_3}$.

Consider a principal $U(1)$ bundle over M with the circle coordinate identified by

$$y \sim y + 2\pi n_5 - n_1 x_3$$

for $n_5 \in \mathbb{Z}$. The global connection one-form on the $U(1)$ bundle can be taken to be

$$A = dy + \frac{1}{2\pi} x_1\, dx_3 + \frac{1}{4\pi} \sin 2x_2 \sin x_3\, dx_1 .$$

The curvature two-form is then given by

$$F_A = dA = -\frac{1}{2\pi} \cos 2x_2 \sin x_3 \; dx_1 \wedge dx_2 + \frac{1}{2\pi}(1 - \frac{1}{2} \sin 2x_2 \cos x_3) dx_1 \wedge dx_3 \,.$$

Moreover,

$$\begin{aligned} d * F_A &= \frac{1}{2\pi} d \left[-\cos 2x_2 \sin x_3 \; dx_3 \wedge dx_4 - (1 - \frac{1}{2} \sin 2x_2 \cos x_3) f \; dx_2 \wedge dx_4 \right] \\ &= \frac{1}{2\pi} d \left[-\cos 2x_2 \sin x_3 \; dx_3 \wedge dx_4 - (3 + 2 \sin 2x_2 \cos x_3) dx_2 \wedge dx_4 \right] = 0 \,. \end{aligned}$$

Thus, F_A satisfies the abelian Yang-Mills equation. However, there does not exist a function B on T^4 such that (A, B) satisfy the cone Yang-Mills equations (2.9)-(2.10). Equation (2.9) implies in the abelian case $d^*(F_A + \zeta B) = 0$. But since we know already $d^* F_A = 0$, this means B is required to satisfy

$$0 = d * (\zeta B) = d(\zeta B) = \zeta \wedge dB \,. \tag{2.21}$$

With ζ being a non-degenerate 2-form, (2.21) gives the condition $dB = 0$.

However, imposing the second cone Yang-Mills equation (2.10) results in a contradiction. For if $dB = 0$, (2.10) reduces to the condition $\zeta^* F_A + \zeta^* \zeta B = 0$. This implies in particular

$$B = \frac{1}{2} \zeta^* \zeta B = -\frac{1}{2} \zeta^* F_A = \frac{1}{4\pi} \cos 2x_2 \sin x_3 \,, \tag{2.22}$$

and clearly, $dB \neq 0$, which contradicts the condition from (2.21). Hence, there is no B that can satisfy the cone Yang-Mills equations.

2.4 Uniqueness of cone Yang-Mills solutions for a fixed connection

In general, given an arbitrary connection A, there does not exist a B that would make (A, B) a cone Yang-Mills solution. To illustrate this in a simple example, let M be a closed manifold and $\zeta = 0$. Then by Lemma 2.2, we know that A must be a Yang-Mills connection and $d_A B = 0$. Hence, there exists no B that would give a cone Yang-Mills solution if A is not Yang-Mills.

However, it is interesting to ask that given a cone Yang-Mills solution (A, B), how many different B's with A fixed would also be a cone Yang-Mills solution? In the case of $\zeta = 0$ and

M closed, if (A, B) is a cone Yang-Mills solution, then so are all $(A, B + \Phi)$ with $d_A\Phi = 0$. For $\zeta \neq 0$, we are able to obtain results in two special cases. First, for two dimensions and suppose A is a Yang-Mills connection, then we have the following:

Proposition 2.7. *Suppose M is a closed Riemann surface and ζ is its volume form. For each Yang-Mills connection A, there exists a unique B such that (A, B) is a cone Yang-Mills solution. In fact, such a pair (A, B) is always cone-flat.*

Proof. Since A is a Yang-Mills connection on a Riemann surface, $F_A = \zeta\Phi$ with $d_A\Phi = 0$. So we can choose $B = -\Phi$ and then (A, B) is a cone-flat solution. We will show below that this is the unique cone Yang-Mills solution for a fixed A, a Yang-Mills connection.

Generally, suppose (A, B) is a cone Yang-Mills solution. As ζ is a volume form, the condition $\zeta^*(F_A + \zeta B) + d_A^* d_A B = 0$ becomes

$$\Phi + B + d_A^* d_A B = 0\,. \tag{2.23}$$

Let d_A act on both sides. This implies $d_A B = -d_A d_A^* d_A B$, and therefore,

$$\langle d_A B, d_A d_A^* d_A B\rangle = -\|d_A B\|^2 \leq 0\,.$$

On the other hand,

$$\langle d_A B, d_A d_A^* d_A B\rangle = \langle d_A^* d_A B, d_A^* d_A B\rangle = \|d_A^* d_A B\|^2 \geq 0.$$

So $\langle d_A B, d_A d_A^* d_A B\rangle$ has to vanish. It follows that $\|d_A B\|^2 = 0$, which implies, $d_A B = 0$, and by (2.23), $B = -\Phi$. So such B is unique when A is Yang-Mills. □

Another special case where we can constrain B is when ζ is the symplectic form on a symplectic manifold. If the structure group is abelian, then B must be unique.

Proposition 2.8. *Suppose A is a connection on the associated adjoint bundle $Ad\,P$ over a symplectic manifold (M^{2n}, ω) and the Riemannian metric is compatible with ω. Take $\zeta = \omega$. Then there is at most one B that satisfies $[B, d_A B] = 0$ and such that (A, B) is a cone Yang-Mills solution. In particular, if the structure group is abelian, there is at most one cone Yang-Mills solution pair (A, B) for any given connection A.*

Proof. Let $\dim M = 2n$. With respect to a compatible metric, $*\,\zeta = *\,\omega = \frac{1}{(n-1)!}\omega^{n-1}$, and $\zeta^*\zeta = n$

Suppose both (A, B_1) and (A, B_2) are cone Yang-Mills solutions. Assume also $[B_1, d_A B_1] = [B_2, d_A B_2] = 0$ which is identically true when the structure group is abelian. Then (2.9)-(2.10) imply for B_1,

$$-*\, d_A(*F_A + \frac{1}{(n-1)!}\omega^{n-1} B_1) = 0\,,$$
$$\zeta^* F_A + nB_1 + d_A^* d_A B_1 = 0\,,$$

and B_2 satisfies identical equations. So by the first equation, we have

$$\omega^{n-1} \wedge d_A B_1 = \omega^{n-1} \wedge d_A B_2.$$

It follows that $d_A B_1 = d_A B_2$ because $\omega^{n-1} : \Omega^1(M) \to \Omega^{2n-1}(M)$ is an isomorphism. With this, the second equation becomes

$$n(B_1 - B_2) = d_A^* d_A(B_2 - B_1) = 0\,,$$

which implies, $B_1 = B_2$. □

The above proposition avoids considering the term $[B, d_A B]$. If $[B, d_A B] = 0$ holds true for all cone Yang-Mills solutions (A, B), then we would have obtained the uniqueness of B when ζ is the symplectic structure. But as we shall see in Example 3.2 in the next section, a cone Yang-Mills solution in general need not satisfy $[B, d_A B] = 0$.

3 Special solutions of the cone Yang-Mills functional

3.1 Two-dimensional solutions with abelian gauge group

Consider cone Yang-Mills solutions on a closed Riemann surface with abelian structure group. In this case, the Euler-Lagrange equations reduce to

$$d^*(F_A + \zeta B) = 0\,, \tag{3.1}$$
$$\zeta^*(F_A + \zeta B) + d^* dB = 0\,. \tag{3.2}$$

The first equation (3.1) implies that $*(F_A + \zeta B)$ is a constant c. So we can assume that

$$F_A + \zeta B = c\,\omega\,, \tag{3.3}$$

where ω is the volume form of M. By Hodge decomposition, we can write

$$\zeta = c'\omega + dd^*(f\omega) = c'\omega - d * df \,,$$

where c' is a constant and f is a function on M.

For any function ϕ on M, we have

$$\begin{aligned}
\langle \phi, \zeta^*(F_A + \zeta B) \rangle &= \langle \zeta\phi, (F_A + \zeta B) \rangle = \langle \phi(c'\omega - d * df), c\,\omega \rangle \\
&= \int_M c\phi(c'\omega - d * df) \\
&= \int_M cc'\phi\,\omega - \int_M d(c\phi * df) + \int_M c\, d\phi \wedge * df \\
&= \langle \phi, cc' \rangle + \langle d\phi, c\, df \rangle \\
&= \langle \phi, cc' + c\, d^* df \rangle
\end{aligned}$$

and therefore,

$$\zeta^*(F_A + \zeta B) = cc' + c\, d^* df \,. \tag{3.4}$$

Plugging this into (3.2), we find that cc' is a d^*-exact constant; hence, cc' must be zero.

If $c' \neq 0$, then c must vanish and then $F_A + \zeta B = 0$. By (3.2), $d^* dB = 0$ which implies $dB = 0$ since M is closed. Thus, we have obtained the following statement.

Proposition 3.1. *Suppose M is a closed Riemann surface, ζ is a non-exact two-form on M, and the structure group is abelian. Then all cone Yang-Mills solutions are cone-flat.*

Now we consider the case $c' = 0$, that is, $\zeta = -d * df$ is a d-exact form. Together, (3.2) and (3.4) imply $d^* d(cf + B) = 0$, and so $cf + B$ is a constant. We can thus write $B = -cf + c''$ for some constant c''. Hence, by (3.3), $F_A = c\,\omega - \zeta B = c\,\omega + (c'' - cf) d * df$. Therefore, we find that the constants c and c'' parametrize the cone Yang-Mills solutions. And finally, in the special case where $f = 0$, i.e. ζ vanishes, the critical points must satisfy $F_A = c\,\omega$ and $B = c''$.

3.2 Three-dimensional solutions from duality relations

Recall in four dimensions, there are special Yang-Mills solutions that satisfy the first-order self-dual/anti-self-dual conditions, $* F_A = \pm F_A$. The intuition that cone Yang-Mills functional can be interpreted as a dimensional reduction of the Yang-Mills functional suggests an analogous duality condition $*_{\mathcal{C}}\, \mathcal{F}_{\mathcal{A}} = \pm\, \mathcal{F}_{\mathcal{A}}$ over 3-manifolds, where $\mathcal{F}_{\mathcal{A}} = (F_A + \zeta B) - \theta d_A B$. With the $*_{\mathcal{C}}$ acting on $\mathrm{Cone}^k(\zeta)$ given by (2.4), we have

$$
\begin{aligned}
*_{\mathcal{C}}\, \mathcal{F}_{\mathcal{A}} &= *_{\mathcal{C}} \left[(F_A + \zeta B) - \theta d_A B\right] \\
&= *(-d_A B) + \theta\left[*(F_A + \zeta B)\right].
\end{aligned}
$$

Hence, we find that $*_{\mathcal{C}}\, \mathcal{F}_{\mathcal{A}} = \mp\, \mathcal{F}_{\mathcal{A}}$ implies

$$
F_A + \zeta B = \pm * d_A B \tag{3.5}
$$

Note that when $\zeta = 0$, the condition becomes the Bogomolny monopole equations.

We will show that a solution of (3.5) is automatically a solution of the cone Yang-Mills equations in two different ways. First, we can check directly that (3.5) implies the cone Yang-Mills equations (2.9)-(2.10). Applying d_A^* to (3.5), we find the following:

$$
\begin{aligned}
d_A^*(F_A + \zeta B) &= \pm\, d_A^* * d_A B \\
&= \pm * d_A d_A B = \pm * [F_A, B] \\
&= [(d_A B - \zeta B), B] = [d_A B, B]
\end{aligned}
$$

where we have used the three-dimensional relations $** = 1$, and $d_A^* = (-1)^k * d_A *$ acting on a k-form. The above implies the first cone Yang-Mills equation (2.9), $d_A^*(F_A+\zeta B)+[B, d_A B] = 0$. Furthermore, it also follows from (3.5) that

$$
\begin{aligned}
d_A^* d_A B &= - * d_A(* d_A B) = \mp * d_A(F_A + \zeta B) \\
&= \mp * \zeta \wedge d_A B = \mp * \zeta\ * (*d_A B) \\
&= -\zeta^*(F_A + \zeta B)
\end{aligned}
$$

which implies the second cone Yang-Mills equation (2.10), $\zeta^*(F_A + \zeta B) + d_A^* d_A B = 0$.

In the second method, analogous to the standard Yang-Mills instanton argument, we can express the three-dimensional cone Yang-Mills functional in the following manner:

$$
\begin{aligned}
\|\mathcal{F}_{\mathcal{A}}\|^2_{\mathcal{C}} &= \int_M c_G \operatorname{tr}[(F_A + \zeta B) \wedge *(F_A + \zeta B)\ +\ d_A B \wedge * d_A B)] \\
&= \int_M c_G \operatorname{tr}\left[(F_A + \zeta B \mp * d_A B) \wedge *(F_A + \zeta B \mp * d_A B)\right]\ \pm\ 2 \int_M c_G \operatorname{tr}\left[(F_A + \zeta B) \wedge d_A B\right] \\
&\geq \pm\, 2 \int_M c_G \operatorname{tr}\left[(F_A + \zeta B) \wedge d_A B\right].
\end{aligned} \tag{3.6}
$$

The equality holds only when $F_A + \zeta B = \pm * d_A B$. Importantly, the bounding integral is a boundary term

$$\begin{aligned} Q &= \int_M c_G \,\mathrm{tr}\,[(F_A + \zeta B) \wedge d_A B] = \int_M c_G \,\mathrm{tr}\left[d\left(F_A\, B + \frac{1}{2}\,\zeta\, B^2\right) - (d_A F_A) B + \zeta \wedge B[A, B]\right] \\ &= \int_{\partial M} c_G \,\mathrm{tr}\left[F_A\, B + \frac{1}{2}\,\zeta\, B^2\right] \end{aligned} \tag{3.7}$$

which is the action of a two-dimensional topological BF-type theory. We note that any infinitesimal local variation of (A, B) away from the boundary does not affect the bound. Hence, $F_A + \zeta B = \pm * d_A B$ must be a critical point of the cone Yang-Mills functional.

The duality-type condition of (3.5) helps simplify the search for cone Yang-Mills solutions in three dimensions. Notably, it is first-order and hence more tractable compared with the second-order cone Yang-Mills equations (2.9)-(2.10). As mentioned, when $\zeta = 0$, the condition becomes the Bogomolny equations and then the known three-dimensional Bogomolny-Prasad-Sommerfeld (BPS) monopole solutions are trivially cone Yang-Mills solutions with Q of (3.7) being proportional to the magnetic monopole charge (for a review, see [9]). More generally, for $\zeta \neq 0$, we can look for self-dual/anti-self-dual Yang-Mills instanton solutions in four dimensions on spaces that can be described as a circle bundle over a three-manifold. If the four-dimensional Yang-Mills instanton solutions are invariant under the S^1 circle action, then we can dimensionally reduce over the circle and obtain solutions that satisfy (3.5). We give such an example below coming from the Taub-NUT gravitational instanton.

Example 3.2. The Taub-NUT gravitational instanton solution can be thought of as a self-dual Yang-Mills solutions of a tangent bundle with structure group $SU(2) \subset SO(4)$. With a point removed, the four-dimensional space can be considered as a circle bundle, $S^1 \to X \to \mathbb{R}^3 - \{0\}$ (see, for example, the description in [5]). Dimensionally reducing this solution leads to a non-abelian cone Yang-Mills solution on $M = \mathbb{R}^3 - \{0\}$ satisfying (3.5).

We start with the Taub-NUT metric written in Gibbons-Hawking form:

$$ds^2_{TN} = e^{2\phi}\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right) + e^{-2\phi}\theta^2\,, \tag{3.8}$$

where $e^{2\phi} = 1 + 2/r$ is a positive function on $\mathbb{R}^3 - \{0\}$ with $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ and the d-closed two-form ζ is defined to be

$$\zeta = d\theta = \pm *_0\, d(e^{2\phi})$$

$$= \pm\, \epsilon_{ijk}\, \partial_k \phi\, e^{2\phi}\, dx^i \wedge dx^j \tag{3.9}$$

with $*_0$ being the Euclidean Hodge star on $M = \mathbb{R}^3 - \{0\}$. The requirement that $d\zeta = 0$ gives the following condition on ϕ on $\mathbb{R}^3 - \{0\}$:

$$\sum_{k=1}^{3} \left[\partial_k^2 \phi + 2(\partial_k \phi)^2\right] = 0\,. \tag{3.10}$$

For the Taub-NUT metric in (3.8), we have the following basis of moving frame (i.e. orthonormal frame) of 1-forms

$$e^i = e^{\phi} dx^i \quad i = 1, 2, 3\,, \qquad \text{and} \qquad e^4 = e^{-\phi}\theta\,.$$

These result in the following connection 1-forms (for a reference, see [8])

$$\omega^i{}_j = -\omega^j{}_i = e^{-\phi}\left(-\partial_i \phi\, e^j + \partial_j \phi\, e^i \mp \epsilon_{ijk} \partial_k \phi\, e^4\right) \tag{3.11}$$

$$\omega^i{}_4 = -\omega^4{}_i = e^{-\phi}\left(\pm \epsilon_{ijk} \partial_j \phi\, e^k + \partial_i \phi\, e^4\right) \tag{3.12}$$

which satisfy the torsionless condition $de^r + \omega^r{}_s \wedge e^s = 0$. (Regarding indices, we will let the indices i, j, k, l, p, q take values in the set $\{1, 2, 3\}$ and r, s, t, u take values in the set $\{1, 2, 3, 4\}$.) Notice that these connection 1-forms are also anti-self-dual/self-dual in the sense that

$$\omega_{rs} = \mp\, \frac{1}{2}\, \epsilon_{rstu}\, \omega_{tu}\,.$$

Hence, the structure group of the bundle reduces to an $SU(2)$ subgroup of $SO(4)$. It can be checked that the resulting Taub-NUT curvature two-form $R^r{}_s = d\omega^r{}_s + \omega^r{}_t \wedge \omega^t{}_s$ is correspondingly anti-self-dual/self-dual.

Now to perform a dimensional reduction, we take the metric on X to be

$$ds_X^2 = e^{2\phi} ds_{TN}^2 = e^{4\phi}\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right) + \theta^2\,, \tag{3.13}$$

conformally scaled by the $e^{2\phi}$ factor in order match the desired metric form of $ds_X^2 = ds_M^2 + \theta^2$ as in (1.5). We note that the four-dimensional Yang-Mills anti-self-dual/self-dual curvature equation is invariant under an overall conformal scaling of the metric. Therefore, for the cone pair (A, B) on $M = \mathbb{R}^3 - \{0\}$, we can just read off from (3.11)-(3.12) by setting $\mathcal{A}^r{}_s = (A + \theta B)^r{}_s = \omega^r{}_s$. We find

$$A^i{}_j = -A^j{}_i = -\partial_i \phi\, dx^i + \partial_j \phi\, dx^j\,, \qquad B^i{}_j = -B^j{}_i = \mp\, \epsilon_{ijk} e^{-2\phi} \partial_k \phi\,, \tag{3.14}$$

$$A^i{}_4 = -A^4{}_i = \pm\,\epsilon_{ijk}\partial_j\phi dx^k\,, \qquad\qquad B^i{}_4 = -B^4{}_i = e^{-2\phi}\partial_i\phi\,. \tag{3.15}$$

These lead to the following:

$$\begin{aligned}
(F_A)^i{}_j &= -\partial_i\partial_k\phi\; dx^k\wedge dx^j + \partial_j\partial_k\phi\; dx^k\wedge dx^i - \epsilon_{ikl}\epsilon_{jpq}\,\partial_k\phi\partial_p\phi\; dx^l\wedge dx^q\\
&\qquad + \partial_i\phi\partial_k\phi\; dx^k\wedge dx^j - \partial_j\phi\partial_k\phi\; dx^k\wedge dx^i - (\partial_k\phi)^2 dx^i\wedge dx^j\,,\\
(F_A)^i{}_4 &= \pm\left(\partial_i\phi\,\epsilon_{jkl}\partial_j\phi\; dx^k\wedge dx^l - \epsilon_{ijk}\partial_j\partial_l\phi\; dx^k\wedge dx^l\right),\\
(d_AB)^i{}_j &= \pm\, e^{-2\phi}\left[2\,\epsilon_{ijk}\,\partial_k\phi\partial_l\phi\; dx^l - \epsilon_{ijk}\partial_k\partial_l\phi\; dx^l + 2\left(\partial_i\phi\,\epsilon_{jkl} - \partial_j\phi\,\epsilon_{ikl}\right)\partial_k\phi\; dx^l\right],\\
(d_AB)^i{}_4 &= e^{-2\phi}\left(-4\,\partial_i\phi\partial_j\phi\; dx^j + 2(\partial_j\phi)^2 dx^i + \partial_i\partial_j\phi\; dx^j\right).
\end{aligned}$$

With (3.9)-(3.10), it can be straightforwardly checked that the above solution satisfies $F_A+\zeta B = \pm * d_AB$ where the Hodge star is defined with respect to the three-dimensional metric

$$ds_M^2 = e^{4\phi}\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right)\,.$$

Hence, (A,B) as defined in (3.14)-(3.15) gives us a highly non-trivial, non-abelian solution of the cone Yang-Mills equations. Moreover, for this solution, it can be checked that $[B, d_AB]\neq 0$.

Remark 3.3. We can consider dimensionally reducing the three-dimensional duality condition $*_{\mathcal{C}}\,\mathcal{F}_{\mathcal{A}} = \mp\,\mathcal{F}_{\mathcal{A}}$ over another circle down to a two-dimensional manifold N. For the metric, we will assume $ds_M^2 = ds_N^2 + \chi^2$ where χ is the global connection one-form (i.e. the global angular form) of the circle bundle M over N. The dimensionally reduced three-dimensional local connection can then be expressed as $A+\chi b$ where $A\in\Omega^1(N, Ad\,P)$ and $b\in\Omega^0(N, Ad\,P)$. The three-dimensional cone curvature then takes the form

$$\mathcal{F}_{\mathcal{A}} = F_{A+\chi b} + \zeta B - \theta d_{A+\chi b}B = F_A + \zeta B + (d\chi)b - \chi d_Ab - \theta d_AB - \theta\chi[b,B]\,,$$

with

$$*_{\mathcal{C}}\,\mathcal{F}_{\mathcal{A}} = \theta\chi\left[*\left(F_A + \zeta B + (d\chi)b\right)\right] - \theta * d_Ab + \chi * d_AB - *[b,B]\,.$$

The condition $*_{\mathcal{C}}\,\mathcal{F}_{\mathcal{A}} = \mp\,\mathcal{F}_{\mathcal{A}}$ then implies the following two equations:

$$F_A + \zeta B + (d\chi)b = \pm * [b,B]\,, \tag{3.16}$$

$$* \, d_AB = \pm\, d_Ab\,. \tag{3.17}$$

In the case where the circle is trivially fibered over N, then we can set $d\chi = 0$. Moreover, when $\zeta = d\chi = 0$, the above equations become equivalent to Hitchin's equations.

4 Classification of cone-flat bundles with respect to a non-degenerate two-form

In this section, we study the question what bundles can carry a pair (A, B) satisfying the cone-flat condition:

$$F_A + \zeta B = 0\,, \qquad d_A B = 0\,. \tag{4.1}$$

Now if $\zeta = 0$, the cone-flat condition reduces to (1) $F_A = 0$, i.e. A is a flat connection, and (2) $d_A B = 0$, that is, B is a covariantly constant section. Since we can always set $B = 0$, the $\zeta = 0$ cone-flat bundles are just the usual flat bundles, and flat bundles are well-known to be classified by the conjugacy classes of homomorphisms from $\pi_1(M)$ to the structure group G of the fiber bundle (see, for example [6]). As the classification for the $\zeta = 0$ case is well-understood, we will in the following always assume that $\zeta \neq 0$.

To simplify our consideration, we will additionally assume that ζ is a non-degenerate two-form. Being d-closed and non-degenerate, ζ is then a symplectic structure and M must be even-dimensional. Additionally, ζ being non-degenerate leads to two simplifications in considering cone-flat pairs (A, B). First, non-degeneracy ensures that ζ nowhere vanishes, and hence, the first cone-flat equation $F_A = -\zeta B$, implies B is determined by the connection A. Second, the second cone-flat equation $d_A B = 0$ is automatically satisfied and redundant when $\dim M \geq 4$. For the first cone-flat equation together with the Bianchi identity imply

$$d_A F_A = -\zeta \wedge d_A B = 0\,. \tag{4.2}$$

If ζ is non-degenerate and $\dim M \geq 4$, then (4.2) implies $d_A B = 0$. In all, ζ is non-degenerate simplifies our consideration of cone-flat pairs (A, B) to checking only that that the curvature is proportional to ζ, i.e. $F_A = -\zeta\, B$, and only if $\dim M = 2$, we need to also check that the resulting B is covariantly constant.

When M is a two-dimensional Riemann surface, ζ being non-degenerate implies that it is the product of a nowhere vanishing function times the volume form. If we take ζ to be the volume form, then we saw in (1.12) that the cone-flat condition (4.1) becomes equivalent to the Yang-Mills equation for A with $B = - * F_A$. Concerning Yang-Mills connections in two dimensions, Atiyah-Bott [2, Theorem 6.7] gave a classification of principal bundles over Riemann surfaces that carry a Yang-Mills connection. So we ask whether Atiyah-Bott's classification of Yang-Mills bundles in two dimensions can be extended to cone-flat bundles in $\dim M = 2n \geq 2$ with respect to a closed, non-degenerate two-form ζ. Indeed, we obtain the following for principal bundles which generalizes Atiyah-Bott's result.

Theorem 4.1. *Let M be a path connected manifold, $\zeta \in \Omega^2(M)$ be a non-zero, non-degenerate, closed two-form, and G be a Lie group. There exists a bijective correspondence between the following sets:*

$$\left\{\begin{array}{c}\text{isomorphism classes of cone-flat connections}\\ \text{with respect to } \zeta \text{ on } G\text{-bundles over } M\end{array}\right\} \simeq \left\{\begin{array}{c}\text{conjugacy classes of}\\ \text{homomorphisms } \rho : \Gamma \to G\end{array}\right\},$$

where Γ is an $\mathbb{R}/\overline{H}$ extension of $\pi_1(M)$ with $\overline{H} \subset \mathbb{R}$ being the closure of the group

$$H := \left\{ \int_{\mathcal{S}} \zeta \;\middle|\; \mathcal{S} \text{ is a representative in } \pi_2(M) \right\}. \tag{4.3}$$

As in the theorem, we will assume in the remainder of this section that M is path-connected. We can give a more explicit description of Γ in terms of path spaces. To do so, we first introduce some notations.

Denote by $\Psi(M)$ the path space consisting of equivalence classes of closed, piecewise smooth paths on M. Two paths $\alpha_1, \alpha_2 : [0,1] \to M$ are considered to be equivalent in $\Psi(M)$ if they have the same image and orientation. Specifically, α_1 and α_2 are equivalent if there exists a piecewise smooth increasing function $\phi : [0,1] \to [0,1]$ with $\phi(0) = 0$ and $\phi(1) = 1$ such that $\alpha_2 = \alpha_1 \circ \phi$.

For two paths $\alpha_1, \alpha_2 : [0,1] \to M$ such that $\alpha_1(1) = \alpha_2(0)$, we define their multiplication by concatenation:

$$(\alpha_2\alpha_1)(s) = \begin{cases} \alpha_1(2s), & s \in [0, \frac{1}{2}]\,, \\ \alpha_2(2s-1), & s \in [\frac{1}{2}, 1]\,. \end{cases}$$

A connected subpath of a path is also a path. We define

$$\alpha_{(t)}(s) = \alpha(s\,t)\,, \qquad t \in [0,1]\,.$$

which is a one-parameter family of subpaths, parametrized by t, all with the same starting point $\alpha(0)$, and ending point being at $\alpha(t)$.

The following subsets of $\Psi(M)$ will be of interest:

- $\Psi_a^b(M)$ be the space of paths from a base point $a \in M$ to another point $b \in M$.
- $\Psi_a^a(M)$ be the semigroup in $\Psi(M)$ that consists of loops with base point a.
- $\Psi_{a,0}^a(M)$ be the subsemigroup of $\Psi_a^a(M)$ that consists of contractible loops.
- $\Psi_{a,\zeta}^a(M)$ be the subsemigroup of $\Psi_{a,0}^a(M)$ that consists of loops which are the boundary of some disk (contractible 2-chain with proper orientation) D in M satisfying $\int_D \zeta = 0$.

Quotienting them leads to the following spaces:

1. $\Psi_a^a(M)/\Psi_{a,0}^a(M) = \pi_1(M)$ is the fundamental group where the identity element is the equivalence class of the constant path at a, and the inverse reverses the path orientation, i.e. $\alpha^{-1}(s) = \alpha(1-s)$.

2. $\Psi_{a,0}^a(M)/\Psi_{a,\zeta}^a(M) \cong \mathbb{R}/H$. As $\zeta \neq 0$, for any real number c there exists a contractible loop γ that is the boundary of some disk D such that $\int_D \zeta = c$. So $\Psi_{a,0}^a(M)/\Psi_{a,\zeta}^a(M)$ is equivalent to a quotient group $\mathbb{R}/H$ by the identification of a loop $\gamma \mapsto [\int_D \zeta]$, where
$$H = \left\{ \int_{\mathcal{S}} \zeta \;\Big|\; \mathcal{S} \text{ is a representative in } \pi_2(M) \right\}.$$
The group H comes from noting that it is possible that the sum of two contractible curves $\partial D_1 + \partial D_2$ can come from the vanishing boundary of a two-sphere formed by two hemisphere disks, D_1 and D_2, glued together at the equator such that $\partial D_1 = -\partial D_2$.

3. $\Psi_a^a(M)/\Psi_{a,\zeta}^a(M) =: \Gamma$. This is the extension of $\pi_1(M)$ by $\mathbb{R}/\overline{H}$ that appears in the statement of the theorem. In the case where H is dense and thus $\mathbb{R}/H$ is not a Lie group, $\Gamma = \pi_1(M)$.

Explicitly, there are three possibilities for H. Let H^+ be the subset of H with positive numbers.

Case 1. When H^+ is empty, $H = 0$ and $\mathbb{R}/H = \mathbb{R}$. In this case, $\Gamma = \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$ is an $\mathbb{R}$-extension of $\pi_1(M)$. (For instance, this occurs for any Riemann surface Σ_g with genus $g \geq 1$, since then $\pi_2(\Sigma_g) = 0$.)

Case 2. When H^+ has a minimal number, $H \simeq \mathbb{Z}$ and $\mathbb{R}/H \simeq S^1$. In this case $\Gamma = \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$ is an S^1-extension of $\pi_1(M)$. (This occurs for Riemann surface of genus $g = 0$ and ζ is not an exact form.)

Case 3. When H^+ is non-empty and has no minimal number, H is dense in $\mathbb{R}$ and $\mathbb{R}/H$ is not a Lie group. In this case, $\Gamma = \pi_1(M)$. This case is impossible when $c\,\zeta$ is an integral cohomology class for some number $c \neq 0$.

We now proceed to prove the classification theorem Theorem 4.1. Our proof will be similar to that given by Morrison [7] for Yang-Mills bundles in the two-dimensional case.

We will first consider the case that $\mathbb{R}/H$ is a Lie group. By the discussion above, $\Gamma = \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$ is either an $\mathbb{R}$ or an S^1 extension of $\pi_1(M)$.

Proof for Case 1 and 2: $\mathbb{R}/H = \mathbb{R}$ or S^1.

Step 1. Given a homomorphism $\rho : \Psi_a^a(M)/\Psi_{a,\zeta}^a(M) \to G$, construct a corresponding principal G-bundle P_ρ with a connection A_ρ.

Let $\Psi_a(M)$ be the space of classes of paths starting at $a \in M$, where any two paths δ_1, δ_2 are identified if $\delta_1(1) = \delta_2(1)$ and $(\delta_2)^{-1}\delta_1 \in \Psi_{a,\zeta}^a(M)$. Notice that $\Psi_a(M)$ is a principal bundle over M. The projection to the base space M, $\tau : \Psi_a(M) \to M$, is given by $\delta \mapsto \delta(1)$, and its structure group is $\Gamma = \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$, since any element $\gamma \in \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$ can act on $\delta \in \Psi_a(M)$ on the right by $\delta \mapsto \delta\gamma$.

Given a homomorphism $\rho : \Gamma \to G$, we define the principal G-bundle P_ρ as an associated G-bundle to $\Psi_a(M)$:

$$P_\rho := \Psi_a(M) \times_\rho G = (\Psi_a(M) \times G) \Big/ (\delta\gamma, g) \sim (\delta, \rho(\gamma)g)$$

where $\gamma \in \Gamma = \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$. To denote the equivalence class, we use the bracket to denote a point $u = [\delta, g] \in P_\rho$. Note that G acts on P_ρ by $[\delta, g]h = [\delta, gh]$ for $h \in G$.

To define a connection on a principal bundle, recall that there are two ways to do so. We can define a connection on P_ρ either as a horizontal subspace, $H_u \subset (TP_\rho)_u$ at all $u \in P_\rho$, or as a one-form $A_\rho \in \Omega^1(P_\rho, \mathfrak{g})$. They are related by $H_u = \ker A_\rho\big|_u$. On P_ρ, we define a horizontal distribution by defining the horizontal lifts for any path α on M as follows. Given an arbitrary point $[\delta, g] \in (P_\rho)_{\alpha(0)}$ on the fiber of base point $\alpha(0) = \delta(1) \in M$, we define the horizontal lift of α starting from $[\delta, g] \in P_\rho$ by $\tilde{\alpha}(t) = [\alpha_{(t)}\delta, g]$ where $\alpha_{(t)}$ denotes the one-parameter family of paths within α starting at $\alpha(0)$ and ending at $\alpha(t)$. It is straightforward to check that this construction is well-defined and satisfies the G invariance condition for a connection on P_ρ. We will not need to explicitly write down A_ρ as a one-form in order to check that A_ρ satisfies the cone-flat condition. We will instead express the curvature in terms of the holonomy group.

Step 2. Verify that the connection A_ρ is cone-flat, that is, $F_{A_\rho} = -\zeta B_\rho$ such that B_ρ is covariantly constant.

Lemma 4.2. *The connection A_ρ constructed above is cone-flat.*

Proof. Let $p \in M$ be an arbitrary point and $v_1, v_2 \in T_pM$ be arbitrary linearly independent vectors. Suppose $\zeta(v_1, v_2) = c \neq 0$. By Darboux's theorem, we can find a local coordinate system $\{x_1, \ldots, x_m\}$ such that $v_1 = \frac{\partial}{\partial x_1}$, $v_2 = \frac{\partial}{\partial x_2}$, and $\zeta = c\, dx_1 \wedge dx_2 + \bar{\zeta}$, where $\bar{\zeta} = dx_3 \wedge dx_4 + \ldots$, if $\dim M \geq 4$. Let D_t be an infinitesimal parallelogram spanned by $\sqrt{t}v_1$ and $\sqrt{t}v_2$ in the local coordinate system, and $\gamma_t = \partial D_t$ be its boundary. Then we have $\int_{D_t} \zeta = c\,t$.

At arbitrary $[\delta, g] \in P_\rho$ on the fiber of p, let v_1^H and v_2^H be the horizontal lift of v_1 and v_2, respectively. Then the curvature

$$F_{A_\rho}\left(v_1^H, v_2^H\right) = \frac{\partial}{\partial t} hol\left(\gamma_t\right)\Big|_{t=0} .$$

Here, $hol\left(\gamma_t\right) \in G$ denotes the holonomy along γ_t at $[\delta, g]$.

On the other hand,

$$[\delta, g] hol\left(\gamma_t\right) = [\gamma_t \delta, g] = [\delta(\delta^{-1}\gamma_t\delta), g] = \left[\delta, g\left(g^{-1}\rho\left(\delta^{-1}\gamma_t\delta\right)g\right)\right] .$$

This implies

$$hol\left(\gamma_t\right) = g^{-1}\rho\left(\delta^{-1}\gamma_t\delta\right)g .$$

Note that $\delta^{-1}\gamma_t\delta$ is contractible and its base point is $a \in M$, i.e. $\delta^{-1}\gamma_t\delta \in \Psi^a_{a,0}(M)$. We can express $\delta^{-1}\gamma_t\delta = \exp(c\,t\,\xi)$, where ξ is an element of the Lie algebra of $\Psi^a_a(M)/\Psi^a_{a,\zeta}(M)$ that generates the subgroup $\Psi^a_{a,0}(M)/\Psi^a_{a,\zeta}(M) \cong \mathbb{R}/H$ and $\int_{D_t} \zeta = \zeta(v_1, v_2)\, t = c\,t \in \mathbb{R}/H$. Note that ξ is independent of the choice of v_1 and v_2. We thus find at $p \in M$

$$F_{A_\rho}\left(v_1^H, v_2^H\right) = \frac{\partial}{\partial t} hol\left(\gamma_t\right)\Big|_{t=0} = c\, \mathrm{Ad}_{g^{-1}} d\rho(\xi) = \zeta(v_1, v_2)\mathrm{Ad}_{g^{-1}} d\rho(\xi) = -\zeta(v_1, v_2) B_\rho ,$$

where $d\rho$ maps the Lie algebra of Γ into $\mathfrak{g} = \mathrm{Lie}(G)$. It is clear that B_ρ as obtained above is covariantly constant as it is a constant when evaluated along horizontally lifted curves.

Thus far, we have assumed $\zeta(v_1, v_2) \neq 0$. If however $\zeta(v_1, v_2) = 0$ which may occur in $\dim M \geq 4$, then we can find a local coordinate system $\{x_1, \ldots, x_m\}$ such that $v_1 = \frac{\partial}{\partial x_1}$, $v_2 = \frac{\partial}{\partial x_3}$, and $\zeta = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \bar{\zeta}$, where $\bar{\zeta}$, if non-zero, is generated by other $dx_i \wedge dx_j$ locally. The proof goes through as above but with c set to zero. $\square$

Step 3. Show that the morphism $\rho \mapsto (P_\rho, A_\rho)$ is invariant under conjugation.

Lemma 4.3. *Suppose ρ and $\bar{\rho}$ are conjugate homomorphisms from $\Gamma = \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$ to G, and also, (P_ρ, A_ρ) and $(P_{\bar{\rho}}, A_{\bar{\rho}})$ are principal bundles with cone-flat connections constructed as above. Then (P_ρ, A_ρ) and $(P_{\bar{\rho}}, A_{\bar{\rho}})$ are equivalent.*

Proof. Suppose ρ and $\bar{\rho}$ are conjugate, i.e. $\bar{\rho} = g_0 \rho g_0^{-1}$ for some $g_0 \in G$. We consider the automorphism on $\Psi_a(M) \times G$ given by $(\delta, g) \mapsto (\delta, g_0 g)$. This automorphism induces the desired bundle isomorphism

$$\begin{aligned} f: \quad \Psi_a(M) \times_\rho G \quad &\longrightarrow \qquad \Psi_a(M) \times_{\bar{\rho}} G \\ (\delta\gamma, g) \sim (\delta, \rho(\gamma) g) &\mapsto \ (\delta\gamma, g_0 g) \sim (\delta, \bar{\rho}(\gamma) g_0 g) = (\delta, g_0 \rho(\gamma) g) \end{aligned}$$

where the second line shows the map on the equivalence class for all $\gamma \in \Gamma = \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$.

Now to show $f^* A_{\bar{\rho}} = A_\rho$, let $[\delta, g]_\rho \in \Psi_a(M) \times_\rho G$ and α on M be an arbitrary path starting at $\alpha(0) = \delta(1)$. As described in Step 1 in defining A_ρ, the horizontal lift of α starting at $[\delta, g]_\rho$ is defined as $\tilde{\alpha}_\rho(t) = [\alpha_{(t)}\delta, g]_\rho$. Likewise, the horizontal lift of α starting at $f\big([\delta, g]_\rho\big) = [\delta, g_0 g]_{\bar{\rho}}$ is defined as $\tilde{\alpha}_{\bar{\rho}}(t) = [\alpha_{(t)}\delta, g_0 g]_{\bar{\rho}}$. Clearly, we have $\tilde{\alpha}_{\bar{\rho}}(t) = f \circ \tilde{\alpha}_\rho(t)$, which implies f_* sends horizontal vectors to horizontal vectors as desired. ☐

By this lemma, given any conjugacy classes $[\rho]$ of the homomorphisms from $\Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$ to G, there is a corresponding G-bundle P_ρ with a cone-flat connection A_ρ. It remains to show that this correspondence is bijective.

Step 4. The morphism $[\rho] \mapsto (P_\rho, A_\rho)$ is injective.

Lemma 4.4. *Let $\rho, \bar{\rho} : \Psi_a^a(M)/\Psi_{a,\zeta}^a(M) \to G$. If (P_ρ, A_ρ) and $(P_{\bar{\rho}}, A_{\bar{\rho}})$ are equivalent, then ρ and $\bar{\rho}$ are conjugate.*

Proof. Suppose $f : P_\rho \to P_{\bar{\rho}}$ is a G-bundle isomorphism and $f^* A_{\bar{\rho}} = A_\rho$. Let us define $h : \Psi_a(M) \to G$ such that

$$f\big([\delta, e]_\rho\big) = [\delta, e]_{\bar{\rho}} h(\delta)$$

where $\delta \in \Psi_a(M)$ and e is the identity of G. Then, for each $\gamma \in \Psi_a^a(M)/\Psi_{a,\zeta}^a(M)$, we have

$$\begin{aligned} [\delta, e]_{\bar{\rho}} h(\delta) \rho(\gamma) &= f\big([\delta, e]_\rho\big)\rho(\gamma) = f\big([\delta, e]_\rho \rho(\gamma)\big) \\ &= f\big([\delta\gamma, e]_\rho\big) = [\delta\gamma, e]_{\bar{\rho}} h(\delta\gamma) = [\delta, e]_{\bar{\rho}} \bar{\rho}(\gamma) h(\delta\gamma)\,. \end{aligned}$$

This implies $\rho(\gamma) = h(\delta)^{-1}\bar{\rho}(\gamma)h(\delta\gamma)$. We will prove that h is a constant.

For arbitrary $\delta \in \Psi_a(M)$, let $\delta_{(t)}$ denote the t-parametrized subpaths of δ starting at $\delta(0) = a$ and ending at $\delta(t)$. By the construction of the connections, $[\delta_{(t)}, e]_\rho$ and $[\delta_{(t)}, e]_{\bar{\rho}}$ are horizontal paths on P_ρ and $P_{\bar{\rho}}$, respectively. The projection of these two paths on M are exactly $\delta_{(t)}$. By the definition of h, we have

$$f\big([\delta_{(t)}, e]_\rho\big) = [\delta_{(t)}, e]_{\bar{\rho}} h(\delta_{(t)}) \,. \tag{4.4}$$

On the other hand, since $f^* A_{\bar{\rho}} = A_\rho$, $f\big([\delta_{(t)}, e]_\rho\big)$ is a horizontal lift of $\delta_{(t)}$ on $P_{\bar{\rho}}$ passing through $f\big([\delta_{(0)}, e]_\rho\big)$. As horizontal subspaces are invariant under right G-action, $[\delta_{(t)}, e]_{\bar{\rho}} h(\delta_{(0)})$ is also a horizontal lift of $\delta_{(t)}$ on $P_{\bar{\rho}}$ passing through $[\delta_{(0)}, e]_{\bar{\rho}} h(\delta_{(0)}) = f\big([\delta_{(0)}, e]_\rho\big)$. Therefore, we have

$$f\big([\delta_{(t)}, e]_\rho\big) = [\delta_{(t)}, e]_{\bar{\rho}} h(\delta_{(0)}) \,. \tag{4.5}$$

Comparing (4.4)-(4.5), we have $h(\delta_{(t)}) = h(\delta_{(0)})$ for all $t \in [0, 1]$. This implies $h(\delta_{(t)})$ is equal to h acting on the constant path at a. So h is a constant, and therefore, ρ and $\bar{\rho}$ are conjugate. □

Step 5. The morphism $[\rho] \mapsto (P_\rho, A_\rho)$ is surjective.

The following lemma leads to surjectivity. It also holds when $\mathbb{R}/H$ is not a Lie group, which we will discuss later.

Lemma 4.5. *Let $\pi : P \to M$ be a principal G-bundle, and let A be a cone-flat connection on P with curvature $F_A = -\zeta B$. Then there exists $\xi \in \mathfrak{g}$ such that for any contractible loop γ starting at $a \in M$ and any oriented disk D in M with $\partial D = \gamma$ (∂D and γ also have the same orientation), the holonomy along γ is*

$$hol(\gamma) = \exp\Big[\Big(\int_D \zeta\Big)\,\xi\Big] \,.$$

Proof. Take a point $u_0 \in P$ on the fiber of $a \in M$. Consider the holonomy bundle

$$\hat{P} = \{u \in P \,|\, \text{there exists a horizontal curve } \tilde{\delta} \text{ such that } \tilde{\delta}(0) = u_0, \tilde{\delta}(1) = u\}.$$

$\hat{P}$ is a principal bundle over M, and its structure group $\hat{G}$ is the holonomy group of P at u_0. Let $\hat{A}$ be the restriction of A on $\hat{P}$. By the holonomy theorem of Ambrose-Singer [1], the Lie algebra of $\hat{G}$ is

$$\hat{\mathfrak{g}} = \operatorname{span}\Big\{F_{\hat{A}}\big(v_1^H, v_2^H\big) \;|\; v_1^H, v_2^H \text{ are horizontal vectors at } u \text{ for some } u \in \hat{P}\Big\} \,.$$

By assumption, $F_A = -\zeta B$, or more precisely $F_A = -(\pi^*\zeta)B$. Since B is covariantly constant, it is equal to some $\xi \in \hat{\mathfrak{g}}$ at any point in $\hat{P}$. Hence, $F_{\hat{A}}\left(v_1^H, v_2^H\right) \in \mathbb{R}\xi$. So $\hat{\mathfrak{g}}$ is 1-dimensional and abelian. Then $\hat{A} = -\theta \otimes \xi$ for some $\theta \in \Omega^1(\hat{P})$ and $d\theta = \hat{\pi}^*\zeta$, where $\hat{\pi} : \hat{P} \to M$ is the projection.

For an arbitrary contractible loop γ and a disk D such that $\partial D = \gamma$, there exists a contractible neighborhood $U \subset M$ of D. Then $\hat{P}|_U = U \times \hat{G}$ is trivial. Let $\sigma : U \to \hat{P}|_U$ be a local section and $\psi : \hat{P}|_U \to U \times \hat{G}$ be a trivialization such that $\psi \circ \sigma(p) = (p, e)$ for $p \in M$. Then $(\psi^{-1})^*\hat{A} = (-\sigma^*\theta \otimes \xi, 0) + (0, MC_{\hat{G}})$, where $MC_{\hat{G}} : T\hat{G} \to \hat{\mathfrak{g}}$ is the Maurer-Cartan form of $\hat{G}$ sending a vector to the corresponding invariant vector field. Observe that $d(\sigma^*\theta) = \zeta$. The horizontal lift $\tilde{\gamma}$ of γ with $\tilde{\gamma}(0) = \sigma(a)$ satisfies $\psi \circ \tilde{\gamma}(t) = (\gamma(t), g(t))$ with $g(t) \in \hat{G}$ and $g(0) = e$. With the horizontal vectors in the kernel space of the connection one-form, we have

$$\begin{aligned} 0 = \hat{A}\big(\tilde{\gamma}'(t)\big)\Big|_{\gamma(t_0)} &= (\psi^{-1*}\hat{A})\big(\gamma'(t), 0\big)\Big|_{\left(\gamma(t_0), g(t_0)\right)} + (\psi^{-1*}\hat{A})\big(0, g'(t)\big)\Big|_{\left(\gamma(t_0), g(t_0)\right)} \\ &= -(\sigma^*\theta)\big(\gamma'(t)\big) \cdot \xi\Big|_{\gamma(t_0)} + MC_{\hat{G}}\big(g'(t)\big)\Big|_{g(t_0)}. \end{aligned}$$

Hence, we find

$$g(t_0) = \exp\Big[\Big(\int_0^{t_0} (\sigma^*\theta)\big(\gamma'(t)\big)dt\Big)\,\xi\Big],$$

and the holonomy along γ is

$$hol(\gamma) = g(1) = \exp\Big[\Big(\int_\gamma \sigma^*\theta\Big)\,\xi\Big] = \exp\Big[\Big(\int_D \zeta\Big)\,\xi\Big].$$

□

Now we proceed to prove surjectivity.

Lemma 4.6. *Let $\pi : P \to M$ be a principal G-bundle with a cone-flat connection A with respect to ζ. Then there exists a homomorphism $\rho : \Psi_a^a(M)/\Psi_{a,\zeta}^a(M) \to G$ such that (P_ρ, A_ρ) and (P, A) are equivalent.*

Proof. For any contractible loop $\gamma \in \Psi_{a,\zeta}^a(M)$ and a disk D such that $\partial D = \gamma$, it follows from the definition of $\Psi_{a,\zeta}^a(M)$ that $\int_D \zeta = 0$. By Lemma 4.5, the holonomy along γ in (P, A) is given by $hol(\gamma) = \exp\Big[\Big(\int_D \zeta\Big)\,\xi\Big] = e$. Hence, we define the following homomorphism:

$$\rho : \Psi_a^a(M)/\Psi_{a,\zeta}^a(M) \to G, \quad \gamma \mapsto \rho(\gamma) = hol(\gamma)\,. \tag{4.6}$$

We will show that the resulting (P_ρ, A_ρ) is equivalent to (P, A). We start by defining a G-bundle isomorphism between the two principal bundles.

Given $[\delta, g] \in P_\rho$, let $\tilde{\delta}$ be the horizontal lift of δ in P with $\tilde{\delta}(0) = u_0$. We define

$$f : P_\rho \to P, \quad [\delta, g] \mapsto \tilde{\delta}(1)\, g\,. \tag{4.7}$$

Note that the definition of f utilizes the horizontal lift determined by the connection A in P. Let us show that this map is well-defined. Suppose $[\delta_1, g_1]$ and $[\delta_2, g_2]$ represent the same class in P_ρ. Then there must exists a $\gamma_0 \in \Psi^a_a(M)/\Psi^a_{a,\zeta}(M)$ such that

$$(\delta_1, g_1) = (\delta_2\gamma_0, \rho(\gamma_0^{-1})g_2) \sim (\delta_2, g_2)\,. \tag{4.8}$$

Thus, $\gamma_0^{-1}\delta_2^{-1}\delta_1 \in \Psi^a_{a,\zeta}(M)$, and its holonomy in (P, A) is trivial according to Lemma 4.5. Moreover, we have

$$hol(\delta_2^{-1}\delta_1) = hol(\gamma_0) = \rho(\gamma_0)\,, \tag{4.9}$$

where the last equality follows from our definition of ρ in (4.6). Now let $\tilde{\delta}_1$ and $\tilde{\delta}_2$ denote the horizontal lift in (P, A) of δ_1 and δ_2, respectively, starting at u_0. Since $\tilde{\delta}_1$ and the horizontal lift of $\delta_2\gamma_0 = \delta_2(\delta_2^{-1}\delta_1)$ starting at u_0 have the same ending point, so does $\tilde{\delta}_1$ and the horizontal lift of δ_2 starting at $u_0 \cdot hol(\gamma_0)$. Hence, the ending points of $\tilde{\delta}_1$ and $\tilde{\delta}_2$ satisfy $\tilde{\delta}_1(1) = \tilde{\delta}_2(1) \cdot hol(\gamma_0)$. Together with (4.7)-(4.9), we find

$$f\left([\delta_1, g_1]\right) = \tilde{\delta}_1(1)\, g_1 = \tilde{\delta}_2(1)\rho(\gamma_0)\, g_1 = \tilde{\delta}_2(1)\, g_2 = f([\delta_2, g_2])\,.$$

Hence, f is well-defined. It is also straightforward to check that f is a G-bundle isomorphism.

Finally, we check that the definition of A_ρ described earlier in Step 1 is consistent with $f^*A = A_\rho$. For $[\delta, g] \in P_\rho$ and an arbitrary path α on M such that $\alpha(0) = \delta(1)$, the horizontal lift of α at $[\delta, g]$ is $\tilde{\alpha}(t) = [\alpha_{(t)}\delta, g]$. Here again, $\alpha_{(t)}$ is the subpath of α starting at $\alpha(0)$ and ending at $\alpha(t)$. Notice that $f\left([\alpha_{(t)}\delta, g]\right) = (\widetilde{\alpha_{(t)}\delta})(1)\, g$ is the ending point of the path on P that is the horizontal lift of $\alpha_{(t)}$ in (P, A) starting at $f([\delta, g]) = \tilde{\delta}(1)\, g$. Clearly then, $f \circ \tilde{\alpha}$ is a horizontal path on (P, A). Hence, f_* sends horizontal vectors to horizontal vectors, and therefore, $f^*(A) = A_\rho$. $\square$

Combining the lemmas above, we have proved that $[\rho] \mapsto (P_\rho, A_\rho)$ is an isomorphism.

Proof for Case 3: $\mathbb{R}/H$ is not a Lie group

We now turn to the case when $\mathbb{R}/H$ is not a Lie group. In this case, H^+ is non-empty and has no minimal number.

Lemma 4.7. *When H^+ is non-empty and has no minimal number, a cone-flat connection is a flat connection.*

Proof. By Lemma 4.5, there exists some $\xi \in \mathfrak{g}$ such that for any contractible loop γ and disk D with $\partial D = \gamma$, we have $hol(\gamma) = \exp\Big[\Big(\int_D \zeta\Big)\,\xi\Big]$.

Let us show that ξ must be zero. For if $\xi \neq 0$, then there exists some small enough t_0 such that for any $0 < t < t_0$, $\exp(t\,\xi) \neq e$ the identity element of G. Now since H^+ has no minimal number, there exists a closed sphere $S \subset M$ such that $\int_S \zeta = t$ for some $0 < t < t_0$. Let γ be a constant loop at some point $p \in S$, and $D = S \setminus \{p\}$. Then $\partial D = \gamma$ so that $hol(\gamma) = \exp\Big[\Big(\int_D \zeta\Big)\,\xi\Big]$. But $hol(\gamma) = e$ as γ is the identity loop. On the other hand, $\int_D \zeta = \int_S \zeta = t$, and therefore, $\exp(t\,\xi) = e$, which gives a contradiction. Thus, we conclude that ξ is zero.

With $\xi = 0$, the holonomy of any contractible loop is trivial. Hence, the curvature vanishes and the connection is flat. □

The classification of G-bundles with flat connections can be represented by the conjugacy classes of homomorphisms from $\pi_1(M) \to G$ (c.f. [6, Theorem 2.9]). So we have proved Theorem 4.1 in this case. This completes the proof of the theorem.

We point out that the classification of cone-flat bundles generally depends on the choice of ζ. In Theorem 4.1, the ζ dependence explicitly appears in the definition of H in (4.3). Below, we will work out the classification and demonstrate its dependence on ζ in the simple example of $U(1)$ bundles over the four-dimensional torus $M = T^4$.

Example 4.8. We describe T^4 as $\mathbb{R}^4/\sim$, with the identification $(x_1, x_2, x_3, x_4) \sim (x_1 + a, x_2 + b, x_3 + c, x_4 + d)$ where $a, b, c, d \in \mathbb{Z}$, and $U(1)$ as $\{z \in \mathbb{C} \mid |z| = 1\}$. Let

$$\zeta = c_1\, dx_1 \wedge dx_2 + c_2\, dx_3 \wedge dx_4$$

be a closed 2-form with $c_1, c_2 \in \mathbb{R} \setminus \{0\}$. By Theorem 4.1, the equivalent classes of $U(1)$ bundles with a cone-flat connection are in 1-1 correspondence with homomorphisms $\rho : \Gamma \to U(1)$, where $\Gamma = \Psi^a_a(M)/\Psi^a_{a,\zeta}(M)$.

Since $\pi_2(T^4)$ is trivial, Γ is an $\mathbb{R}$-extension of $\pi_1(T^4)$. To describe its group structure explicitly, let a_i for $i = 1, 2, 3, 4$ be the straight line path in $\mathbb{R}^4$ starting at the origin and ending at the point where the i-th coordinate, $x_i = 1$, and $x_j = 0$ for $j \neq i$. When projected to T^4, $\{a_1, a_2, a_3, a_4\}$ become the generators of $\pi_1(T^4)$. Although $\pi_1(T^4)$ is abelian, the elements of $\{a_1, a_2, a_3, a_4\}$ may no longer commute in Γ.

For a contractible loop b, there is an oriented disk D such that $\partial D = b$ and ∂D has the same orientation as b. Let $|b| = \int_D \zeta$, and we note that $|b|$ is independent of the choice of D. Now recall that $\Gamma = \Psi^a_a(M)/\Psi^a_{a,\zeta}(M)$ is a quotient of loops by contractible ones whose integral, $|b| = \int_D \zeta = 0$. Moreover, contractible loops b and b' would represent different classes in Γ if and only if $|b| \neq |b'|$. So, a class of contractible loops $b \in \Gamma$ can be represented by the real number $|b|$.

On T^4 then, we can think of Γ as being generated by $\{a_1, a_2, a_3, a_4, |b|\}$, with multiplication defined by

$$a_i|b| = |b|a_i \qquad |b'b| = |b| + |b'|\,.$$

However, $a_j^{-1}a_i^{-1}a_ja_i$ can represent some non-trivial class of a contractible loop. Specifically, we have

$$\begin{aligned}
&|a_2^{-1}a_1^{-1}a_2a_1| = c_1,\\
&|a_4^{-1}a_3^{-1}a_4a_3| = c_2,\\
&|a_j^{-1}a_i^{-1}a_ja_i| = 0, \text{ for other } i, j,
\end{aligned}$$

Now, the possible homomorphisms of $\rho : \Gamma \to U(1)$ is dependent on whether $\frac{c_1}{c_2} \in \mathbb{Q}$. Since $U(1)$ is abelian,

$$\rho(c_1) = \rho(a_2^{-1})\rho(a_1^{-1})\rho(a_2)\rho(a_1) = \rho(a_2)^{-1}\rho(a_2)\rho(a_1)^{-1}\rho(a_1) = 1.$$

Similarly $\rho(c_2) = 1$, so $\rho(pc_1 + qc_2) = 1$ for any $p, q \in \mathbb{Z}$.

When $\frac{c_1}{c_2} \in \mathbb{Q}$, $\{(pc_1 + qc_2)\,|\,p, q \in \mathbb{Z}\}$ has a minimal positive number c_0. Then $\rho(b)$ must have the form $e^{2\pi i\frac{n|b|}{c_0}}$ for some $n \in \mathbb{Z}$. In this case, the Euler class of the circle bundle is $\frac{n}{c_0}\zeta$, and the choice of $\rho(a_i)$ for $i = 1, \ldots, 4$ determines the connection.

When $\frac{c_1}{c_2} \notin \mathbb{Q}$, $\{(pc_1 + qc_2)\,|\,p, q \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. So $\rho(b)$ must be 1 for any $|b| \in \mathbb{R}$. Thus, the classification is only dependent on $\rho(a_i)$, and becomes equivalent to the classification of flat connections. Actually, the Euler class in this case is $c\,\zeta$ for some $c \in \mathbb{R}$. But the Euler class is an integral class, and hence, the only possible c is 0, i.e. every cone-flat connection is flat.

Remark 4.9. When M is simply-connected, the Hurewicz homomorphism $\pi_2(M) \to H_2(M, \mathbb{Z})$ is surjective. Assuming ζ is not d-exact, H is discrete if and only if there exists some non-zero constant c such that $c[\zeta] \in H^2(M, \mathbb{Z})$. Therefore, if such c exists, then $\Gamma = \mathbb{R}/\mathbb{Z}$ and the classification of isomorphism classes of cone-flat connections on G-bundles is given by the conjugacy classes of $\mathrm{Hom}(S^1, G)$. If such c does not exist, then Γ is trivial and then there does not exist non-trivial cone-flat connections.

As an application of this observation, let M be simply-connected and closed. If M is also a projective manifold, then we can choose ζ to be the Kähler form which is an integral class, and so, $\zeta \in H^2(M, \mathbb{Z})$. The isomorphism classes of cone-flat connections on G-bundles is then given by the conjugacy classes of $\mathrm{Hom}(S^1, G)$. If however M is a non-projective Kähler manifold and we still let ζ be the Kähler form, then there is no non-zero constant c such that $c[\zeta] \in H^2(M, \mathbb{Z})$ (see, for a reference, [4, Corollary 5.3.3]) and the isomorphism classes of cone-flat connections would be trivial.

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