An introduction to algebraic fundamental groups

1 Notations and Statement of the Main Results

Throughout the talk, all schemes are locally Noetherian. All maps are of locally finite type.

There two main motivations of Grothendieck to develop the étale cohomology: proving the Weil Conjecture and introducing the fundamental groups to the category of schemes. As we all know, the Zariski topology is too coarse to consider any reasonable loop on it. On the other hand, we are indeed looking for something involving the scheme structure. So, one should define the covering on the level of rings.

Also, we need to give some new understanding of the fundamental group. An alternative way to interpret fundamental group is to view it as the transformation group of the universal covering space. Say, $\widetilde{X}$ is the universal covering of X. Then we can reasonably define $\pi_1(X) = \mathrm{Aut}_X(\widetilde{X})$. However, this brings another problem: if we begin with an elliptic curve $X = \mathbb{C}/\mathbb{Z}^2$, we should expect its universal covering to be $\mathbb{C}$. Unfortunately, there is no morphism between $\mathbb{C}$ and X in the category of schemes because, essentially, there is no infinite cover in the category of schemes.

So, the way to solve the problem is to define what is visible in the category of schemes to be the universal covering (or equivalently the fundamental groups), namely, the "collection" of all coverings which corresponding to the subgroups of fundamental group of finite index. To be more precise, we expect the following diagram

Topology and Geometry Side	Algebraic Geometry Side
Coverings in the topological sense	Étale morphisms
Finite index subgroups of π_1^{geo}	Finite étale morphisms
π_1^{geo}	$\pi_1^{\text{alg}} = \varprojlim_{H \lhd G} G/H$

Definition 1.1. We say a homomorphism $f : A \to B$ is *étale* if

(i) it is of finite type and **flat**;

(ii) it is unramified, namely, for any prime $\mathfrak{q} \in \operatorname{Spec} B$ and $\mathfrak{p} = \mathfrak{q} \cap A \in \operatorname{Spec} A$, we have $B/\mathfrak{p}B$ is a finite **separable field** extension of $A/\mathfrak{p}A$.

What we are interested in are finite étale maps. They are **both open and closed map**. Here is a list of general properties of étale maps.

- Locally, étale map looks like $\operatorname{Spec} B = \operatorname{Spec}\big(A[T]/f(T)\big)_g \to \operatorname{Spec} A$, where $f(T)$ is a unitary polynomial and $f'(T)$ is invertible in B. Or, more generally, $\operatorname{Spec}\big(\frac{A[T_1,\cdots,T_n]}{f_1(T),\cdots,f_n(T)}\big)_g$, where the Jacobian $\mathrm{Jac}(\partial f_i/\partial T_j)$ is invertible. (Note: we have same n here)
- $f : X \to S$ is unramified at $x \in X \Leftrightarrow \Omega_{X/S,x} = 0 \Leftrightarrow \Delta : X \to X \times_S X$ is an open immersion in a neighborhood of x. (In affine case, this means that the diagonal map maps X to a "connected component" of $X \times_S X$.)
- The lifting problem for closed subscheme Z_0 of Z defined by ideal sheaf $\mathcal{I}$ such that $\mathcal{I}^2 = 0$

(or equivalently, $\mathcal{I}^N = 0$ for some N)

$$\begin{array}{ccc} X & \longleftarrow & Z_0 \\ {\scriptstyle f}\downarrow & \nwarrow & \downarrow \\ Y & \longleftarrow & Z \end{array}$$

has a (resp. unique) solution if and only if f is smooth (resp. étale).

- The étaleness is preserved under base change and composition. Moreover, for $X \xrightarrow{f} Y \xrightarrow{g} Z$, if g and $g \circ f$ are étale, then f is.

Now, we will state the main result of this talk.

Theorem 1.2 (Fundamental Group). *Let X be a* ***connected*** *locally noetherian scheme and $\mathbf{FEt}(X)$ be the category of finite étale cover of X. Fix a geometric point $\bar{x} \in X$, then there exists an profinite group $G = \pi_1(X, \bar{x})$ and an equivalence of category $\mathbf{FEt}(X) \xrightarrow{\sim} G\text{−sets}^{\text{cont.}}$. For a different choice $\bar{x}' \in X$, there is a continuous isomorphism $\pi_1(X, \bar{x}) \cong \pi_1(X, \bar{x}')$ and the isomorphism is unique up to an inner automorphism.*

A special case of this theorem is the category of finite étale covers of $\text{Spec}(k)$ for k a field. In that case, $G = \text{Gal}(k^{\text{sep}}/k)$.

Theorem 1.3 (First Homotopy Exact Sequence). *Let Y be a* ***connected*** *locally noetherian scheme. Let $f : X \to Y$ satisfy the following (technical) condition:*

- *f is proper;*
- *f is separable in the sense that* ***every*** *fiber is geometrically reduced;*
- *$f_*(\mathcal{O}_X) \xleftarrow{\sim} Y$.*

Let $\bar{a}$ be a geometric point of X and $\bar{y}$ its image in Y. Denote the geometric fiber by $X_{\bar{y}}$. Then, we have an exact sequence of fundamental groups:

$$\pi_1(X_{\bar{y}}, \bar{a}) \xrightarrow{\phi} \pi_1(X, \bar{a}) \xrightarrow{\psi} \pi_1(Y, \bar{y}) \longrightarrow (1)$$

2 Construction of Fundamental Groups

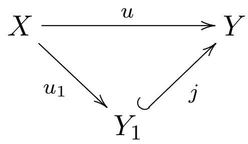
Throughout this chapter, S denotes a connected scheme. We will use the formalism of Galois category. The category $\mathcal{C} = \mathbf{FEt}(S)$ is called Galois if it satisfies the following conditions:

(G0) $\mathcal{C}$ has an initial object ϕ (the empty scheme) and a final object S.

(G1) Finite fibre products exist in $\mathcal{C}$.

(G2) If $X, Y \in \mathcal{C}$, then the disjoint union $X \coprod Y \in \mathcal{C}$.

(G3) Any morphism $u : X \to Y$ in $\mathcal{C}$ admits a (unique) factorization of the form



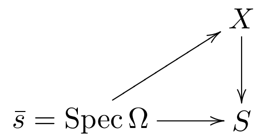
where u_1 is effective epimorphism (i.e., surjective étale) and $Y = Y_1 \coprod Y_2$ for $Y_2 \in \mathcal{C}$.

In fact, this follows from that a finite étale map is both closed and open.

(G4) If $X \in \mathcal{C}$ and G is a finite group acting on the right ($g^* \circ h^* = (hg)^*$). Then the quotient X/G exists and $X \to X/G$ is effective epimorphism (surjective étale).

Since étaleness can be descent under faithfully flat map, by passing to the strictly henselization of $\mathcal{O}_{S,s}$ for any $s \in S$, we are reduced to the trivial case.

Next, I will define a covariant functor F from $\mathcal{C} = \mathbf{FEt}(S)$ to the category of finite sets, called the fundamental functor. This map will become the map of equivalence of category between $\mathcal{C}$ and G−sets$^{\text{cont.}}$. Fix a geometric point $\bar{s} = \operatorname{Spec}\Omega$. For any $X \in \mathbf{FEt}(S)$, we define $F(X)$ to be the set of commutative diagrams (or equivalently, the geometric points of X above $\bar{a}$)



It is obvious that $|F(X)| = \operatorname{rank}(X/S)$. Moreover, given a morphism $X \to Y$ in the category $\mathcal{C}$, we have a natural map $F(X) \to F(Y)$. The functor satisfies the following axioms:

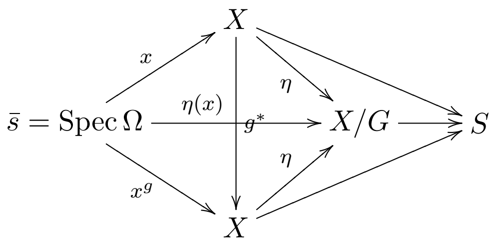
(F0) $F(X) = \phi \Longleftrightarrow X = \phi$. And $F(S) = \{e\}$ is a one pointed set.

(F1) $F(X \times_Z Y) = F(X) \times_{F(Z)} F(Y)$, $\forall X, Y, Z \in \mathcal{C}$.

(F2) $F(X_1 \coprod X_2) = F(X_1) \coprod F(X_2)$.

(F3) If $X \to Y$ is an effective epimorphism in $\mathcal{C}$, then the map $F(X) \to F(Y)$ is **onto**.

(F4) Let $X \in \mathcal{C}$ and G acts as S-automorphism on the right. Then G acts naturally on the right of $F(X)$ via the following commutative diagram:



Moreover, we have a natural bijection $\tilde{\eta} : F(X)/G \to F(X/G)$.

The bijectivity follows from the following facts: (i) for any point $\mathfrak{p} \in X/G$, the action of G on the fiber is transitive and (ii) for any point $\mathfrak{m} \in \eta^{-1}(\mathfrak{p})$ in the fiber, the decomposition group $G_{\mathfrak{p}}$ at $\mathfrak{p}$ surjectively maps to $\operatorname{Gal}(k(\mathfrak{m})/k(\mathfrak{p}))$.

(F5) If $u : X \to Y$ is a morphism in $\mathcal{C}$ such that the corresponding map $F(u) : F(X) \to F(Y)$ is bijective, then u is isomorphism.

Example 2.1. Before going further, we first study what happens to the classical case of covering of locally arcwise connected topological spaces S. The category $\mathcal{C}$ consists of all topological coverings X of S and all maps are covering maps. Pick a point $s \in S$, define $F(X) = u^{-1}(s)$ for $u : X \to S$. Then, as well known, we have a universal covering $\tilde{S}$, satisfying the following two properties: $\pi_1(S, s) = \mathrm{Aut}_S(\tilde{S})$ and $\mathrm{Hom}_{\mathcal{C}}(\tilde{S}, X) \xrightarrow{\sim} F(X)$, $\forall X \in \mathcal{C}$.

Definition 2.2. A covariant functor $F : \mathcal{C} \to \mathbf{Sets}$ is called representable if there exists $X \in \mathcal{C}$ and $\xi \in F(X)$ such that $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to F(Y)$, $f \mapsto F(f)(\xi)$ is bijective, i.e., $\mathrm{Hom}(X, -) \to F$ is an isomorphism of functors.

As indicated by the example, we will prove that F is pro-presentable, i.e., there exists a "compatible" family $\{S_i, \xi_i\}_{i \in I}$ for $S_i \in \mathcal{C}$, $\xi_i \in F(S_i)$, such that $F(X) = \varinjlim_{i \in I} \mathrm{Hom}(S_i, X)$, $\forall X \in \mathcal{C}$.

I claim without proof that this is not too different from the classical case except that we do not have a universal covering, but inductive limit instead. In practise, one can show that, it is enough to take all the connected objects $S_i \in \mathcal{C}$ and all $\xi_i \in F(S_i)$ with effective epimorphisms between them. We then have pro-representability.

Next, we will try to extract the group from the information given by S_i. We note that for any connected $X \in \mathcal{C}$, $\sigma \in \mathrm{Aut}(X)$ acts on $\xi \in F(X)$ via $\xi \mapsto \sigma \circ \xi$. This gives a map $\mathrm{Aut}(X) \hookrightarrow F(X)$.

Definition 2.3. A connected object $X \in \mathcal{C}$ is called *galois* if for any $\xi \in F(X)$, the map $\mathrm{Aut}(X) \to F(X)$ defined by $u \mapsto u \circ \xi$ is a **bijection**.

Lemma 2.4. *For any $\eta \in F(Y)$, we have a galois object X, $\xi \in F(X)$ and $u \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ such that $F(u)(\xi) = \eta$.*

This lemma is crucial in the proof of the existence of fundamental group. Granted by this lemma, we may assume that the S_i representing F are given by galois ones. Moreover, we can take only one fixed $\tau_i \in F(S_i)$ but not all the elements in $F(S_i)$.

Let $G_i = \mathrm{Aut}(S_i)$ and τ_i be the fixed element. We have a bijection $\theta_i : G_i \to F(S_i)$ defined by $u \mapsto u \circ \tau_i$. Denote the transition map of S_i to be $\phi_{ij} : S_j \to S_i$, i.e., $F(\phi_{ij})(\tau_j) = \tau_i$. Therefore, we can define ψ_{ij} to be

$$G_j \xrightarrow{\theta_j} F(S_j) \xrightarrow{F(\phi_{ij})} F(S_i) \xrightarrow{\theta_i^{-1}} G_i.$$

In other word, we have the commutative diagram

$$\begin{array}{ccc} S_i & \overset{\psi_{ij}(u)}{\dashrightarrow} & S_i \\ \uparrow{\scriptstyle \phi_{ij}} & & \uparrow{\scriptstyle \phi_{ij}} \\ S_j & \xrightarrow{u} & S_j \end{array}$$

Therefore, we obtain the fundamental group as a profinite group

$$\pi_1(S, \bar{s}) = \varprojlim_{\psi_{ij}} G_i.$$

One can show that the functor F establishes an equivalence of category of $\mathbf{FEt}(S)$ and $G-\mathrm{sets}^{\mathrm{cont.}}$, where $G = \pi_1(S, \bar{s})$. Moreover, from the construction, it depends on the choice of τ_i, which means that the fundamental group is defined up to an inner automorphism.

3 First Homotopy Exact Sequence

From the construction of fundamental group, we see that it is functorial in the sense that given a morphism $f : X \to Y$, mapping $\bar{x}$ to $\bar{y}$, we have a functor $f^* : \mathbf{FEt}(Y) \to \mathbf{FEt}(X)$. This gives rise to a natural continuous homomorphism $f_* : \pi_1(X, \bar{x}) \to \pi_1(Y, \bar{y})$.

We can interpret the property of the map f_* in terms of the property of the corresponding functor $f^* : \mathbf{FEt}(Y) \to \mathbf{FEt}(X)$.

- f_* is surjective if and only if for any connected $Y' \in \mathbf{FEt}(Y)$, $Y' \times_Y X \in \mathbf{FEt}(X)$ is also connected.
- f_* is trivial if and only if for any $X \in \mathbf{FEt}(Y)$, $Y' \times_Y X$ is isomorphic to disjoint union of finite copies of X.

Now, under the prescribe condition of Theorem 1.3, we will show the surjectivity of the last map and the triviality of the composition in the sequence of fundamental groups.

$$\pi_1(X_{\bar{y}}, \bar{a}) \xrightarrow{\phi} \pi_1(X, \bar{a}) \xrightarrow{\psi} \pi_1(Y, \bar{y}) \longrightarrow (1)\ .$$

To see that ψ is surjective, it is suffice to look at the connected étale base change $Y' \to Y$:

$$\begin{array}{ccc} X & \longleftarrow & X \times_Y Y' = X' \\ \Big\downarrow f & & \Big\downarrow f'\text{(proper)} \\ Y & \longleftarrow & Y' \end{array}$$

Since $\mathcal{O}_Y = f_*(\mathcal{O}_X)$, $f'_*(\mathcal{O}_{X'}) = f'_*(\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'}) = (f'_*\mathcal{O}_X) \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'} = \mathcal{O}_{Y'}$. Therefore, X' is connected by Zariski connected theorem.

As to the composition, we note that for any base change $Y' \to Y$, $Y' \times_Y \overline{k(y)} = \coprod_{\text{finite}} \overline{k(y)}$. Therefore, $\phi \circ \psi = 0$ follows.

The hard part of the proof is the exactness at the middle term which I will omit, which involves a beautiful use of theorem of cohomology of base change.

Remark 3.1. If we drop the assumptions $f_*(\mathcal{O}_X) = \mathcal{O}_Y$ and Y connected, we have a long exact sequence:

$$\pi_1(X_{\bar{y}}, \bar{a}) \xrightarrow{\phi} \pi_1(X, \bar{a}) \xrightarrow{\psi} \pi_1(Y, \bar{y}) \longrightarrow \pi_0(X_{\bar{y}}, \bar{a}) \xrightarrow{\phi} \pi_0(X, \bar{a}) \xrightarrow{\psi} \pi_0(Y, \bar{y}) \longrightarrow (1)$$

where $\pi_0(-, *)$ denotes the (pointed) sets of (connected) components of the corresponding schemes.

4 Existence Theorem and Its Application

Theorem 4.1 (Existence Theorem). *Let S be a locally noetherian scheme and for simplicity $S_0 = S_{\text{red}}$. Then we have an equivalence of category $\mathbf{FEt}(S) \xrightarrow{\sim} \mathbf{FEt}(S_0)$. More precisely,*

- $\forall X, Y \in \mathbf{FEt}(S)$, *we have* $\operatorname{Hom}_S(X, Y) = \operatorname{Hom}_{S_0}(X_0, Y_0)$ *and*
- *if* $X_0 \in \mathbf{FEt}(S_0)$, *then* $\exists X \in \mathbf{FEt}(S)$ *such that* $X \times_S S_0 \xrightarrow{\sim} X_0$.

The first assertion follows immediately from the lifting property of étale map. But the second needs hard descent theory, which I will not talk about. This theorem tells us that somehow the fundamental group "depends only on the topological space".

An application of the existence theorem is the following

Theorem 4.2 (Second Homotopy Exact Sequence). *Let A be a* ***complete noetherian local ring*** *and $S = \operatorname{Spec} A$ with $\bar{s}_0$ as the geometric point at the closed point. Let X be a* ***proper*** *S-scheme and $\bar{X}_0 = X \times_S \bar{s}_0$ its geometric special fiber. Assume that $\bar{X}_0$ is connected. Then the following sequence is exact.*

$$(1) \longrightarrow \pi_1(\bar{X}_0, \bar{a}_0) \longrightarrow \pi_1(X, a_0) \longrightarrow \pi_1(S, \bar{s}_0) \longrightarrow (1)$$

Moreover, we have isomorphism $\pi_1(S, \bar{s}_0) \cong \mathrm{Gal}(k(\bar{s}_0)/k(s_0))$.

Now, let $\bar{a}_1$ be an arbitrary geometric point of X and $\bar{x}_1$ its image in S. Denote $\bar{X}_1 = X \times_S \bar{s}_1$. Then we get a (**non-exact**) sequence

$$\pi_1(\bar{X}_1, \bar{a}_1) \longrightarrow \pi_1(X, a_1) \longrightarrow \pi_1(S, \bar{s}_1) \longrightarrow (1)$$

such that the composition is trivial. And we have the following diagram:

$$\begin{array}{ccccccccc}
 & & \pi_1(\bar{X}_1, \bar{a}_1) & \longrightarrow & \pi_1(X, a_1) & \longrightarrow & \pi_1(S, \bar{s}_1) & \longrightarrow & (1) \\
 & & & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} & & \\
(1) & \longrightarrow & \pi_1(\bar{X}_0, \bar{a}_0) & \longrightarrow & \pi_1(X, a_0) & \longrightarrow & \pi_1(S, \bar{s}_0) & \longrightarrow & (1)
\end{array}$$

This is **not** necessarily commutative; but it is **commutative up to an inner automorphism of** $\pi_1(S, \bar{s}_0)$.

Therefore, we obtain a continuous homomorphism $\pi_1(\bar{X}_1, \bar{a}_1) \to \pi_1(\bar{X}_0, \bar{a}_0)$ determined up to an inner automorphism of $\pi_1(X)$. This is called the **homomorphism of specialization of the fundamental group**.

In particular, if X is **separable** over S, the first row is exact, and hence $\pi_1(\bar{X}_1, \bar{a}_1) \to \pi_1(\bar{X}_0, \bar{a}_0)$ is **surjective**. In general, for S arbitrary locally noetherian scheme and X with universally connected fibers, if $\bar{s}_0 \rightsquigarrow \bar{s}_1$, the specialization map is surjective. We say that the fundamental group also satisfies the semi-continuity property.

References

[Mur] J. P. Murre, *Lectures on an introduction to Grothendieck's theory of the fundamental group*, Tata Institute of Fundamental Research, 1967.