APPENDIX: TENSOR BEING CRYSTALLINE IMPLIES EACH FACTOR BEING CRYSTALLINE UP TO TWIST

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Let p be a prime number. Let K be a finite extension of $\mathbb{Q}_p$. Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of K and let G_K denote the absolute Galois group. For E a finite extension of $\mathbb{Q}_p$, we use $\mathbf{Rep}_E(G_K)$ to denote the category of continuous representations of G_K on a finite dimensional E-vector space.

Let $\mathbb{C}_p$ denote the p-adic completion of $\overline{\mathbb{Q}}_p$. We do not recall the definition of Fontaine rings $\mathbb{B}_{\mathrm{dR}}$, $\mathbb{B}_{\mathrm{cris}}$, and etc, for which one may consult [1]. The aim of this appendix is to give a simpler proof of Theorem 1 below, which is proved by Di Matteo [4]. We hope that the proof presented here is more accessible to readers who are less familiar with p-adic Hodge theory, and hence makes Theorem 5 at the end of the appendix less mysterious.

Theorem 1. *Let $V, W \in \mathbf{Rep}_E(G_K)$ be two p-adic representations of G_K which are Hodge-Tate. If $W \otimes_E V$ is de Rham, then so are V and W themselves.*

Proof. We warn the readers that one has to be very careful when dealing with the coefficient fields. We start by some simple reductions dealing with coefficients. Since being de Rham or Hodge-Tate is preserved when replacing K and E by finite extensions, we may assume that $K = E$ is a Galois extension of $\mathbb{Q}_p$.

For $\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)$ and $U \in \mathbf{Rep}_K(G_K)$, we denote $\mathbf{D}_{\mathrm{dR},\sigma}(U) = (V \otimes_{K,\sigma} \mathbb{B}_{\mathrm{dR}})^{G_K}$, where $\otimes_{K,\sigma}$ means that the tensor is taken along the homomorphism $K \xrightarrow{\sigma} K \to \mathbb{B}_{\mathrm{dR}}$. We say U is *σ-de Rham* if $\dim_K \mathbf{D}_{\mathrm{dR},\sigma}(U) = \dim_K U$. Then U is de Rham if and only if it is σ-de Rham for all $\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)$. Theorem 1 will then following from Proposition 6, which requires a few lemmas first. $\square$

Lemma 2. *Assume that K is Galois over $\mathbb{Q}_p$. Let $V \in \mathbf{Rep}_K(G_K)$ be a Hodge-Tate representation. Then $\mathbf{D}_{\mathrm{dR},\sigma}(V) \neq 0$ for any $\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)$.*

Proof. We fix $\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)$. Since V is Hodge-Tate, we have $V \otimes_{K,\sigma} \mathbb{C}_p = \mathbb{C}_p(n_1) \oplus \cdots \oplus \mathbb{C}_p(n_d)$, for integers $n_1 \leq \cdots \leq n_d$. We first note that basic property of continuous group cohomology theory tells us

$$H^i(G_K, V \otimes t^n \mathbb{B}_{\mathrm{dR}}^+) = 0 \text{ for } i = 0, 1 \text{ and } n \gg 0.$$

Consider the long exact sequence

$$\begin{aligned} 0 &\to \mathrm{H}^0(G_K, V \otimes_{K,\sigma} t^{n+1}\mathbb{B}_{\mathrm{dR}}^+) \to \mathrm{H}^0(G_K, V \otimes_{K,\sigma} t^n\mathbb{B}_{\mathrm{dR}}^+) \to \mathrm{H}^0(G_K, V \otimes_{K,\sigma} \mathbb{C}_p(n)) \\ &\to \mathrm{H}^1(G_K, V \otimes_{K,\sigma} t^{n+1}\mathbb{B}_{\mathrm{dR}}^+) \to \mathrm{H}^1(G_K, V \otimes_{K,\sigma} t^n\mathbb{B}_{\mathrm{dR}}^+) \to \mathrm{H}^1(G_K, V \otimes_{K,\sigma} \mathbb{C}_p(n)). \end{aligned}$$

By easy induction, we know that $H^i(G_K, V \otimes_{K,\sigma} t^n\mathbb{B}_{\mathrm{dR}}^+) = 0$ for $i = 0, 1$ and $n > -n_1$. When $n = -n_1$, the first several terms of the long exact sequence above

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becomes

$$0 \to 0 \to \mathrm{H}^0(G_K, V \otimes_{K,\sigma} t^{-n_1}\mathbb{B}_{\mathrm{dR}}) \to \mathrm{H}^0(G_K, \mathbb{C}_p(n+n_1) \oplus \cdots \oplus \mathbb{C}_p(n+n_d)) \to 0.$$

Here, the relevant H^1-term vanishes because $n+1 > -n_1$. This forces $H^0(G_K, V \otimes_{K,\sigma} t^{-n_1}\mathbb{B}_{\mathrm{dR}}) \neq 0$ and therefore $\mathbf{D}_{\mathrm{dR},\sigma}(V) \neq 0$. □

Lemma 3. *If D is a $\mathbb{B}_{\mathrm{dR}}$-vector space of finite dimension d, with a semilinear action of G_K. Then the natural map $D^{G_K} \otimes_K \mathbb{B}_{\mathrm{dR}} \to D$ is always injective. In particular,* $\dim_K(D^{G_K}) \leq d$.

Proof. This is a standard B-admissibility argument; for completeness, we reproduce it here. We need only to show that a K-linearly independent subset of D^{G_K} is also $\mathbb{B}_{\mathrm{dR}}$-linearly independent. Suppose not, we pick a counterexample of minimal number of (K-linearly independent) elements in D^{G_K}; in other words, $e_1, \dots, e_r \in D^{G_K}$ are K-linearly independent but $\alpha_1 e_1 + \cdots + \alpha_r e_r = 0$ for $\alpha_i \in \mathbb{B}_{\mathrm{dR}} \backslash \{0\}$. Since $\mathbb{B}_{\mathrm{dR}}$ is a field, we may moreover assume that $\alpha_1 = 1$. Applying $g \in G_K$ to this equality, we have

$$e_1 + g(\alpha_2)e_2 + \cdots + g(\alpha_r)e_r = 0 \quad \Rightarrow (g\alpha_2 - \alpha_2)e_2 + \cdots + (g\alpha_r - \alpha_r)e_r = 0.$$

By the minimality of the linear relation, we conclude that $g\alpha_i = \alpha_i$ for $i = 2, \dots, d$. Hence each $\alpha_i \in \mathbb{B}_{\mathrm{dR}}^{G_K} = K$. But $e_1, \dots, e_r$ were assumed to be K-linearly independent. We arrive at a contradiction. This proves the lemma. □

Proposition 4. *Assume that K is Galois over $\mathbb{Q}_p$. Fix $\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)$. Let $V, W \in \mathbf{Rep}_K(G_K)$ be Hodge-Tate representations. If $W \otimes_K V$ is σ-de Rham, then so are V and W.*

Proof. Denote $n = \dim_K V$ and $m = \dim_K W$. By Lemma 2, we know that $\mathbf{D}_{\mathrm{dR},\sigma}(V) = (V \otimes_{K,\sigma} \mathbb{B}_{\mathrm{dR}})^{G_K} \neq 0$. Let r denote the dimension of this K-vector space. Consider the quotient Q of the injective map (by Lemma 3)

$$\mathbf{D}_{\mathrm{dR},\sigma}(V) \otimes_K \mathbb{B}_{\mathrm{dR}} \hookrightarrow V \otimes_{K,\sigma} \mathbb{B}_{\mathrm{dR}};$$

it is a vector space over $\mathbb{B}_{\mathrm{dR}}$ of dimension $n-r$ with continuous action of G_K. Now, taking the G_K-invariants of the following exact sequence

$$0 \to W \otimes_{K,\sigma} \mathbf{D}_{\mathrm{dR},\sigma}(V) \otimes_K \mathbb{B}_{\mathrm{dR}} \to (W \otimes_K V) \otimes_{K,\sigma} \mathbb{B}_{\mathrm{dR}} \to W \otimes_{K,\sigma} Q \to 0,$$

we obtain

$$0 \to \mathbf{D}_{\mathrm{dR},\sigma}(W) \otimes_K \mathbf{D}_{\mathrm{dR},\sigma}(V) \to \mathbf{D}_{\mathrm{dR},\sigma}(W \otimes_K V) \to \big(W \otimes_{K,\sigma} Q\big)^{G_K}.$$

By Lemma 3, we know that the dimensions of the first and third terms are at most rm and $(n-r)m$ respectively; whereas the dimension of the middle term is nm since $W \otimes_K V$ is σ-de Rham. Therefore, $\dim_K \mathbf{D}_{\mathrm{dR},\sigma}(W) \otimes_K \mathbf{D}_{\mathrm{dR},\sigma}(V) = rm$, yielding W being σ-de Rham. A symmetric argument proves that V is also σ-de Rham. □

As a corollary of Theorem 1, together with the well-known "black box": de Rham implies potentially semistable, it is not hard to deduce Theorem 5 below.

Theorem 5. *Let $V, W \in \mathbf{Rep}_E(G_K)$ be two p-adic representations of G_K. If $W \otimes_E V$ is crystalline, then there exists a finite extension F of E and a continuous character $\eta : G_K \to F^\times$ such that $V \otimes_E F(\eta)$ and $W \otimes_E F(\eta^{-1})$ are both crystalline,* *where $F(\eta)$ and $F(\eta^{-1})$ are 1-dimensional p-adic representations over F associated to η and η^{-1}, respectively.*

Proof. The reduction to Theorem 1 is carried out in [4]. For the convenience of the readers, we sketch the idea.

(As shown in [4, Section 2],) one can twist V and W by a character (coming from Lubin-Tate module of $\mathcal{O}_K$ to change the generalized Hodge-Tate weights (see [3]) to be integers; this is essentially because those Hodge-Tate weights from V and from W pairwise adds up to integers. One also easily see that the actions of the Tate-Sen operator on V and W are semisimple because their tensor product is. Hence, V and W are Hodge-Tate up to twists.

Now, by Theorem 1, one concludes that, up to the same twist, V and W are de Rham. By the main theorem of [2], they are (up to twist) potentially semistable. Consider the associated Deligne-Weil representation. This question essentially reduces to representation question over $\mathbb{C}$-vector spaces, and is discussed in [4, Theorem 1.4]. $\square$

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