RAMIFICATION OF HIGHER LOCAL FIELDS, APPROACHES AND QUESTIONS

LIANG XIAO AND IGOR ZHUKOV

Abstract. A survey paper includes facts, ideas and problems related to ramification in finite extensions of complete discrete valuation fields with arbitrary residue fields. Some new results are included.

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This is yet another attempt to organize facts, ideas and problems concerning ramification in finite extensions of complete discrete valuation fields with arbitrary residue fields.

We start in Section 3 with a rather comprehensive description of the classical ramification theory describing the behavior of ramification invariants in the case of perfect residue fields. This includes some observations that could be not published earlier, e.g., Prop. 3.3.2 and 3.5.1. We proceed in Section 4 with the detailed study of an example showing that almost the entire classical theory breaks down if we admit inseparable extensions of residue field and this cannot be easily repaired.

The remaining part of the survey describes several approaches aimed to reproduce parts of the classical theory in the non-classical setting.

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Before discussing general constructions of the upper ramification filtration, in Section 5 we consider separately abelian extensions starting with an important case of m-dimensional local fields (with finite last residue fields). The study of this case can be helpful in development of appropriate intuition, especially for those familiar with higher local class field theory. Introduction of m-dimensional local fields both determined interest to generalization of classical ramification theory and suggested tools for this; development of each of the main approaches to higher local class field theory (by Parshin, Kato, Fesenko) was complemented by studies of ramification theory for abelian extensions of such fields. We continue with a discussion of Kato's generalization of Swan conductors, which defines an upper ramification filtration for an abelian extension of any complete discrete valuation field.

Section 6 is devoted to the description of upper ramification filtrations in the general case. This section includes very different approaches: that of Abbes and T. Saito using rigid analytic geometry, and their reinterpretation by means of l-adic sheaves; that of Kedlaya and the second author using p-adic differential equations; that of Borger using generic perfection; and that of Boltje, Cram and Snaith. We list the basic properties of the ramification filtrations first, and then discuss how to prove the properties using specific constructions. We give references on the comparison results among these constructions. At the end, we introduce the notion of irregularities with properties analogous to those of ramification.

The next section starts with the observation that we still do not have a "fully satisfactory" ramification theory since the upper ramification filtration does not give us enough information about "naïve" invariants including the lower ramification filtration; we sketch some requirements for a "satisfactory theory". We proceed to describe an approach based on the theory of elimination of wild ramification. It results in a construction bearing some properties of the classical theory and giving additional information on the ramification of the given extension. This approach still does not fill the gap but gives some room for further developments as mentioned at the end of the section.

Sections 8 and 9 are devoted to the approach of Deligne who started to analyze 2-dimensional ramification problems by looking at all their 1-dimensional restrictions. This makes sense in the context of 2-dimensional schemes, and we suggest to study ramification in an extension of 2-dimensional local fields by "globalizing" the setting, i.e., constructing a sufficiently nice morphism of complete 2-dimensional local rings which serves as a model for given extension. For such morphisms Deligne's idea is applicable: we can look at the induced morphisms of algebroid curves on spectra of 2-dimensional rings and use the classical ramification invariants for them. This study is at a very beginning stage, with some initial observations and a lot of open questions.

In Section 10, we discuss the ramification theory in a semi-local or a global geometric context, for the l-adic and p-adic realizations as well as for the analogous algebraic D-module case. We will focus on the study of behavior of local information: Abbes-Saito ramification filtration, in a global context. The goal of the latter is to compute the Euler characteristic in all three situations in terms of the (local) ramification data, in hope to generalize the Grothendieck-Ogg-Shafarevich formula. Furthermore, we hope to describe or even define log-characteristic cycles using the ramification data.

The last section includes some open questions which we find curious and which are not covered in the previous text.

We almost do not touch here asymptotic properties of ramification numbers in infinite extensions and related notions of deeply ramified or arithmetically profinite extensions except for Subsection 3.10; our subject is restricted to the area of finite extensions of complete fields which still remains full of mystery.

We understand that the subject is not fashionable and in many aspects looks elementary. For this reason, various interesting results, observations, conjectures and questions have good chances to remain unpublished or tend to be forgotten; some of the included questions can already have answers. We would be happy to learn more about what is known and *what is unknown*; please do not hesitate to send us your comments and suggestions.

Anyway, we were concentrated mostly on the current state of the subject and even more on open questions (chosen according to our personal tastes); we did not aim to give a historical survey of the subject and apologize for obvious incompleteness (and possible bias) of the presenting traces of historical information.

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Notation

If K is a complete discrete valuation field of characteristic 0 or p with the residue field of characteristic $p > 0$, the following notation is used.

- $\pi = \pi_K$: an arbitrary uniformizing element of K;
- $v = v_K$: the valuation on K as well as its (non-normalized) extension to the algebraic closure of K; we normalize it so that $v(\pi_K) = 1$;
- $\mathcal{O}_K$: ring of valuation in K;
- $\mathfrak{m}_K = \{ a \in \mathcal{O}_K : v(a) > 0 \}$: the maximal ideal of $\mathcal{O}_K$;
- $U_K = \mathcal{O}_K^*$;
- $U_{i,K} = 1 + \mathfrak{m}_K^i$, $i \geq 1$;
- $|\cdot|$: the norm on K given by $|\pi|^{v(\cdot)}$; when K is of mixed characteristic, we require that $|p| = p^{-1}$;
- $\overline{K}$: the residue field of K;
- $\overline{a}$: the residue class in $\overline{K}$ of $a \in \mathcal{O}_K$;
- $e = e_K = v_K(p)$: the absolute ramification index of K;
- K^{alg}: an algebraic closure of K;
- K^{ab}: the maximal abelian extension of K inside a given K^{alg};
- G_K: the absolute Galois group of K (often abbreviated to G when there is no confusion);
- ζ_{p^n}: a primitive p^nth root of unity in K^{alg} (assuming $\operatorname{char} K = 0$).

For any integral scheme S, $k(S)$ is the field of rational functions on S. For an integral domain A, $Q(A)$ is its fraction field.

A representation of G_K is always assumed to be continuous.

1. Basic definitions

1.1. **Ramification invariants.**

Here we recall various ramification invariants associated with a finite extension L/K where K is a complete discrete valuation field with the residue field $\overline{K}$ of characteristic $p > 0$. We shall make a distinction between the classical case when $\overline{K}$ is perfect (or at least when $\overline{L}/\overline{K}$ is separable) and the non-classical case when this assumption is omitted.

We mention without reference facts proved in [Se68] or [FV]; in other cases, proofs or references are usually included.

The most well-known ramification invariants are:

• the *ramification index* $e(L/K) = v_L(\pi_K)$;

• the *different* $\mathcal{D}_{L/K}$, which can be defined, e.g., as the annihilator ideal of the $\mathcal{O}_L$-module of Kähler differentials $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$;

• the *depth of ramification*

$$d_M(L/K) = \inf_{a \in L}(v_M(\mathrm{Tr}_{L/K}\, a) - v_M(a)),$$

where M is any finite extension of K.

These three invariants are related by a simple formula ([Hy, formula (1-4)]):

$$v_L(\mathcal{D}_{L/K}) = e(L/K) - 1 + d_L(L/K). \tag{1}$$

One of the fundamental properties of the depth is its additivity [Hy, Lemma (2-4)]. Namely, for an intermediate field K' in L/K we have

$$d_M(L/K) = d_M(L/K') + d_M(K'/K). \tag{2}$$

We have

$$[L:K] = e(L/K)f(L/K) = e_t(L/K)e_w(L/K)f_s(L/K)f_i(L/K) = e_te_wf_sf_i,$$

where $(e_t, p) = 1$, $e_w = p^N$ for some $N \geq 0$, $f_s = [\overline{L} : \overline{K}]_{sep}$, $f_i = [\overline{L} : \overline{K}]_{ins}$.

A finite extension L/K is said to be:

• *unramified*, if $[L : K] = f_s$;

• *totally ramified*, if $f_s = f_i = 1$;

• *tame*, if $e_w = f_i = 1$;

• *wild*, if $[L : K] = e_w$;

• *ferocious*[1], if $[L : K] = f_i$;

• *weakly unramified*, if $e_t = e_w = 1$;

• *completely ramified*, if $e_t = f_s = 1$.

Note that L/K is tame if and only if $d_L(L/K) = 0$ [Hy, Remark (2-12)].

If L/K is a Galois extension with Galois group G, for any $\sigma \in G$ one defines the *Artin* and *Swan ramification numbers* by the formulas

$$i(\sigma) = i_G(\sigma) = \inf_{a \in \mathcal{O}_L} v_L(\sigma(a) - a);$$
$$s(\sigma) = s_G(\sigma) = \inf_{x \in L^*} v_L(\sigma(a)a^{-1} - 1).$$

In particular, our convention says $i_G(1) = s_G(1) = \infty$.

If $\mathcal{O}_L$ is generated by $x_1, \dots, x_n$ as an $\mathcal{O}_K$-algebra, we have

$$i_G(\sigma) = \inf_i v_L(\sigma(x_i) - x_i);$$
$$s_G(\sigma) = \inf_i v_L(\sigma(x_i)x_i^{-1} - 1).$$

[1]Such extensions are more often referred to as fiercely ramified; this is a translation of original French expression “ferocement ramifié”. However, John Coates told one of the authors that the English word “ferocious” is more appropriate here than “fierce”.

In the classical case we have ([Sn], 6.1.4):

$$s_G(\sigma) = \begin{cases} i_G(\sigma) - 1, & i_G(\sigma) > 0; \\ 0, & i_G(\sigma) = 0. \end{cases}$$

On the other hand, if L/K is ferocious, then $s_G(\sigma) = i_G(\sigma)$ for any $\sigma \in G$.

For an integer $i \geq -1$ the ith (*"lower"*) *ramification subgroup* is defined as

$$G_i = \{\sigma \in G : i_G(\sigma) \geq i + 1\}. \tag{3}$$

More generally, for non-negative integers n and i, the (n, i)*th ramification subgroup* is defined as

$$G_{n,i} = \{\sigma \in G : v_L(\sigma(x) - x) \geq n + i \text{ for all } x \in \mathfrak{m}_L^i\}.$$

It is a normal subgroup in G. There is a need to consider $G_{n,i}$ with $i > 0$ only in the non-classical case. Indeed, in the classical case $G_{n,i} = G_{n-1}$ if $p|i$, and $G_{n,i} = G_n$ otherwise ([dS, §2]).

The subgroups $G_n = G_{n+1,0}$ and $H_n := G_{n,1}$ form a filtration on G ([dS, Prop. 2.2–2.3]):

$$G \supseteq G_0 \supseteq H_1 \supseteq G_1 \supseteq H_2 \supseteq \cdots \supseteq \{1\}$$

Here $G/G_0 \simeq \mathrm{Gal}(\overline{L}/\overline{K})$ and $(G : G_0) = f_s(L/K)$; G_0/H_1 is a cyclic group of order $e_t(L/K)$; H_1 is a p-group of order $e_w(L/K)f_i(L/K)$. The subgroups G_0 and H_1 will be referred to as the *inertia subgroup* and the *wild ramification subgroup* of G respectively.

For $i > 1$, the subgroups $G_{n,i}$ are non-informative, since

$$G_{n,i} = \begin{cases} H_n, & p \nmid i, \\ G_{n-1}, & p \mid i, \end{cases} \quad \text{when } n > 1,$$

and (when $n = 1$) $G_{1,i}/H_1$ is exactly the kernel of multiplication by i in the cyclic group G_0/H_1 (see [dS, Prop. 2.3]).

All elements of $\{\, s(\sigma) : \sigma \in G_0, \sigma \neq 1\,\}$ are called the ("lower") *ramification breaks* of L/K. If L/K is an inseparable normal extension, the ramification breaks of L/K are defined as $\{$the breaks of $L_0/K\} \cup \{\infty\}$ where L_0/K is the maximal separable subextension of L/K.

In the classical case the breaks are exactly the nonnegative integers i with $G_i \neq G_{i+1}$. If $(G_i : G_{i+1}) = p^m$, then i is called a ramification break of *multiplicity* m.

For the rest of the subsection, we assume that $\overline{L}/\overline{K}$ is separable. For a Galois extension L/K, the *Hasse-Herbrand function* $\varphi_{L/K} : [-1, \infty) \to [-1, \infty)$ is a piecewise linear map defined by the formula

$$\varphi_{L/K}(u) = \int_0^u \frac{dt}{(G_0 : G_t)};$$

here it is assumed that $G_t = G_{[t]+1}$ for non-integral t, i.e., in the formula (3) we allow real numbers t, and $(G_0 : G_t) = 1$ for $t < 0$. Since $\varphi_{L/K}$ is strictly increasing, the inverse function $\psi_{L/K}$ is well defined.

It is known that, for a normal subextension M/K, we have

$$\varphi_{L/K} = \varphi_{M/K} \circ \varphi_{L/M}.$$

(It is essential here that we consider the classical case!) Therefore, $\varphi_{L/K}$ can be defined for an arbitrary finite separable extension L/K by the formula $\varphi_{L/K} = \varphi_{L'/K} \circ \psi_{L'/L}$, where L'/K is any finite Galois extension containing L/K.

Using the Hasse-Herbrand function, we define the "*upper*" *ramification subgroups*

$$G^u = G_{\psi_{L/K}(u)} \text{ for all } u \geq -1.$$

The non-negative rational numbers u such that $G^v \neq G^u$ for any $v > u$ are called the *upper ramification breaks* of L/K. The biggest such u is called the *highest ramification break*, denoted by $b(L/K)$.

The upper ramification breaks are exactly the ordinates of points on the graph of $\varphi_{L/K}$ where the slope changes, whereas the lower ramification breaks are their abscissas. The number 0 is a break if and only if $e_t \neq 1$; the other breaks are called *wild*. A change of slope by a factor p^m corresponds to a wild break of multiplicity m. This property can be used as a definition of lower and upper breaks for non-Galois finite extensions L/K. (In this case even the lower breaks need not be integral.)

1.1.1. **Example.** Let L/K be a totally ramified cyclic extension of degree p^n, and let $s_1 < \cdots < s_n$ be all Swan ramification numbers of L/K. Then L/K have n upper breaks $h_1 < \cdots < h_n$, all of multiplicity 1, and

$$h_r = s_1 + \sum_{i=2}^{r} \frac{s_i - s_{i-1}}{p^{i-1}} = \sum_{i=1}^{r} \frac{p-1}{p^i} s_i + \frac{1}{p^r} s_r. \tag{4}$$

1.2. m**-dimensional complete discrete valuation fields.** We give only definitions; see [HLF, Ch. I] for more information.

For K a field, a structure of an *m-dimensional complete discrete valuation field* (m-CDVF) on K is a sequence of fields $k_m = K$, k_{m-1}, ..., k_0 such that k_i is a complete discrete field with the residue field k_{i-1}, $1 \leq i \leq m$. The field k_{m-1} (resp. k_0) is referred to as the *first* (resp. the *last*) *residue field* of K.

If the last residue field is perfect, K is said to be an m-dimensional local field. (NB: often it is required that the last residue field is finite.)

A *system of local parameters* of K is any m-tuple $t_1, \ldots, t_m$ such that each t_i is a lifting to K of some uniformizing element of k_i.

Fix a system of local parameters $t_1, \ldots, t_m$ and consider the map

$$\mathbf{v}_K = (v_1, \ldots, v_m) \colon K^* \to \mathbb{Z}^m,$$

where $v_m = v_{k_m}$, $v_{m-1}(\alpha) = v_{k_{m-1}}(\alpha_{m-1})$, α_{m-1} is the class of $\alpha t_m^{-v_m(\alpha)}$ in k_{m-1}, and so on. Then $\mathbf{v}_K$ is a discrete valuation of rank m; here it is assumed that $\mathbb{Z}^m$ is lexicographically ordered as follows: $\mathbf{i} = (i_1, \ldots, i_m) < \mathbf{j} = (j_1, \ldots, j_m)$, if and only if

$$i_l < j_l,\ i_{l+1} = j_{l+1}, \ldots, i_m = j_m \text{ for some } l \leq m.$$

If we change the system of local parameters, the valuation is replaced by an equivalent one. Thus, $\mathbf{v}_K$ is defined up to equivalence.

For any finite extension L/K, there exists a unique structure of an m-dimensional complete discrete valuation field on L compatible with that on K; the non-normalized ($\mathbb{Q}^m$-valued) extension of $\mathbf{v}_K$ on L is also denoted by $\mathbf{v}_K$.

The notion of depth of ramification can be generalized as follows ([Hy, (1-3)]):

$$\mathbf{d}_M(L/K) = \inf_{a \in L} (\mathbf{v}_M(\mathrm{Tr}_{L/K}\, a) - \mathbf{v}_M(a)),$$

where both L and M are finite extensions of K.

2. Cyclic extensions of degree p and genome

2.1. Cyclic extensions of degree p. Here we look carefully at the case of a Galois extension L/K with $[L:K]=p$ (see also [Hy, Lemma (2-16)]). This is important for discussing examples in the subsequent sections.

We fix a generator σ of the Galois group $G=\mathrm{Gal}(L/K)$; then $i(\sigma)$ and $s(\sigma)$ are independent of the choice of σ; so we put $s(L/K)=s(\sigma)$.

Since $[L:K]=e_te_wf_sf_i$, and e_t is prime to p, there are 3 cases.

Case U (unramified): $f_s=p$, $e_w=f_i=1$. In this case, $i(\sigma)=s(\sigma)=0$.

Case W (wild): $e_w=p$, $f_s=f_i=1$. Set $s=v_L(\sigma(\pi_L)/\pi_L-1)$. Then $\mathcal{O}_L=\mathcal{O}_K[\pi_L]$ immediately implies $i(\sigma)=s+1$ and $s(\sigma)=s$.

Case F (ferocious): $f_i=p$, $f_s=e_w=1$. Choose any $t\in\mathcal{O}_L$ such that $\overline{t}\notin\overline{K}$. Set $s=v_L(\sigma(t)/t-1)$. Then $\mathcal{O}_L=\mathcal{O}_K[t]$ and $i(\sigma)=s(\sigma)=s$.

In all three cases we have $d_L(L/K)=(p-1)s(L/K)$.

Let us compute the ramification invariants for specific constructions of cyclic extension of degree p, i.e., for Artin-Schreier and Kummer extensions.

1°. $\operatorname{char}K=p$. In this case $L=K(x)$ for x satisfying $x^p-x=a\in K$. We put $\wp(X)=X^p-X$. We have $v(a)\le 0$ since $\mathfrak{m}_K\subset\wp(K)$ by Hensel's lemma. Choose an equation with maximal possible $v(a)$.

If $v(a)=0$, the Hensel's lemma implies $\overline{a}\notin\wp(\overline{K})$, and we are in Case U.

If $v(a)<0$ and $p\nmid v(a)$, we are obviously in Case W, and $s(L/K)=-v(a)$.

If $v(a)<0$ and $p\mid v(a)$, the maximality of $v(a)$ implies that $\overline{\pi^{-v(a)}a}\notin\overline{K}^p$. It follows that we are in Case F, and $s(L/K)=-v(a)/p$.

2°. $\operatorname{char}K=0$, $\zeta_p\in K$. In this case $L=K(x)$ for x satisfying $x^p=a\in K$. We can choose a with $v(a)=1$ or $v(a)=0$; in the latter case we require that $l=v(a-1)$ is maximal. Then we can distinguish 5 cases.

A. $v(a)=1$. Here we are in Case W, and
$$s(L/K)=v_L(\zeta_p-1)=\frac{e_L}{p-1}=\frac{pe}{p-1}.$$

B. $v(a)=0$ and $\overline{a}\notin\overline{K}^p$. This is Case F, and
$$s(L/K)=v_L(\zeta_p-1)=\frac{e_L}{p-1}=\frac{e}{p-1}.$$

C. $v(a)=0$, $\overline{a}=1$, $l<\frac{pe}{p-1}$, $p\nmid l$. This is Case W, and $s(L/K)=\frac{pe}{p-1}-l$.

D. $v(a)=0$, $\overline{a}=1$, $l<\frac{pe}{p-1}$, $p\mid l$. From the maximality of l it follows that this is Case F, and $s(L/K)=\frac{1}{p}\big(\frac{pe}{p-1}-l\big)$.

E. $v(a)=0$, $\overline{a}=1$, $l\ge\frac{pe}{p-1}$. It follows from Hensel's lemma that in fact $l=\frac{pe}{p-1}$, and this is Case U.

2.2. Genome of an extension. Let L/K be a cyclic extension of degree p^n. It can be uniquely written as a tower $L=M_n/M_{n-1}/\dots/M_1/M_0=K$ of cyclic extensions of degree p. The *genome* of L/K is defined to be the word $T_1\dots T_n$, where
$$T_i=\begin{cases}W, & \text{if } M_i/M_{i-1} \text{ is wild,}\\ F, & \text{if } M_i/M_{i-1} \text{ is ferocious.}\end{cases}$$

However, it is not clear how to define the genome for a general Galois extension of degree p^n.

2.2.1. **Question.** Let L/K be a completely ramified Galois extension. Can we define a tower $L = M_n/M_{n-1}/\dots/M_1/M_0 = K$ of cyclic extensions of degree p in an "almost canonical" way so that the word $T_1 \dots T_n$ as above is well defined?

3. What is nice in the classical case

Throughout this section we consider only the case when $\overline{K}$ is perfect. We list various facts which are sometimes referred to as "beautiful ramification theory" in the classical case. (Probably the whole collection of facts has not been ever included in one text.)

3.1. **Factor groups.** Let K' be an intermediate field in L/K. Then the ramification invariants of K'/K can be described in terms of those of L/K. More specifically, let L/K be a finite Galois extension with $G = \mathrm{Gal}(L/K)$, and K' an intermediate extension corresponding to a normal subgroup H. Then for any $\sigma \in G/H$, $\sigma \neq 1$, the Herbrand's theorem (see [Se68, Ch. IV, Prop. 3]) says

$$i_{G/H}(\sigma) = \frac{1}{e_{L/K'}} \sum_{\tau H = \sigma} i_G(\tau). \tag{5}$$

It follows that we have the following statement comparing the lower and the upper ramification filtrations on G/H with those on G.

3.1.1. **Proposition.** *1. For any $v \geq -1$ we have $(G/H)_v = G_{\psi_{L/K'}(v)} H/H$.*

2. For any $v \geq -1$ we have $(G/H)^v = G^v H/H$.

3.1.2. **Corollary.** *If $H = G_j$ for some j, then*

$$(G/H)_i = \begin{cases} G_i/H, & i \leq j, \\ \{1\}, & i \geq j. \end{cases}$$

One of the nice consequences of Prop. 3.1.1 is that we can define the upper ramification filtration for an infinite Galois extensions L/K by the formula

$$\mathrm{Gal}(L/K)^v = \varprojlim_{\substack{L'/K \text{ finite} \\ L' \subset L}} \mathrm{Gal}(L'/K)^v.$$

In particular, we have an upper ramification filtration on the whole absolute Galois group.

3.2. **Subgroups.** Let L/K be a finite Galois extension, and K'/K any subextension. Put $G = \mathrm{Gal}(L/K)$ and $H = \mathrm{Gal}(L/K')$. Obviously, $H_i = G_i \cap H$ for any i. Therefore,

$$H^i = H_{\psi_{L/K'}(i)} = G_{\psi_{L/K'}(i)} \cap H = G^{\varphi_{L/K} \circ \psi_{L/K'}(i)} \cap H = G^{\varphi_{K'/K}(i)} \cap H.$$

3.3. **Base change.** Here we observe how the ramification invariants change as one passes from L/K to LK'/K' for some finite extension K'/K linearly disjoint with L/K. We start with the basic case of two Galois extensions of degree p.

3.3.1. **Lemma.** *1. Let L_1/K and L_2/K be Galois extensions of degree p with positive $s_1 = s(L_1/K)$ and $s_2 = s(L_2/K)$, and $s_1 < s_2$. Then $s(L_1L_2/L_2) = s_1$, and $s(L_1L_2/L_1) = s_1 + p(s_2 - s_1)$.*

2. Let L_1/K and L_2/K be linearly disjoint Galois extensions of degree p such that $s = s(L/K) > 0$ is the same for any subextension L/K of degree p in L_1L_2/K. Then $s(L_1L_2/L_2) = s(L_1L_2/L_1) = s$.

Proof. Set $L = L_1L_2$ and $G = \mathrm{Gal}(L/K)$.

Assume first that L/K has two distinct lower ramification breaks $s'_1 < s'_2$. Put $H_2 = G_{s'_1+1}$, $K' = L^{H_2}$. Then by Cor. 3.1.2 we have

$$\mathrm{Gal}(K'/K)_i = \begin{cases} \mathrm{Gal}(K'/K), & i \le s'_1, \\ \{1\}, & i > s'_1, \end{cases}$$

whence $s(K'/K) = s'_1$.

Let K''/K be any other subextension of degree p in L/K. Put $H = \mathrm{Gal}(L/K'')$. Let σ_0 be any element of G outside H. Note that $\sigma_0 H$ contains a unique element of H_2 whose Artin number is $s'_2 + 1$. By (5),

$$i_{G/H}(\sigma_0|_{K''}) = \frac{1}{p}((p-1)\cdot(s'_1+1) + 1\cdot(s'_2+1)) = s'_1 + \frac{s'_2 - s'_1}{p} + 1.$$

It follows that $s(K''/K) = s'_1 + \frac{s'_2-s'_1}{p}$. Since s_1 and s_2 are among $s(K'/K)$ and (all) $s(K''/K)$, and $s_1 < s_2$, we conclude that $s_1 = s'_1$, $s_2 = s'_1 + \frac{s'_2-s'_1}{p}$.

In the remaining case when L/K has one break s' of multiplicity 2, the same computation shows that $s(K''/K) = s'$ for any subextension K''/K of degree p in L/K. □

This can be generalized as follows.

3.3.2. **Proposition.** *Let L/K and K'/K be finite Galois p-extensions. Assume that L/K have upper ramification breaks $h_1, \dots, h_r$ with multiplicities $m_1, \dots, m_r$. Assume that all the upper ramification breaks of K'/K are distinct from $h_1, \dots, h_r$. Then the upper ramification breaks of LK'/K' are $\psi_{K'/K}(h_1), \dots, \psi_{K'/K}(h_r)$ and their multiplicities are $m_1, \dots, m_r$.*

Proof. For $[L : K] = [K' : K] = p$, this is the first part of Lemma 3.3.1. The general case follows by double induction on $[L : K]$ and $[K' : K]$. □

3.3.3. **Question.** If L/K and K'/K are Galois extensions of degree p with the same ramification break, we cannot determine the ramification invariants of LK'/K' in general. However, in view of the second part of Lemma 3.3.1, we can do this if we know the ramification breaks of all subextensions of degree p in LK'/K.

How can this observation be generalized to arbitrary finite Galois p-extensions L/K and K'/K?

3.4. **Filtration on the group of units and the norm map.** For a finite extension L/K, consider the norm map $N_{L/K} : L^* \to K^*$ and its interaction with the filtration on K^* given by the subgroups $U_{i,K}$ for $i \ge 1$, and the similar filtration on L^*. For any $i \ge 1$, define $f(i)$ by the conditions

$$N_{L/K}U_{i,L} \subset U_{f(i),K}, \quad N_{L/K}U_{i,L} \not\subset U_{f(i)+1,K}.$$

Then the map $f = f_{L/K}$ can be computed from the ramification breaks of L/K and vice versa, at least if the residue field $\overline{K}$ is infinite. Indeed, [FV, Prop. (3.1)] states essentially the following.

3.4.1. **Proposition.** *Assume that $\overline{K}$ is infinite. Let L/K be a finite Galois extension. Put $\psi = \psi_{L/K}$. Then for any positive integer j we have $f(i) = j$, if $\psi(j-1) + 1 \le i \le \psi(j)$.*

3.4.2. **Remark.** Thus, for infinite $\overline{K}$, $f_{L/K}(i)$ is equal to the minimal integer not less than $\varphi_{L/K}(i)$. If $\overline{K}$ is finite, $f_{L/K}(i)$ can "jump" at the lower ramification breaks of L/K.

3.4.3. **Question.** How to express $f_{L/K}$ in terms of $\varphi_{L/K}$ when $\overline{K}$ is finite?

3.4.4. **Question.** What is the exact relation between $f_{L/K}$ and $\varphi_{L/K}$ for a non-Galois extension L/K?

3.5. **Artin-Schreier and Kummer filtrations and the embedding map.** First assume that $\operatorname{char} K = p$. Then we have a filtration on $K/\wp(K)$ by the subgroups

$$C_{i,K} = (\mathfrak{m}_K^i + \wp(K))/\wp(K), \quad i \le 0.$$

(Recall that $\mathfrak{m}_K \subset \wp(K)$ by Hensel lemma.) Then, for a finite extension L/K, we can consider the interaction of this filtration with a similar one on $L/\wp(L)$. For $i \ge 0$, let $g(i)$ denote the unique integer such that $\varepsilon(C_{-i,K}) \subset C_{-g(i),L}$ and $\varepsilon(C_{-i,K}) \not\subset C_{-g(i)+1,L}$, where $\varepsilon : K/\mathfrak{p}(K) \to L/\mathfrak{p}(L)$ is the natural map.

In the same spirit, if $\operatorname{char} K = 0$, $\zeta_p \in K$, we can consider the filtration on $K^*/(K^*)^p$ given by the subgroups

$$C^*_{i,K} = U_{i,K}(K^*)^p/(K^*)^p, \quad 1 \le i \le \frac{pe}{p-1}.$$

(Recall that $U_{\frac{pe}{p-1}+1,K} \subset (K^*)^p$.) For a finite extension L/K and a positive integer $i < \frac{pe}{p-1}$, we let $g(i)$ denote the unique integer such that $\varepsilon\big(C^*_{\frac{pe}{p-1}-i,K}\big) \subset C^*_{\frac{pe}{p-1}-g(i),L}$ and $\varepsilon\big(C^*_{\frac{pe}{p-1}-i,K}\big) \not\subset C^*_{\frac{pe}{p-1}-g(i)+1,L}$, where $\varepsilon : K^*/(K^*)^p \to L^*/(L^*)^p$ is the natural map.

The function $g = g_{L/K}$ in both cases is closely related to $\psi = \psi_{L/K}$. Namely, Prop. 3.3.2 and explicit computation of the ramification break for an Artin-Schreier or Kummer extension immediately imply the following

3.5.1. **Proposition.** *Let i be a positive integer not divisible by p and distinct from any upper ramification break of L/K. (We also require $i < \frac{pe}{p-1}$ if $\operatorname{char} K = 0$.) Then $g(i) = \psi(i)$.*

If $\overline{K}$ is infinite, we can use the second part of Lemma 3.3.1 to prove

3.5.2. **Proposition.** *When $\overline{K}$ is infinite and when i is a positive integer not divisible by p (provided $i < \frac{pe}{p-1}$ if $\operatorname{char} K = 0$), we have $g(i) = \psi(i)$.*

Since the upper breaks are always prime to p, this means that g determines the ramification invariants of L/K whenever $\operatorname{char} K = p$ or $\overline{K}$ is infinite.

Similarly, if $\zeta_{p^n} \in K$, one can define an explicit filtration on $K^*/(K^*)^{p^n}$ compatible with the upper ramification filtration on the maximal abelian extension of K of exponent p.

3.5.3. **Question.** Can we recover $\psi_{L/K}$ from the filtrations on $K(\zeta_{p^n})^*/(K(\zeta_{p^n})^*)^{p^n}$ for all n, thus eliminating the condition $i < \frac{pe}{p-1}$ in Prop. 3.5.2?

If $\operatorname{char} K = p$, the explicit form of the filtration on $W_r(K)/\wp(W_r(K))$ compatible with the ramification filtration is given in [Br, §1]. Here W_r denotes the group Witt vectors of length r, and

$$\wp((x_0, \dots, x_{r-1})) = (x_0^p, \dots, x_{r-1}^p) -_{W_r(K)} (x_0, \dots, x_{r-1});$$

note that Brylinski uses a different notation. For a new proof and very clear treatment of related questions, see [Th].

3.6. **Hasse-Arf theorem.**

3.6.1. **Theorem.** *Let L/K be a finite abelian extension. Then all upper ramification breaks of L/K are integral.*

See [Se68, Ch. IV, §3], [FV, Ch. III, (4.3)].
An inverse result is due to Fesenko [Fe95b]:

3.6.2. **Proposition.** *Let L/K be a totally ramified finite Galois extension such that for any totally ramified finite abelian extension K'/K all upper ramification breaks of LK'/K' are integral. Then L/K is abelian.*

3.6.3. **Question.** Can we replace the class of all abelian extensions K'/K by a smaller class here, e.g., by the class of all elementary abelian extensions, at least in the case $\operatorname{char} K = p$?

3.6.4. **Question.** For a finite Galois extension L/K, can we determine $\mathrm{Gal}(L/K)$, if we know all upper ramification breaks of LK'/K' for all abelian extensions K'/K?

One of the related results is the following Sen congruence (see, e.g., [Sn, Theorem 6.1.34]).

3.6.5. **Proposition.** *Let L/K be a finite Galois extension, $\sigma \in \mathrm{Gal}(L/K)$ such that $s(\sigma) > 0$ and $\sigma^{p^n} \neq 1$. Then*

$$s\big(\sigma^{p^{n-1}}\big) \equiv s\big(\sigma^{p^n}\big) \mod p^n.$$

3.7. **Artin and Swan representations.** (See [Se68, Ch. VI], [Se77] as well as the discussion in [Sn, 6.1].) Fix a finite Galois extension L/K, and put $G = \mathrm{Gal}(L/K)$. We define the *Artin and Swan central function* a_G, $sw_G : G \to \mathbb{Z}$ by formulas

$$a_G(\sigma) = \begin{cases} -f \cdot i_G(\sigma), & \sigma \neq 1, \\ f \sum_{\tau\neq 1} i_G(\tau), & \sigma = 1, \end{cases}$$

$$sw_G(\sigma) = \begin{cases} -f \cdot s_G(\sigma), & \sigma \neq 1, \\ f \sum_{\tau\neq 1} s_G(\tau), & \sigma = 1, \end{cases}$$

where $f = f(L/K)$.

The Serre's theorem on the existence of Artin representations ([Se77, p. 68]) claims:

3.7.1. **Proposition.** *The central functions a_G and sw_G are characters of certain complex representations of G.*

For the corresponding representations A_G and SW_G we have the following explicit formulas in the ring of complex representations $R(G)$ (cited from [Sn, 6.1]):

$$A_G = \sum_{i=0}^{\infty} [G_0 : G_i]^{-1} Ind_{G_i}^G (Ind_{\{e\}}^G(1) - 1)$$

and

$$SW_G = A_G + Ind_{G_0}^G(1) - Ind_{\{e\}}^G(1),$$

where $Ind_H^G(V)$ denotes the representation of G induced by the representation V of H, and 1 is the class of 1-dimensional trivial representation of the corresponding group.

For a normal subgroup H of G it follows from Herbrand's theorem that

$$SW_{G/H} \simeq SW_G \otimes_{\mathbb{C}[G]} \mathbb{C}[G/H]. \tag{6}$$

For the character χ of a complex representation V of G, the *Artin conductor* of χ (or V) is defined as

$$\mathrm{Ar}_K(\chi) = \mathrm{Ar}_K(V) = \langle a_G, \chi \rangle_G = \frac{1}{|G|} \sum_{g \in G} a_G(g) \overline{\chi(g)}.$$

Similarly, the *Swan conductor* of χ (or V) is

$$\mathrm{Sw}_K(\chi) = \mathrm{Sw}_K(V) = \langle sw_G, \chi \rangle_G = \frac{1}{|G|} \sum_{g \in G} sw_G(g) \overline{\chi(g)};$$

we have

$$\mathrm{Sw}_K(V) = \mathrm{Ar}_K(V) + \dim V^{G_0} - \dim V.$$

3.7.2. **Example.** (see [Se68, Ch. VI, Prop. 5]) Let L/K be a totally ramified cyclic extension of degree p^n, and χ the character of any faithful (i.e., injective) representation of $G = \mathrm{Gal}(L/K) = \langle g \rangle$. Let $s_1 < \cdots < s_n$ be all Swan ramification numbers of L/K. Then

$$\begin{aligned}
\mathrm{Sw}_K(\chi) &= \frac{1}{p^n} \sum_{i=1}^{p^n} \zeta^i sw_G(g^i) \\
&= \frac{1}{p^n} \sum_{r=0}^{n} \sum_{v_p(i)=r} \zeta^i sw_G(g^i) \\
&= -\frac{1}{p^n} \sum_{r=0}^{n-1} s_{r+1} \sum_{v_p(i)=j} \zeta^i + \frac{1}{p^n} \sum_{i=1}^{p^n-1} s_G(g^i) \\
&= \frac{1}{p^n} \Big(s_n + \sum_{r=0}^{n-1} (p^{n-r} - p^{n-r-1}) s_{r+1} \Big) \\
&= b(L/K)
\end{aligned}$$

in view of (4), where ζ is a primitive p^nth root of unity in $\mathbb{C}$.

3.7.3. **Remark.** This is the simplest case of the following fact (see [Se68, Ch. VI, §2, Ex. 2]). Let V be an irreducible representation of G of dimension d. Then $\mathrm{Ar}_K(V) = d(b(L/K) + 1)$, where $b(L/K)$ is the highest (upper) ramification break defined in Section 1.1.

As a consequence of this fact, we may define the Artin conductor and Swan conductor of a finite dimensional complex representation V of G to be

$$\mathrm{Ar}_K(V) = \sum_{a \geq -1} (a+1) \cdot \dim V^{G^{a+}} / V^{G^a}, \quad \mathrm{Sw}_K(V) = \sum_{a \geq 0} a \cdot \dim V^{G^{a+}} / V^{G^a}.$$

Note that one can recover the ramification filtration on G from Artin conductors of all its irreducible representations. (The same does not hold for Swan conductors since Swan conductor measures only *wild* ramification and does not know anything about $(G_0 : G_1)$.)

In a similar way, one can define Swan conductors for $\mathbb{F}_l$-representations; this version of Swan conductor is used in the Grothendieck-Ogg-Shafarevich formula (see Subsection 3.11 below).

There is an alternative and equivalent way of stating Proposition 3.7.1.

3.7.4. **Proposition.** *For all finite dimensional complex representation V of G, the Artin conductor $\mathrm{Ar}_K(V)$ and the Swan conductors $\mathrm{Sw}_K(V)$ are non-negative integers.*

Applying this to all one-dimensional representations of G and using the above explicit description of Artin and Swan conductors (Remark 3.7.3), we obtain that $b(L/K)$ is always an integer for an abelian extension L/K. Thus, we recover the original Hasse-Arf Theorem 3.6.1. So sometimes the above proposition will be also referred to as the *Hasse-Arf theorem.*

3.8. **Local class field theory.** Let K be a complete discrete valuation field of any characteristic with a quasi-finite residue field of prime characteristic. (A field F is called *quasi-finite* if $G_F \simeq \widehat{\mathbb{Z}}$.)

The central theorem of local class field theory states that there exists a homomorphism $\Theta_K : K^* \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ uniquely determined by the following two properties.

1. For any finite abelian extension L/K, Θ_K induces an isomorphism $\Theta_{L/K} : K^*/N_{L/K}L^* \to \mathrm{Gal}(L/K)$.

2. For any prime element π_K, the restriction of $\Theta_K(\pi_K)$ on the maximal unramified extension of K is the Frobenius automorphism.

It appears that the reciprocity map transforms the valuation filtration on the multiplicative group into the upper ramification filtration on (abelian) Galois group. More precisely, we have the following results. ([Se68], Ch. XV, Th.1 with Cor. 3 and Th. 2. Note that $N_{L/K}U_{\psi(n),L} \subset U_{n,K}$ by Prop. 3.4.1.)

3.8.1. **Proposition.** *Let L/K be a finite abelian extension. Put $\psi = \psi_{L/K}$.*

1. For any positive integer n, the canonical map $U_{n,K}/N_{L/K}U_{\psi(n),L} \to K^*/N_{L/K}L^*$ is injective.

2. The reciprocity map $\Theta_{L/K}$ transforms the filtration on $K^*/N_{L/K}L^*$ by subgroups $U_{n,K}/N_{L/K}U_{\psi(n),L}$ into the filtration on $G = \mathrm{Gal}(L/K)$ by G^n.

3.8.2. **Proposition.** *Let L/K be a possibly infinite abelian extension with Galois group $G = \mathrm{Gal}(L/K)$. Then for any positive integer n, the image of $\Theta_K(U_{n,K}) \subset \mathrm{Gal}(K^{\mathrm{ab}}/K)$ in G is dense in G^n (and is equal to G^n if the residue field $\overline{K}$ is finite).*

In characteristic 0, provided $\zeta_p \in K$, this implies the self-duality of the valuation filtration on $K^*/(K^*)^p$ with respect to the Hilbert symbol. In characteristic p, we have a duality between the valuation filtration on $K^*/(K^*)^{p^r}$ and the Brylinski filtration on $W_r(K)/\wp(W_r(K))$, see [Br, Theorem 1].

For Fesenko's non-abelian reciprocity map [Fe01], compatibility with the ramification filtration was established in [IS].

3.9. **Local anabelian geometry.** Let K_1 and K_2 be local fields (complete discrete valuation fields with *finite* residue fields) such that there exists an isomorphism between absolute Galois groups of K_1 and K_2 preserving the ramification filtration. Then this isomorphism is induced by an isomorphism between K_1 and K_2.

This was first proved in the characteristic 0 case by Sh. Mochizuki [Mo-S]. A proof suitable for any characteristic was given by Abrashkin [Abr00, Abr10].

3.10. **A theorem of Deligne.** Let K and K' be two complete discrete valuation fields (typically with large absolute ramification indices in the case of mixed characteristic). Assume that there exists $b \in \mathbb{N}$ such that there is an isomorphism $\mathcal{O}_K/\pi_K^b\mathcal{O}_K \cong \mathcal{O}_{K'}/\pi_{K'}^b\mathcal{O}_{K'}$ as *rings*. Deligne [De84] proved the following result.

3.10.1. **Proposition.** *Keep the notation as above. If $\overline{K}$ has a perfect residue field, then there is a canonical isomorphism*

$$G_K/G_K^b \cong G_{K'}/G_{K'}^b. \tag{7}$$

In other words, the quotient Galois groups above depend only on the truncated discrete valuation rings $\mathcal{O}_K/\pi_K^b\mathcal{O}_K \cong \mathcal{O}_{K'}/\pi_{K'}^b\mathcal{O}_{K'}$. Note that there were no assumptions on the characteristics of K and K'. In particular, they could be *different*, which may be used to build a connection between the mixed characteristic fields and the equal characteristic fields on the aspect of ramification theory.

Deligne's theorem provides an alternative way to understand the field of norms of Fontaine and Wintenberger [FW1, FW2] (which precedes Deligne's work).

Put $K_n = \mathbb{Q}_p(\zeta_{p^n})$ for $n \in \mathbb{N}$ and $K_\infty = \cup_{n\in\mathbb{N}} K_n$. We take the uniformizer π_{K_n} to be $\zeta_{p^n} - 1$. Then the tower $(K_n)_{n\in\mathbb{N}}$ is APF (short for arithmetically profinite) in the sense of [FW1, FW2]. The following statement is a special case of the main result of Fontaine-Wintenberger [FW1, FW2] (exposed also in [FV, Ch. III, Theorem 5.7]).

3.10.2. **Theorem.** *There is a canonical isomorphism between the absolute Galois group of K_∞ and that of the equal characteristic field $\mathbb{F}_p((T))$.*

One can give a heuristic proof using Deligne's theorem as follows. For each n, we put $r_n = p^{n-1}(p-1)$ so that $\mathcal{O}_{K_n}/\pi_{K_n}^{r_n} \cong \mathbb{F}_p[\![T]\!]/(T^{r_n})$. Deligne's theorem then implies that we have an isomorphism

$$G_{\mathbb{F}_p((T))}/G_{\mathbb{F}_p((T))}^{r_n} \cong G_{K_n}/G_{K_n}^{r_n}. \tag{8}$$

An easy computation shows that $\varphi_{K_n/K}(n) = r_n$. The basic property in Subsection 3.2 implies that $G_{K_n}^{r_n} = G_{\mathbb{Q}_p}^n \cap G_{K_n}$. Thus, taking the inverse limit of (8) gives an isomorphism between $G_{\mathbb{F}_p((T))}$ and G_{K_∞}.

We expect that the same proof works for general complete discrete valuation field K in place of $\mathbb{Q}_p$, at least when the residue field $\overline{K}$ is perfect, and hence we could reprove the main result of [FW1, FW2] this way. The APF condition is expected to ensure that the inverse limit of (8) as $n \to \infty$ gives the isomorphism between the Galois group of K_∞ and that of $\overline{K}((T))$. Unfortunately, we do not know if such a proof exists in the literature.

3.11. **Global formulas.** Let $\mathcal{X}$ be a smooth projective curve over an algebraically closed field and let $\mathcal{Y}$ be its normalization in a finite extension of $k(\mathcal{X})$. Riemann-Hurwitz formula compares the genera of these curves:

$$2g_{\mathcal{Y}} - 2 = [k(\mathcal{Y}) : k(\mathcal{X})](2g_{\mathcal{X}} - 2) + \sum_{Q} v_Q(\mathcal{D}_{\mathcal{Y}/\mathcal{X}}),$$

where Q runs over all closed points of $\mathcal{Y}$.

Let U be a dense open subset of $\mathcal{X}$, $\bar{\eta}$ a geometric generic point of $\mathcal{X}$, and $\mathcal{F}$ a locally constant sheaf of $\mathbb{F}_l$-modules of finite rank on $U_{\text{ét}}$. Then the geometric generic fiber $M = \mathcal{F}_{\bar{\eta}}$ is a finite-dimensional $\mathbb{F}_l$-representation of $\mathrm{Gal}(k(\mathcal{X}))$; it factors through $\mathrm{Gal}(L/k(\mathcal{X}))$, where $L/k(\mathcal{X})$ is some finite Galois extension.

For a closed point P of $\mathcal{X}$, the Swan conductor $\mathrm{Sw}_P\,\mathcal{F}$ is defined as the Swan conductor of M considered as $\mathrm{Gal}(L_w/k(\mathcal{X})_v)$-module, where v corresponds to P, and w is any extension of v to L. Independence of L follows from an $\mathbb{F}_l$-analog of (6). Then the Grothendieck-Ogg-Shafarevich formula for $\mathcal{F}$ reads:

$$\chi_c(U, \mathcal{F}) = \chi_c(U, \mathbb{F}_l)\,\mathrm{rank}\,\mathcal{F} - \sum_{P \in \mathcal{X} \setminus U} \mathrm{Sw}_P\,\mathcal{F}, \tag{9}$$

where $\chi_c(U, \cdot)$ is the Euler-characteristic of the corresponding étale sheaf. (This can be obtained from the shape of G.-O.-S. formula in [Mil] as follows. Let $u : U \hookrightarrow \mathcal{X}$, $\mathcal{F}_0$ a constant sheaf on $U_{\text{ét}}$ of rank equal to $\mathrm{rank}\,\mathcal{F}$. Apply the formula in [Mil, Ch. V, Th. 2.12] to both $u_!\mathcal{F}$ and $u_!\mathcal{F}_0$ and compute the difference.)

See [Kö] for equivariant versions of Riemann-Hurwitz and Grothendieck-Ogg-Shafarevich formulas.

3.11.1. **Remark.** We point out that there is an analogous statement for lisse $\mathbb{Q}_l$-sheaves instead of lisse $\mathbb{F}_l$-sheaves.[2] In fact the formula for the former reduces to that of the latter, as we explain now.

A lisse $\mathbb{Q}_l$-sheaf $\mathcal{F}$ corresponds to a representation $\rho : \pi_1(U) \to \mathrm{GL}_d(\mathbb{Q}_l)$. Since the fundamental group is profinite and hence compact, the image $\rho(\pi_1(U))$ lands in $\mathrm{GL}_d(\mathbb{Z}_l)$ (up to conjugation). This integral representation ρ° gives rise to a lisse $\mathbb{Z}_l$-sheaf $\mathcal{F}^\circ$. Put $\bar{\rho} = \rho^\circ \bmod l$ and $\bar{\mathcal{F}} = \mathcal{F}^\circ/l$. It is not difficult to show that the Euler characteristic of $\mathcal{F}$ agrees with that of $\bar{\mathcal{F}}$. We need to match the Swan conductors.

Note that, for each point $P \in \mathcal{X}\setminus U$, the wild ramification group W_P at P is a pro-p group; but the kernel of $\mathrm{GL}_d(\mathbb{Z}_l) \to \mathrm{GL}_d(\mathbb{F}_l)$ is a pro-l group. Hence the image $\rho(W_P)$ has trivial intersection with $\mathrm{Ker}(\mathrm{GL}_d(\mathbb{Z}_l) \to \mathrm{GL}_d(\mathbb{F}_l))$; consequently, we have an isomorphism $\rho(W_P) \cong \bar{\rho}(W_P)$. From this it is clear that $\mathrm{Sw}_P\mathcal{F} = \mathrm{Sw}_P\bar{\mathcal{F}}$, since both sides depend only on the action of the wild inertia group.

3.12. **Completeness.** Given a finite Galois extension of complete discrete valuation fields L/K with $\mathrm{Gal}(L/K) = G$, we have a number of ramification invariants occurring in various formulas: $e(L/K)$, $v_L(\mathcal{D}_{L/K})$, G_i and G^i for $i \geq 0$, $\mathrm{Ar}_K(V)$ and $\mathrm{Sw}_K(V)$ for a complex representation V of G. However, there is a sufficient system of ramification invariants, namely, the lower ramification filtration, which "describes the ramification completely": all the other ramification invariants (including local terms of classical global formulas) can be expressed in terms of it.

[2] We can of course consider a finite extension of $\mathbb{Q}_l$ in place of $\mathbb{Q}_l$; the argument goes through with no essential changes.

(Upper ramification filtration is a sufficient system of invariants as well. The same is true for Artin conductors of all complex representations of G.) For example,

$$e(L/K) = |G_0|;$$

$$v_L(\mathcal{D}_{L/K}) = \sum_{i=0}^{\infty} |G_i| - 1; \tag{10}$$

and

$$\mathrm{Sw}_K(V) = \sum_{i=1}^{\infty} \frac{1}{(G : G_i)} \dim_{\mathbb{C}}(V/V^{G_i}),$$

where V is a finite-dimensional complex representation of G.

4. What is missing in the non-classical case

This section is devoted to the detailed study of an example of extension L/K with $\mathrm{Gal}(L/K) \simeq (\mathbb{Z}/p)^2$ for which Lemma 3.3.1 (as well as any reasonable analog of it) fails. Furthermore, the example exhibits obstacles to extension of the most part of the classical theory to the general case.

Let K be a complete discrete valuation field of characteristic $p > 0$ with imperfect residue field. Fix a prime element π and $t \in \mathcal{O}_K$ such that $\bar{t} \notin \overline{K}^p$. Take some positive integers $N > n > m$ such that $N \equiv n \equiv -1 \pmod p$. Now we define L_1/K and L_2/K by Artin-Schreier equations:

$$\begin{aligned} K_1 &= K(x_1), \quad x_1^p - x_1 = a_1 = \pi^{-n} + \pi^{-m} t, \\ K_2 &= K(x_2), \quad x_2^p - x_2 = \pi^{-N}, \end{aligned} \tag{11}$$

and set $L = K_1 K_2 = K(x_1, x_2) = K_1(x_2) = K_2(x_1)$.

In view of the considerations in Section 2, both K_1/K and K_2/K are wild, and $s(K_1/K) = n$, $s(K_2/K) = N$. Note also that for any subextension K'/K of degree p in L/K we have $s(K'/K) = N$ unless $K' = K_1$.

$$\begin{tikzcd}
 & L \arrow[dl, dash] \arrow[dr, dash, "?"] & \\
K_1 \arrow[dr, dash, "n"'] & & K_2 \arrow[dl, dash, "N"] \\
 & K &
\end{tikzcd}$$

Let us compute $s(L/K_2)$. Put $N = pD - 1$. Then $\pi_2 = x_2 \pi^D$ is a uniformizer of K_2. The equation

$$(\pi^D x_2)^p - \pi^{(p-1)D}(\pi^D x_2) = \pi$$

implies that

$$\pi = \pi_2^p - \pi_2^{(p-1)pD+1} + \cdots,$$

where the dots denote terms of higher order. Thus,

$$\begin{aligned}
a_1 &= (\pi_2^p - \pi_2^{(p-1)pD+1} + \cdots)^{-n} + (\pi_2^p - \pi_2^{(p-1)pD+1} + \cdots)^{-m} t \\
&= \pi_2^{-pn}(1 - \pi_2^{(p-1)pD-p+1} + \cdots)^{-n} + \pi_2^{-pm}(1 - \pi_2^{(p-1)pD-p+1} + \cdots)^{-m} t \\
&= \pi_2^{-pn}(1 + n\pi_2^{(p-1)N} + \cdots) + \pi_2^{-pm}(1 + m\pi_2^{(p-1)N} + \cdots) t \\
&= \pi_2^{-pn} + n\pi_2^{-pn+(p-1)N} + \cdots + \pi_2^{-pm} t + \cdots \\
&\equiv \underbrace{\pi_2^{-n}}_{-n} + n\underbrace{\pi_2^{-pn+(p-1)N}}_{-pn+(p-1)N} + \cdots + \underbrace{\pi_2^{-pm} t}_{-pm} + \cdots \quad (\operatorname{mod} \wp(K_2)),
\end{aligned}$$

where the numbers under the braces denote the corresponding values of v_{K_2}.

Assume further that $m > \frac{n}{p}$. Since $-n < -pn + N(p-1)$, the valuation of the sum is $-pm$. We can conclude that L/K_2 is ferocious, and $s(L/K_2) = m$. Note that the latter number is not determined by the values of $n = s(K_1/K)$ and $N = s(K_2/K)$. (However, if $m < \frac{n}{p}$, the valuation of the sum is $-n$, the extension L/K_2 is wild and $s(L/K_2) = m$. In fact, we are in the classical case here.)

We see that an analog of Lemma 3.3.1 is not true in the general case: we cannot predict $s(L/K_2)$ even having known the $s(K'/K)$ for any subextension K'/K of degree p in L/K.

Next, the "compatibility with factor groups" property also fails in the general case. Indeed, from the depth additivity (2) we have

$$d_L(L/K) = d_L(L/K_2) + d_L(K_2/K) = (p-1)m + (p-1)N,$$

and

$$d_L(L/K_1) = d_L(L/K) - d_L(K_1/K) = (p-1)(m+N) - (p-1)n,$$

whence $s(L/K_1) = m+N-n$. Therefore, the two breaks of the (lower) ramification filtration of L/K are m and $m+N-n$, and these two numbers do not give enough information to determine, say, $s(K_1/K) = n$.

Essentially, this example shows that we cannot give a suitable definition of "upper ramification filtration" based on the usual (Artin or Swan) ramification numbers, and consequently we lose all constructions and facts using this upper filtration: Hasse-Arf theorem, Artin and Swan representations, global formulas etc.

Also, we do not have any "completeness" for the known systems of invariants. In particular, one of the motivating goals in the development of a "non-classical" ramification theory could be to obtain an explicit form for the order of different (or, equivalently, for the depth of ramification) in terms of suitable lower or upper ramification breaks, i.e., an analog of (10).

For more examples showing "mysterious behavior" of ramification invariants in the non-classical case, see [Hy], [Sn, 6.2], [Lo].

5. Upper ramification filtration: abelian extensions

As we could see in the previous section, the classical ramification invariants behave poorly when the residue field $\overline{K}$ is no longer perfect. In particular, we cannot expect any theory of upper ramification filtration based on usual ramification numbers. However, one can be interested in an "independent" construction of an upper filtration *per se* with properties analogous to some properties of the upper filtrations in the classical case, e.g., to some of those stated in Subsections 3.1, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11.

Fortunately, there is a quite satisfactory theory for the upper ramification filtrations, which now becomes standard. The general construction will be addressed in the next section, and here we are concentrated on the case of abelian extensions.

5.1. **Upper filtration via class field theory.** Note that for abelian extensions of usual local fields the upper ramification filtration can be recovered from the filtration on the multiplicative group by Prop. 3.8.1. In the same way one could *define* an upper ramification filtration in the situations where some class field theory is available, e.g., for abelian extensions of higher local fields with finite last residue field (see [HLF] for the basic facts about higher local fields and [Fe96] for a survey of various versions of higher local class field theory). This approach was explored in several papers starting from [Lo]. For example, Hyodo [Hy] defines ("upper") ramification breaks for a finite abelian extension L/K of m-dimensional local fields (with finite last residue field) as m-tuples

$$\mathbf{j}_{L/K}(l) = \begin{cases} \max\{\mathbf{i} \in \mathbb{Z}_+^m : |\Theta_{L/K}(U_\mathbf{i} K_m^{\mathrm{top}} K)| \geq p^l\}, & \text{if such } \mathbf{i} \text{ exists,} \\ 0, & \text{otherwise,} \end{cases} \tag{12}$$

for all $l \geq 1$, where $\Theta_{L/K} : K_m^{\mathrm{top}} K \to \mathrm{Gal}(L/K)$ is the reciprocity map and $(U_\mathbf{i} K_m^{\mathrm{top}} K)$ is the standard filtration on $K_m^{\mathrm{top}} K$ defined by means of the valuation of rank m:

$$U_\mathbf{i} K_m^{\mathrm{top}} K = \langle\, \{u, x_2, \ldots, x_m\} \,|\, u, x_2, \ldots, x_m \in K^*,\ \mathbf{v}_K(u-1) \geq \mathbf{i}\,\rangle.$$

In other words, for $\mathbf{i} = (i_1, \ldots, i_m) > 0$, the subgroup $G^\mathbf{i}$ in $G = \mathrm{Gal}(L/K)$ is defined as $\Theta_{L/K}(U_\mathbf{i} K_m^{\mathrm{top}} K)$ assuming that the last residue field of K is finite. (Recall that $\Theta_{L/K}$ induces an isomorphism from $K_m^{\mathrm{top}} K / N_{L/K} K_m^{\mathrm{top}} L$ onto $\mathrm{Gal}(L/K)$.) If we are not interested in multi-index numbering, we can put

$$G^i = \bigcup_{i_m = i} G^\mathbf{i}$$

for any positive integer i.

For the case of arbitrary perfect last residue field, see [Fe95a, §4].

Using this definition, one can translate questions concerning ramification in abelian extensions of m-dimensional local fields into questions about natural "valuation" filtration on groups $K_m^{\mathrm{top}} K$. In particular, the behavior of the upper filtration on $\mathrm{Gal}(L/K)$ with respect to the restriction to a subgroup $\mathrm{Gal}(L/K')$ is related to the action of the norm map $N_{K'/K} : K_m^{\mathrm{top}} K' \to K_m^{\mathrm{top}} K$ on the valuation filtration.

5.2. **Kato-Swan conductor.** In a compatible manner with the above construction, Kato [Ka89] introduced a notion of a conductor for one-dimensional representations of $\mathrm{Gal}(L/K)$, where L/K is a finite extension of a complete discrete valuation field with *any* residue field.

We do not include Kato's definition, since it is difficult to do this in a self-contained manner; see, e.g., [Sn, 6.2]. However, his conductor $\mathrm{KSw}(\chi)$ can be characterized by either of the following two properties ([Sp99], Prop. 3.3.10 and Cor. 3.3.11).

5.2.1. **Proposition.** *Let $\chi \in H^1(K)$ be a character of $G^{\mathrm{ab}} = \mathrm{Gal}(K^{\mathrm{ab}}/K)$; denote by L_χ the subfield in K^{ab} fixed by χ.*

1. $\mathrm{KSw}(\chi)$ *is the smallest integer* $n \geq 0$ *such that* $\{\chi_{L_0}, u\} = 0$ *in* $\mathrm{Br}\, L_0$ *for any* $u \in U_{n+1,L_0}$*, where* L_0 *is the maximal unramified subextension in* L_χ/K*.*

2. $\mathrm{KSw}(\chi)$ *is the smallest integer* $n \geq 0$ *such that* $U_{n+1,K} \subset N_{L_\chi/K} L_\chi^*$*.*

Here $H^1(K) = \mathrm{Hom}(G^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z})$; the braces denote the cohomological pairing $H^1(K) \times K^* \to H^2(K) \cong \mathrm{Br}\, K$.

From this, one can define a filtration $G^{\mathrm{ab},\bullet}$ on G^{ab} so that, for any character χ of G^{ab}, we have

$$\mathrm{KSw}(\chi) = \inf\{a > 0 \,|\, G^{\mathrm{ab},a} \subseteq \mathrm{Ker}\chi\};$$

we call this filtration the *Kato filtration* on G^{ab}.

For an m-dimensional local field K with finite last residue field and $\chi \in H^1(K)$, $\mathrm{KSw}(\chi)$ is exactly the smallest integer $n \geq 0$ such that $\Theta_{L/K}(U_{\mathbf{i}} K_m^{\mathrm{top}} K)$ acts trivially on L_χ whenever $i_m > n$, see [Sp99, 3.4]. In other words, $\mathrm{KSw}(\chi)$ is the last component of the maximal break $\mathbf{j}(1)$ for L_χ/K in Hyodo's notation (12).

In the classical case this Kato-Swan conductor coincides with the usual Swan conductor. This relation between KSw and the usual (Swan) ramification numbers is in force also in the so-called Case II (cf. Subsection 7.2), [Ka89, prop. 6.8, p.12]:

5.2.2. **Proposition.** *Let* L/K *be a finite Galois extension and* $\chi : G_{L/K} \to \mathbb{C}^*$ *a one-dimensional representation. Assume that either* $\overline{L}/\overline{K}$ *is separable or* $e(L/K) = 1$ *and* $\overline{L}/\overline{K}$ *is generated by one element. Then*

$$\mathrm{KSw}(\chi) = -\frac{1}{e(L/K)} \sum_{\sigma \in G_{L/K}} s(\sigma)\chi(\sigma),$$

where we use the convention that $s(1) = -\sum_{\sigma \in G_{L/K}, \sigma \neq 1} s(\sigma)$*. (See Subsection 1.1 for the definition of* $s(\sigma)$*.)*

6. Upper ramification filtration: general case

In this section, we discuss a few approaches which generalize the ramification filtration constructed by Kato to the whole Galois group. Before giving the constructions, we list their properties in the first three subsections, provided with typical examples. We then turn into various constructions and related topics on the subject in the following subsections.

6.1. **Basic properties.** Let K be a complete discrete valuation field as before with possibly imperfect residue field $\overline{K}$ of characteristic p. In particular, K could be of either mixed characteristic or equal characteristic. Let $G = G_K$ be the Galois group of K. There exist two ramification filtrations $G^\bullet_{\mathrm{nlog}}$ and $G^\bullet_{\mathrm{log}}$ on G, indexed by non-negative rational numbers; they are called the *(upper) non-logarithmic ramification filtration* and *(upper) logarithmic ramification filtration* of G, respectively. Roughly speaking, the adjectives "non-logarithmic" and "logarithmic" refer to different normalizations to balance the "wild part" and the "ferocious part" of the ramification. In particular, when $\overline{K}$ is perfect, both of these filtrations are the same (up to a shift of indexing) as the usual upper ramification filtration. (See property (5) below.)

We use the standard convention for ramification filtrations: for $a \in \mathbb{R}_{\geq 0}$, we write G^a_{nlog} to mean the closure of $\cup_{b \geq a, b \in \mathbb{Q}_{\geq 0}} G^a_{\mathrm{nlog}}$ and G^{a+}_{nlog} to mean the closure of $\cup_{b > a, b \in \mathbb{Q}_{\geq 0}} G^a_{\mathrm{nlog}}$; and the same for the logarithmic ramification filtration. For

L a finite Galois extension of K, we use $b_{\mathrm{nlog}}(L/K)$ to denote the *highest non-logarithmic ramification break* $\inf\{b \mid G^b_{\mathrm{nlog}} \subseteq G_L\}$; and the same for the logarithmic ramification filtration.

The basic properties are listed as follows (proved in [AS02]).

(1) Both filtrations are left continuous, with rational breaks.
(2) For $0 < a \leq 1$, G^a_{nlog} is the inertia subgroup of G (inverse limit of inertia subgroups over finite subextensions).
(3) $G^{1+}_{\mathrm{nlog}} = G^{0+}_{\mathrm{log}}$ is the wild ramification subgroup of G (inverse limit of wild ramification subgroups over finite subextensions).
(4) For any $a > 0$, we have inclusions $G^{a+1}_{\mathrm{nlog}} \subseteq G^a_{\mathrm{log}} \subseteq G^a_{\mathrm{nlog}}$ (which are strict inclusions if $\overline{K}$ is not perfect).
(5) If $\overline{K}$ is perfect, we have $G^{a+1}_{\mathrm{nlog}} = G^a_{\mathrm{log}} = G^a$ for all $a \geq 0$; here (G^a) is usual upper ramification filtration;
(6) If K'/K is a finite unramified extension, then both filtrations on $G_{K'}$ are induced by those on G_K;
(7) If K'/K is a finite tame extension with $e(K'/K) = m$, then $(G_{K'})^{ma}_{\mathrm{log}} = (G_K)^a_{\mathrm{log}}$ for any $a > 0$;
(8) If K'/K is any finite extension with $e(K'/K) = m$, then $(G_{K'})^{ma}_{\mathrm{log}} \subset (G_K)^a_{\mathrm{log}}$ for any $a > 0$.

The following is a typical example of ramification breaks.

6.1.1. **Example.** Let $K = \overline{K}((\pi))$ be an equal characteristic complete discrete valuation field and let $L = K(z)$ be an Artin-Schreier extension given by $z^p - z = a\pi^{-n}$ for $a \in \overline{K}[\![\pi]\!]^*$ and $n \in \mathbb{N}$. We assume that the generator z is chosen so that n is minimal (see §2). The Galois group $\mathrm{Gal}(L/K)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

(1) If $p \nmid n$, then we have $b_{\mathrm{nlog}}(L/K) = n + 1$ and $b_{\mathrm{log}}(L/K) = n$.
(2) If $p|n$, then we have $b_{\mathrm{log}}(L/K) = b_{\mathrm{nlog}}(L/K) = n$.

Another important property of these two upper ramification filtrations is the integrality of the associated Artin and Swan conductors. For a finite dimensional representation $\rho : G_K \to \mathrm{GL}(V)$ with finite image, we put $b_{\mathrm{nlog}}(\rho) = b_{\mathrm{nlog}}(V) := b_{\mathrm{nlog}}(L/K)$ and $b_{\mathrm{log}}(\rho) = b_{\mathrm{log}}(V) := b_{\mathrm{log}}(L/K)$, where L is the finite extension of K corresponding to the kernel of ρ. Under very mild technical restrictions, these ramification filtrations enjoy the following Hasse-Arf property, as proved in [X10, X12a].

6.1.2. **Theorem.** *Assume either K is of equal characteristic, or $p > 2$ and K is* not *absolutely unramified (i.e. p is* not *an uniformizer). Let $\rho : G_K \to \mathrm{GL}(V)$ be an irreducible representation with finite image. Then the Artin conductor* $\mathrm{Art}(\rho) := b_{\mathrm{nlog}}(\rho) \cdot \dim \rho$ *and the Swan conductor* $\mathrm{Sw}(\rho) := b_{\mathrm{log}}(\rho) \cdot \dim \rho$ *are integers.*

6.2. **Refined Artin/Swan conductors.** It is a natural question to ask whether one can obtain information about the graded pieces of the ramification filtrations. The following theorem is proved with some restrictions in [AS03], [Sa09], [X12a] and, in full generality, in [Sa12].

6.2.1. **Theorem.** *The graded pieces* $\mathrm{gr}^a G^\bullet_{\mathrm{nlog}} := G^a_{\mathrm{nlog}}/G^{a+}_{\mathrm{nlog}}$ *($a > 1$) and* $\mathrm{gr}^a G^\bullet_{\mathrm{log}} := G^a_{\mathrm{log}}/G^{a+}_{\mathrm{log}}$ *($a > 0$) are abelian groups of exponent p. Moreover, there is a natural* injective *homomorphism*

$$\mathrm{rsw} : \mathrm{Hom}(\mathrm{gr}^a G^\bullet_{\mathrm{log}}, \mathbb{F}_p) \hookrightarrow \Omega^1_{\mathcal{O}_K}(\log) \otimes_{\mathcal{O}_K} \mathfrak{m}^{-a}_{K^{\mathrm{alg}}}/\mathfrak{m}^{(-a)+}_{K^{\mathrm{alg}}}, \quad a \in \mathbb{Q}_{>0},$$

where $\Omega^1_{\mathcal{O}_K}(\log) := \Omega^1_{\mathcal{O}_K} + \mathcal{O}_K\frac{d\pi_K}{\pi_K}$, $\mathfrak{m}^{-a}_{K^{\mathrm{alg}}} := \{x \in K^{\mathrm{alg}} \,|\, v_K(x) \geq -a\}$, *and* $\mathfrak{m}^{(-a)+}_{K^{\mathrm{alg}}} := \{x \in K^{\mathrm{alg}} \,|\, v_K(x) > -a\}$.

Following Kato, the map above is called the *refined Swan conductor homomorphism.* When K is of equal characteristic, there is an analogous natural injective homomorphism, called the *refined Artin conductor homomorphism*

$$\mathrm{rar} : \mathrm{Hom}(\mathrm{gr}^a G^{\bullet}_{\mathrm{nlog}}, \mathbb{F}_p) \hookrightarrow \Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathfrak{m}^{-a}_{K^{\mathrm{alg}}}/\mathfrak{m}^{(-a)+}_{K^{\mathrm{alg}}}, \quad a \in \mathbb{Q}_{>1}.$$

See [X12b] for more details. The analogous refined Artin conductor homomorphism is also expected in the mixed characteristic case, using a variant of the argument of [Sa12].

When $\overline{K}$ is finite and a a positive integer, the rsw map is compatible with the natural homomorphism in local class field theory in the following way.

$$\begin{array}{ccc}
\mathrm{Hom}((G^{\mathrm{ab}})^a/(G^{\mathrm{ab}})^{a+}, \mathbb{F}_p) & \longrightarrow & \mathrm{Hom}(\mathrm{gr}^a G^{\bullet}_{\log}, \mathbb{F}_p) \\
\big\downarrow \scriptstyle{\mathrm{LCFT}} & & \scriptstyle{\mathrm{rsw}} \big\downarrow \\
\mathrm{Hom}(U_{a,K}/U_{a+1,K}, \mathbb{F}_p) & \xrightarrow{\log^\vee} & \Omega^1_{\mathcal{O}_K}(\log) \otimes_{\mathcal{O}_K} \mathfrak{m}^{-a}_{K^{\mathrm{alg}}}/\mathfrak{m}^{(-a)+}_{K^{\mathrm{alg}}}
\end{array}$$

where G^{ab} denotes the abelianized Galois group with the induced filtration, the left vertical map is the isomorphism from the local class field theory, and the map $\log^\vee$ is characterized below. For a homomorphism $\eta : U_{a,K}/U_{a+1,K} \to \mathbb{F}_p$, its image $\log^\vee(\eta)$ is the element $w_\eta \pi_K^{-a}\frac{d\pi_K}{\pi_K}$ for $w_\eta \in \overline{K}$ such that

$$\eta\big(1 + x\pi_K^a\big) = \mathrm{tr}_{\overline{K}/\mathbb{F}_p}(xw_\eta).$$

6.2.2. **Example.** Continuing with the setup in Example 6.1.1, we fix a generator z. Fixing the isomorphism $K \cong \overline{K}((\pi))$, we have

$$\Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \overline{K} \cong \Omega^1_{\overline{K}} \oplus \overline{K}d\pi \quad \text{and} \quad \Omega^1_{\mathcal{O}_K}(\log) \otimes_{\mathcal{O}_K} \overline{K} \cong \Omega^1_{\overline{K}} \oplus \overline{K}\frac{d\pi}{\pi}.$$

Let $d\bar{a}$ be the usual differential of $\bar{a}$ in $\Omega^1_{\overline{K}}$; it is zero if and only if $\bar{a}$ is a pth power in $\overline{K}$. We can in turn view this element in $\Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \overline{K}$ and $\Omega^1_{\mathcal{O}_K}(\log) \otimes_{\mathcal{O}_K} \overline{K}$ using the direct sum decomposition above.

There is a natural isomorphism $\rho : \mathrm{Gal}(L/K) \to \mathbb{F}_p$ given by $\sigma \mapsto \sigma(z) - z \in \mathbb{F}_p$. This ρ induces a homomorphism from $\mathrm{gr}^{b_{\mathrm{nlog}}(L/K)} G^{\bullet}_{K,\mathrm{nlog}}$ or $\mathrm{gr}^{b_{\log}(L/K)} G^{\bullet}_{K,\log}$ to $\mathbb{F}_p$, which we still denote by ρ. Then the images of ρ under the refined Artin and Swan conductor homomorphisms are as follows.

In case (1), $\mathrm{rar}(\rho) = \pi^{-n-1} n a d\pi$ and $\mathrm{rsw}(\rho) = \pi^{-n}(na\frac{d\pi}{\pi} + d\bar{a})$.

In case (2), $\mathrm{rar}(\rho) = \pi^{-n} d\bar{a}$ and $\mathrm{rsw}(\rho) = \pi^{-n} d\bar{a}$. (They are *not* literally the same because they live in different spaces.)

One can check that the refined Swan and Artin conductors do not depend on the choice of z.

6.2.3. **Question.** When $\overline{K}$ is perfect, one can check that the refined Swan conductor homomorphism is in fact an isomorphism. (This is a folklore result, and, to our best knowledge, it has not appeared in the literature.) When $\overline{K}$ is not perfect, is the refined Swan conductor homomorphism still an isomorphism? What about the analogous refined Artin conductor homomorphism? This appears to be a very deep question regarding the structure of the Galois group G_K.

6.3. **Multi-index filtration for higher dimensional fields.** Using the refined Swan conductor, one can naturally associate a multi-index (upper) filtration for an m-CDVF K as follows. We will only treat the case with logarithmic ramification filtration and when the last residue field k_0 is perfect to simplify the notation; one can easily modify the construction to adapt to the general case and to the non-logarithmic case.

Let K be an m-CDVF with the first residue field k_{m-1}. Assume the last residue field k_0 is perfect. We fix a system of local parameters $t_1, \dots, t_m$. In this case, we have

$$\Omega^1_{\mathcal{O}_K}(\log) \otimes_{\mathcal{O}_K} k_{m-1} = \bigoplus_{i=1}^m k_{m-1} \frac{dt_i}{t_i}.$$

For $i_m \in \mathbb{Q}_{>0}$ and for $\lambda = \sum_{i=1}^m \alpha_i \frac{dt_i}{t_i} \in \Omega^1_{\mathcal{O}_K}(\log) \otimes_{\mathcal{O}_K} t_m^{-i_m} k_{m-1}^{\mathrm{alg}}$, we set

$$\mathbf{v}_{\log}(\lambda) = \min\{\mathbf{v}(\alpha_1), \dots, \mathbf{v}(\alpha_m)\}.$$

This gives a multi-index valuation on $\Omega^1_{\mathcal{O}_K}(\log) \otimes_{\mathcal{O}_K} t_m^{-i_m} k_{m-1}^{\mathrm{alg}}$.

We put $\mathbb{Q}^m_{>\mathbf{0}} = \{\mathbf{i} \in \mathbb{Q}^m \,|\, i_m > 0\}$. For $\mathbf{i} = (i_1, \dots, i_m) \in \mathbb{Q}^m_{>\mathbf{0}}$, we can define a filtration on $G := G_K$ by the following characterization:

$$G^{\mathbf{i}}_{\log} := \big\{\sigma \in G^{i_m}_{\log} \,\big|\, \chi(\sigma) = 0 \text{ for all } \chi : \mathrm{gr}^{i_m} G^{\bullet}_{\log} \to \mathbb{F}_p \text{ such that } \mathbf{v}_{\log}(\mathrm{rsw}(\chi)) > -\mathbf{i}\big\}.$$

6.3.1. **Question.** When K has finite last residue field, does this multi-index filtration on G_K^{ab} agree with the one defined by (12) (with $l = 1$), which uses the Milnor K-group $K_m^{\mathrm{top}} K$? This amounts to comparing the refined Swan conductor homomorphism with the one defined by Kato for characters of G_K^{ab}. The comparison is expected by experts. In the equal characteristic case, this is proved in [AS09] and also appears implicitly in Chiarellotto and Pulita [ChP]. But in the mixed characteristic, to our best knowledge, it has not appeared in the literature.

6.4. **Construction of the filtration d'après Abbes and T. Saito.** Now we proceed to describe the construction of the upper ramification filtrations in the general case developed by Abbes and T. Saito [AS02].

Abbes and T. Saito made use of rigid analytic spaces. (We refer to [BGR] for basics of rigid analytic spaces.) Their construction is motivated by the following crucial but easy proposition in the case of perfect residue field.

6.4.1. **Proposition.** *Let K be a complete discrete valuation field with perfect residue field. Let L be a finite Galois extension of K with Galois group $G_{L/K}$. We know that $\mathcal{O}_L$ is generated as an $\mathcal{O}_K$-algebra by one element x. Let $P(u)$ denote the minimal polynomial of x.*

(i) Let $b(L/K)$ be the highest ramification break as defined just before Example 1.1.1. We assume that L/K is not unramified so that $b(L/K) \geq 0$. Then

$$b(L/K) = \frac{1}{e(L/K)} \Big(\sum_{\sigma \in G_{L/K}, \sigma \neq 1} v_L(\sigma(x) - x) + \max_{\sigma \in G_{L/K}, \sigma \neq 1} v_L(\sigma(x) - x) \Big).$$

(ii) Consider the rigid analytic space for each positive rational number a:

$$X^a = \big\{ u \in K^{\mathrm{alg}} \ : \ |u| \leq 1, \ |P(u)| \leq |\pi_K|^a \big\}.$$

Then X^a has $[L : K]$ geometric connected components if and only if $a > b(L/K)$.

Proof. The first statement is straightforward if one unwinds the definition of the upper ramification filtration.

A rigorous proof of (ii) can be found in [AS02, Lemma 6.6]. We will give a rough idea of why this is true. The picture here is that, if a is very large, we confine u in some very small neighborhoods of the roots of $P(u) = 0$, or equivalently the conjugates of x. The rigid space X^a is expected to be geometrically a disjoint union of very small disks centered at each of the conjugates of x. In other words, X^a should have $[L : K]$ geometric connected components. In contrast, when $a \to 0^+$, the condition $|P(u)| < |\pi_K|^a$ is significantly weakened, and X^a is almost the whole disk $|u| \leq 1$.

When the rational number a decreases from a big starting value, the disks grow larger. Consider the first moment such that some of the $[L : K]$ disks clash together, and the number of geometric connected components decreases. We need to show that the rational number a at this moment is exactly the highest ramification break $b(L/K)$. Indeed, the cut-off condition is obviously $|u-x| < \min_{\sigma\in G_{L/K},\sigma\neq 1} |\sigma(x)-x|$ (or with a conjugate of x in place of x). This implies that $|u - \sigma(x)| = |\sigma(x) - x|$ for $\sigma \neq 1$. Thus

$$|P(u)| = \prod_{\sigma\in G_{L/K}} |u - \sigma(x)| = |u - x| \prod_{\sigma\in G_{L/K}} |\sigma(x) - x| < |\pi_K|^{b(L/K)}.$$

In fact, this explanation can be turned into a complete proof if it is argued more carefully. □

Imitating this description in the general case, Abbes and T. Saito gave the following construction. Let K be a complete discrete valuation field and L a finite Galois extension of K. Suppose that $\mathcal{O}_L$ is generated by $x_1, \dots, x_r$ as an $\mathcal{O}_K$-algebra. Then we may write $\mathcal{O}_L$ as the quotient $\mathcal{O}_K[u_1, \dots, u_r]/(f_1, \dots, f_s) \simeq \mathcal{O}_L$, where the isomorphism sends u_i to x_i, and $\{f_1, \dots, f_s\}$ is some set of generators of the ideal. For a positive rational number a, consider the following rigid analytic space

$$X^a_{L/K} := \left\{ \underline{u} = (u_1, \dots, u_r) \in (K^{\mathrm{alg}})^r \ : \ \begin{array}{c} |u_1| \leq 1, \dots, |u_r| \leq 1; \\ |f_1(\underline{u})| \leq |\pi_K|^a, \dots, |f_s(\underline{u})| \leq |\pi_K|^a \end{array} \right\}.$$

Put $G = G_K$ for simplicity. Inspired by Prop. 6.4.1, we want to define the (upper) ramification filtration G^b_{nlog} of G so that $X^a_{L/K}$ has $[L : K]$ geometric connected components if and only if $a > b_{\mathrm{nlog}}(L/K)$. It is not difficult to see that the space $X^a_{L/K}$ does not depend on the choice of f_i's, and the set of geometric connected components $\pi_0^{\mathrm{geom}}(X^a_{L/K})$ does not depend on the choice of u_i's (because adding a new generator is equivalent to changing $X^a_{L/K}$ to a fiber bundle over $X^a_{L/K}$ whose fibers are disks). Thus, our construction is well defined, depending only on L.

Abbes and T. Saito [AS02] prove the existence of such ramification filtration using certain abstract framework of "Galois functor" by studying functors for all rational a that take every finite Galois extension L of K to the set of geometric connected components $\pi_0^{\mathrm{geom}}(X^a_{L/K})$. They call this filtration the *non-logarithmic ramification filtration* G^a_{nlog} for $a \in \mathbb{Q}_{\geq 0}$. They also give a logarithmic variant of the construction which defines the *logarithmic ramification filtration* $G^a_{\log}$ for $a \in \mathbb{Q}_{\geq 0}$. For details, we refer to [AS02]. For later reference when comparing different definitions of the filtrations, we will refer to these two filtrations as the *Abbes-Saito filtrations*.

The following comparison theorem is proved partially in [ChP] and in full in [AS09].

6.4.2. **Theorem.** *Kato filtration on G^{ab} agrees with the filtration induced from the Abbes-Saito non-logarithmic filtration on $G = G_K$. Moreover, the refined Swan conductor defined in* [Ka89] *is compatible with the refined Swan conductor homomorphism defined in Theorem 6.2.1.*

We also mention that Abbes-Saito's construction can be applied to finite flat group schemes over $\mathcal{O}_K$ and it defines a ramification filtration on the group schemes. For progress along this line, see [AM, Ha12, Ha12+]. This result may be used to prove the existence of canonical subgroups for a p-divisible group with small degree; see [Ti].

6.5. **Construction of the filtrations by p-adic differential equations.** Another useful equivalent definition of the ramification filtration is based on the theory of p-adic differential equations.

We first consider the case when $K = \overline{K}((\pi))$ is of equal characteristic and $\overline{K}$ is perfect. Put $F = W(\overline{K})[\frac{1}{p}]$. Consider the following bounded Robba ring, for $r \in (0,1) \cap p^{\mathbb{Q}}$:

$$\mathcal{R}^r_{\mathrm{bdd}} := \Big\{ \sum_{n \in \mathbb{Z}} a_n T^n \,\Big|\, a_n \in F, |a_i| \text{ is bounded, and } \lim_{i \to -\infty} |a_i| \cdot r^i = 0 \Big\}.$$

It is the ring of analytic functions on the annulus $r \leq |T| < 1$ which take bounded values.

Let V be an irreducible p-adic representation of $G = G_K$ with finite image. The theory of Fontaine (see, e.g., [Ke05, Section 4]) associates V with a *differential module* over $\mathcal{R}^r_{\mathrm{bdd}}$ for some positive rational number r sufficiently close to 0, that is a finite free module $\mathcal{F} = \mathcal{F}_V$ over $\mathcal{R}^r_{\mathrm{bdd}}$ equipped with a connection

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{R}^r_{\mathrm{bdd}}} \Omega^1_{\mathcal{R}^r_{\mathrm{bdd}}/F}.$$

This is equivalent to giving a derivation $\partial = \frac{d}{dT}$ on $\mathcal{F}$ (satisfying Leibniz rule).

This construction gives the access to the full power of the theory of p-adic differential equations in the study of the ramification of G. For $r' \in p^{\mathbb{Q}}$ with $r' \in [r, 1)$, we use $F(T)^{(r')}$ to denote the completion of $F(T)$ with respect to the r'-Gauss norm, that is the norm extending the following norm $|\cdot|_{(r')}$ on $F[T]$:

$$\Big|\sum_{n \geq 0} a_n T^n\Big|_{(r')} = \max_{n \geq 0}\{|a_n| r'^n\}.$$

We pick a norm $|\cdot|_{\mathcal{F},(r')}$ on $\mathcal{F}^{(r')} := \mathcal{F} \otimes_{\mathcal{R}^r_{\mathrm{bdd}}} F(T)^{(r')}$ and consider the *spectral norm*

$$|\partial|_{\mathrm{sp},\mathcal{F},(r')} := \lim_{n \to \infty} |\partial^n|^{1/n}_{\mathcal{F},(r')},$$

where $|\partial^n|_{\mathcal{F},(r')}$ is the operator norm of ∂^n. The spectral norm does not depend on the chosen norm $|\cdot|_{\mathcal{F},(r')}$ on $\mathcal{F}^{(r')}$. This is one of the key invariants of a p-adic differential equation. It was explained by Kedlaya in [Ke05] (based on the work of Christol-Mebkhout, Crew, Matsuda, Tsuzuki) that the highest ramification break $b(V)$ has the following characterization by spectral norms:

$$\text{for } r' \text{ sufficiently close to } 1^-, \ |\partial|_{\mathrm{sp},\mathcal{F},(r')} = p^{-1/(p-1)} \cdot (r')^{-b(V)-1}.$$

A generalization of this approach without the perfectness of $\overline{K}$ is introduced by Kedlaya in [Ke07]. Assume that $\overline{K}$ has a finite p-basis (as the general case reduces to this case). The construction works formally the same except the following changes:

- The field F is taken to be the fraction field of a Cohen ring of $\overline{K}$; here the Cohen ring is an absolutely unramified complete discrete valuation ring with residue field $\overline{K}$; we refer to [Wh] for a functorial construction of Cohen rings.
- We have the derivation $\partial_0 = \frac{d}{dT}$ as well as other derivations $\partial_1, \dots, \partial_n$ coming from a chosen p-basis of $\overline{K}$. Using this, Kedlaya defines the *logarithmic differential ramification filtration* such that for r' sufficiently close to 1^-,

$$\max\left\{|\partial_0|_{\mathrm{sp},\mathcal{F},(r')} \cdot r', |\partial_1|_{\mathrm{sp},\mathcal{F},(r')}, \dots, |\partial_n|_{\mathrm{sp},\mathcal{F},(r')}\right\} = p^{-1/(p-1)} \cdot (r')^{-b_{\log}(V)}, \tag{13}$$

  where, as before, $b_{\log}(V)$ is the highest ramification break defined by the logarithmic differential ramification filtration.

A different normalization in the above formula by removing the factor r' in the first term of (13) gives rise to the differential non-logarithmic ramification filtration.

The differential ramification filtrations enjoy the following properties.

(1) Kedlaya [Ke07] proves the Hasse-Arf property (as in Theorem 6.1.2), using the integrality of Newton polygons. (One can alternatively deduce this by reducing to the perfect residue field case.)
(2) It is proved in [X10] that Kedlaya's differential ramification filtration agrees with Abbes-Saito filtration; this then proves Theorem 6.1.2 in the equal characteristic by transferring the Hasse-Arf property through the comparison. Same result for one-dimensional representations was priorly obtained by Chiarellotto and Pulita [ChP].
(3) In the equal characteristic case, [X12b] realizes the refined Swan conductor homomorphism using p-adic differential modules; this is related to the eigenvalues of the matrices for the differential operators $\partial_0, \dots, \partial_n$, acting on an appropriate basis of $\mathcal{F}$. [X12b] further relates the refined Swan conductor homomorphism to the variation of Swan conductor (see Subsection 10.2).
(4) When K is of mixed characteristic under some mild condition, it is proved in [X12a] that one can "fake" the Robba ring construction above and apply recent results [Ke10a, KeX] on p-adic differential equations to deduce the Hasse-Arf theorem.

6.5.1. **Question.** Can we realize the refined Swan conductor homomorphism in the mixed characteristic case, using certain fake Robba ring construction?

6.6. **Geometric construction based on Abbes-Saito's original definition.** Soon after the introduction of Abbes-Saito filtrations, Abbes and Saito gave the following geometric reinterpretation of the definition, which aims at a more global application.

To start, we first assume that K is of equal characteristic and satisfies the following condition:

(Geom) There exist a smooth scheme X over a field k and an irreducible divisor D smooth over k with the generic point η, such that K is isomorphic to the completion of $k(X)$ with respect to the valuation given by η.

Properties for a general equal characteristic field K may be reduced to the case with this condition by taking certain limit.

Now, given a finite dimensional irreducible l-adic representation ρ of G_K, we may realize it as an l-adic sheaf $\mathcal{F} = \mathcal{F}_\rho$ over $U := X \backslash D$, possibly after shrinking X. Using vanishing cycles, T. Saito [Sa09] gives a construction that can detect the highest logarithmic ramification break $b := b_{\log}(\rho)$, which we review here.

Let $\mathcal{I}_D$ denote the ideal sheaf for the closed immersion $D \subset X$. Let $(X \times X)'$ be the blow-up of $X \times X$ along $D \times D$. Let $(X \times X)^\sim$ denote the complement of the proper transform of $(X \times D) \cup (D \times X)$. Let $\tilde{u} : (X \times X)^\sim \to X$ denote the natural projection to the first factor. The diagonal embedding $U \to U \times U \subset (X \times X)^\sim$ extends to a natural embedding $\tilde{\delta} : X \to (X \times X)^\sim$. Let $\mathcal{J}_X$ denote the ideal sheaf for this closed immersion. Let $\tilde{j} : U \times U \to (X \times X)^\sim$ denote the natural inclusion.

For $a \in \mathbb{Q}_{\geq 0}$, we use $(X \times X)^{(a)}$ denote the normalization of the scheme associated to the quasi-coherent sub-$\mathcal{O}_{(X \times X)^\sim}$-modules

$$\sum_{n \in \mathbb{N}} \tilde{u}^*(\mathcal{O}_X(\lfloor na \rfloor D)) \cdot \mathcal{J}_X^n \subset \tilde{j}_* \mathcal{O}_{U \times U}. \quad \text{(Here, } \lfloor \cdot \rfloor \text{ is the floor function).}$$

When a is a positive integer, this is one of the open charts for the blow-up of $(X \times X)^\sim$ along the ideal sheaf $\tilde{u}^*(\mathcal{I}_D)^a + \mathcal{J}_X$.

We use the following notation for morphisms:

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "\delta"] & X \arrow[d, "\delta^{(a)}"] \\ U \times U \arrow[r, "j^{(a)}"] & (X \times X)^{(a)} \end{tikzcd}$$

Put $\mathcal{H} := \mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^*\mathcal{F})$. T. Saito [Sa09] proves that

6.6.1. **Proposition.** *The highest log ramification break $b_{\log}(\rho) \leq a$ if and only if the base change map*

$$\delta^{(a)*} j_*^{(a)} \mathcal{H} \to j_* \mathcal{E}nd(\mathcal{F})$$

is an isomorphism at the generic point η of D.

When the condition of the proposition is satisfied, the restriction of $j_*^{(a)}\mathcal{H}$ on the complement $(X \times X)^{(a)} \backslash (U \times U)$ is a direct sum of the Artin-Schreier sheaves defined by certain linear forms. These linear forms give rise to the refined Swan conductor homomorphism. See [Sa09] for more details.

When the field K is of mixed characteristic, T. Saito [Sa12] imitates the equal characteristic construction to make sense of $X \times_k X$ using infinitesimal deformations. It would be interesting to see if one can put T. Saito's construction in a more global setting for complete regular rings of mixed characteristic, and obtain global results similar to those in [Sa09].

6.7. **Alternative constructions of upper filtrations.** We now explain some other constructions of upper ramification filtrations.

Borger [Bo04, Bo02] constructs a non-logarithmic ramification filtration using a "generic residual perfection" process. His result is based on the following observation: taking $\mathcal{O}_K = \mathbb{F}_p(x)[\![\pi]\!]$ as an example, a naïve idea would be to reduce the definition of ramification filtrations to the case of perfect residue field, by adjoining p^∞-roots of x. Note that x should be thought of as a lift of the x of the residue field.

But there is no canonical such lift, as one could choose, for example, $x+\pi$ instead and adjoin all p^∞-roots of $x+\pi$. Borger's idea is to introduce an indeterminate u_1 and consider $\mathbb{F}_p(x,u_1)[\![\pi]\!]$; he then adjoins all p^∞-roots of $x+u_1\pi$. Next, he has to deal with p-power roots of u_1. For this, he adjoins another indeterminate u_2 and all p^∞-roots of $u_1+u_2\pi$. Continuing this process and "taking limit" gives a "generic perfection of $\mathcal{O}_K$".

To present this observation systematically, Borger showed that there is a moduli space $\mathrm{Spf}(A^{\mathrm{u}})$ that parameterizes the ways of modifying $\mathcal{O}_K$ so that its residue field is perfect. In the example above,

$$A^{\mathrm{u}} = \mathbb{F}_p(x)[u_1,u_2,\dots][\![\pi]\!][(x+u_1\pi)^{1/p^\infty},(u_1+u_2\pi)^{1/p^\infty},\dots].$$

Let A^{g} denote the the completion of A^{u} at the generic point of its special fiber. Then $Q(A^{\mathrm{g}})$ is a complete discrete valuation field with *perfect* residue field.

We may then use the natural map $G_K \to G_{Q(A^{\mathrm{g}})}$ to define an (upper) ramification filtration on G_K as the preimage of the ramification filtration on the latter group. Borger [Bo04] proves that the Artin conductor given by his non-logarithmic ramification filtration is compatible with the "non-logarithmic" (Artin-like) version of Kato conductor. It is later proved in [X10] that, when K is of equal characteristic, Borger's (non-logarithmic) filtration agrees with Abbes-Saito non-logarithmic filtration. In the mixed characteristic case, a similar argument used in [X10] relates Abbes-Saito non-logarithmic filtration with a variant of Borger's filtration (see [X12a, Remark 3.2.14]). It would be interesting to see if the two filtrations are exactly the same.

Boltje, Cram and Snaith (see [BCS], [Sn, 6.3]) define a conductor in the general case by means of explicit Brauer induction. This results in a conductor compatible with Swan conductor and Kato-Swan conductor in the cases where those are defined. As of yet, we are not aware of any attempt to compare the approach of Boltje-Cram-Snaith with other constructions mentioned above.

One more approach is initiated in [Z13], [Z14]. It is based on consideration of composites of a given finite extension with various (infinite) elementary abelian extensions.

6.8. **A generalization of the theorem of Deligne.** As it was discussed in Subsection 3.10, one expects to be able to associate quotient of Galois groups to truncated discrete valuation rings. More concretely, consider two complete discrete valuation fields K and K' and assume that there exists $b\in\mathbb{N}$ such that there is an isomorphism $\mathcal{O}_K/\pi_K^b\mathcal{O}_K \cong \mathcal{O}_{K'}/\pi_{K'}^b\mathcal{O}_{K'}$ as rings. Unlike in Subsection 3.10, we do not assume that the residue field $\overline{K}=\overline{K}'$ is perfect.

6.8.1. **Question.** Does this isomorphism of rings still imply that $G_K/G^b_{K,\mathrm{nlog}} \cong G_{K'}/G^b_{K',\mathrm{nlog}}$ and $G_K/G^b_{K,\log} \cong G_{K'}/G^b_{K',\log}$? Are these isomorphisms of quotient groups canonical? Moreover, are they compatible with the refined Swan/Artin conductor homomorphisms?

In the non-logarithmic case, it appears that Hiranouchi and Taguchi [HT] have started a project towards proving the isomorphism of quotients of Galois groups. See also the survey paper [Hi].

6.9. **Vector bundles with irregular singularities.** It is quite well known that there is a strong analogy between representations of G_K (when $\overline{K}$ is perfect) and differential modules over $\mathbb{C}((T))$, that is finite dimensional vector spaces V over $\mathbb{C}((T))$ equipped with a derivation $\partial = \frac{d}{dT}$ (i.e. $\partial(av) = \partial(a)v + a\partial(v)$ for $a \in \mathbb{C}((T))$ and $v \in V$). Such a module is called *regular* if $T\partial$ preserves a $\mathbb{C}[\![T]\!]$-lattice Λ of V. For $P \in \mathbb{C}((T))$, we can define a rank one differential module $E(P) = \mathbb{C}((T)) \cdot e$ such that $\partial(e) = Pe$.

The Turrittin-Levelt-Hukuhara Theorem (see, e.g., [Ke10a, Section 7.5]) says that there exists $n \in \mathbb{N}$ such that we have a decomposition

$$V \otimes_{\mathbb{C}((T))} \mathbb{C}((T^{1/n})) \cong \oplus_{i=1}^r V_i,$$

where each V_i is of the form $V_i = E(P_i) \otimes R_i$ for an element $P_i \in \mathbb{C}((T^{1/n}))$ and a regular module R_i over $\mathbb{C}((T^{1/n}))$.

The analogous invariant of ramification break is just $\max\{0, -v_{\mathbb{C}((T))}(P_i)\}$. We define the *irregularity* of V to be

$$\mathrm{Irr}(V) := \sum_{i=1}^r \dim V_i \cdot \max\{0, -v_{\mathbb{C}((T))}(P_i)\}.$$

We can give an interpretation of this invariant in terms of the spectral norms of the differential operators ∂. For details, see [Ke10a, X12b].

In the general case when $K = \overline{K}((T))$ with $\overline{K}$ of characteristic zero, there might be additional derivations $\partial_1, \dots, \partial_n$ on $\overline{K}$. For example, when $\overline{K} = \mathbb{C}(x, y)$, we may consider the derivations $\partial_1 = \frac{d}{dx}$ and $\partial_2 = \frac{d}{dy}$. We consider a differential module V over $\overline{K}((T))$, that is a finite dimensional vector space $\overline{K}((T))$ equipped with commuting actions of $\partial_0 = \frac{d}{dT}, \partial_1, \dots, \partial_n$. When V is irreducible, one can define the irregularity of V by taking the maximum among all irregularities computed by the spectral norms of all differential operators. For general V, its irregularity is defined to be the sum of the irregularities over all Jordan-Hölder constituents. For the details, we refer to [X12b].

Similarly, one can define a refined irregularity as an analog of the refined Swan conductor for Galois representations. This is explained in [X12b].

7. Elimination theory

7.1. **The expectations.** We see that the Kato-Swan conductor as well as the Abbes-Saito ramification filtration work perfectly in all the situations where one needs the ramification invariants that "live downstairs", i.e., for an extension L/K, those invariants that are more closely attached to K than to L. These include multiple questions related to the absolute Galois group of a complete discrete valuation field, or, in algebraic geometry, to the étale site of an algebraic or arithmetic variety.

In other words, probably we have the best possible "upper ramification filtration"[3]. However, in general we cannot recover the usual (lower) ramification filtration from it. There are no Hasse-Herbrand functions, and we cannot write down any analogs with functorial properties as in Subsections 3.2–3.5. The reason for this is rather fundamental: any single ramification filtration as well as any theory of Swan-type conductor describes the ramification of an extension of degree p with

[3] The terminology is absolutely misleading! The upper ramification breaks live downstairs, and the lower ones live upstairs.

just *one number*. But we saw in the example in Section 4 that a "comprehensive" ramification theory should provide more information in this case. Indeed, in (11) we have to know not only n and N but also m.

Also, we have no formula for the order of different (or depth)[4] in terms of the upper breaks which would be a substitute for (10). The best possible estimates in the case of an n-dimensional local field (with finite last residue field) are given by Hyodo inequalities (see [Hy], Th. (1-5), Prop. (3-4), Ex. (3-5)):

$$(p-1)\sum_{l\geq 1}\frac{\mathbf{j}_{L/K}(l)}{p^l}\leq \mathbf{d}_K(L/K)\leq \frac{p-1}{p}\sum_{l\geq 1}\mathbf{j}_{L/K}(l), \tag{14}$$

where $\mathbf{j}_{L/K}(l)$ are defined in (12).

A possible distant goal for further investigations of ramification in the imperfect residue field case could be to construct a certain system of invariants $\Sigma(L/K)$ for any finite extension L/K which would completely describe the ramification of L/K. This vague desire can be made more specific by listing at least the following requirements.

(1) "Naïve" ramification invariants (ramification index, order of different, genome, Artin and Swan ramification numbers) as well as other important invariants (such as Abbes-Saito conductor) can be expressed in terms of $\Sigma(L/K)$.

(2) Ramification of intermediate extensions (i.e., $\Sigma(L/M)$ and $\Sigma(M/K)$) can be expressed in terms of $\Sigma(L/K)$; reasonable base change properties in spirit of Prop. 3.3.2 are available.

(3) Local terms of appropriate global formulas can be expressed in terms of $\Sigma(L/K)$.

Of course, it would be nice to have more explicit set of requirements, which could possibly take the form of a certain "axiomatic ramification theory". However, we have a lot to learn at phenomenological level before this becomes feasible.

7.2. **Background.** Here we discuss a theory producing some additional ramification information that can be organized in analogs of the lower and upper filtrations. The approach, orginated in [Ka87], is based on two observations.

1. The Herbrand theorem (5) is true not only in the classical case but, more generally, in all the *monogenic* cases, i.e., when $\mathcal{O}_L=\mathcal{O}_K[x]$ for some x. Consequently, the ramification invariants of monogenic extensions, defined in a usual manner, possess all the usual functorial properties. The inverse statement is also true; more precisely ([Sp99, Prop. 1.5.2]):

7.2.1. **Proposition.** *Let L/K be a finite Galois p-extension. Then the following properties are equivalent:*

(i) $\mathcal{O}_L=\mathcal{O}_K[x]$ for some x;

(ii) for every normal subgroup H of G the Herbrand property (5) *holds;*

(iii) the Hilbert formula holds:

$$v_L(\mathcal{D}_{L/K})=\sum_{\sigma\neq 1}i_G(\sigma)=\sum_{i\geq 0}(|G_i|-1).$$

[4] The order of different and the depth can be considered as invariants that "live in the middle".

In [Sp99] such extensions are called *well ramified*. There are three types of well ramified extensions.

Case I. All the extensions with separable $\overline{L}/\overline{K}$.

Case II. All the weakly unramified extensions such that $\overline{L}/\overline{K}$ is generated by 1 element. (In particular, if K is a two-dimensional local field, or, more generally, if $[\overline{K}:\overline{K}^p]=p$, then all weakly unramified extensions of K are well ramified.)

Case III. Those well ramified extensions that belong neither to Case I nor to Case II. Spriano showed that for any L/K from Case III there exists an intermediate field M such that M/K is in Case I, and L/M is in Case II. A general description of Case III extensions was given in [HLF, Sect. I, §18] and [Sp00].

For us, the above remark on two-dimensional fields is important.

2. Let L/K be any finite Galois extension of complete discrete valuation fields with imperfect residue fields of characteristic $p>0$, and let k be a constant subfield of K, i.e., a maximal complete subfield of K with perfect residue field. (If $\operatorname{char} K = 0$, such a subfield is unique.) Epp's theorem on elimination of wild ramification [E] (corrections in [P] and [Kuhl]) asserts that there exists a finite extension k'/k such that $k'L/k'K$ is weakly unramified. The paper [KZ] contains various refined versions of Epp's theorem, with applications to classification of higher local fields.

7.3. **Construction.** Now we are ready to describe the construction from [Z03] and [HLF, Sect. I, §17]. For a given complete two-dimensional[5] discrete valuation field K, fix a constant subfield k. An extension L/K is said to be *constant* if $L=k'K$ and *almost constant*, if $L\subset k'K_u$, where k'/k is a finite extension, and K_u/K is an unramified extension. We say that a field L is *standard* if a prime element of its constant subfield is also a prime element of L. The choice of a constant subfield k in K determines a constant subfield l in L which is algebraic over k.

For any finite Galois extension L/K denote by L_0 the inertia subfield in L/K and by L_c/K the maximal almost constant subextension in L/K. The idea is to induce:

(1) the ramification filtration on $\operatorname{Gal}(L_c/L_0)$ by the filtration for the corresponding constants subfields;

(2) the ramification filtration on $\operatorname{Gal}(L/L_c)$ by the filtration on an isomorphic group $\operatorname{Gal}(k'L/k'L_c)$, where k'/k is a finite extension that makes $\operatorname{Gal}(k'L/k'L_c)$ weakly unramified by Epp's theorem (and even ferocious in view of the definition of L_c).

Namely, introduce a set

$$\mathbb{I}=\{-1,0\}\cup\{(\mathbf{c},i)|i\in\mathbb{Q},\ i>0\}\cup\{(\mathbf{c},\infty)\}\cup\{(\mathbf{i},i)|i\in\mathbb{Q},\ i>0\}$$

with linear order

$$\begin{aligned} -1<0<&(\mathbf{c},i)<(\mathbf{i},j) &&\text{for any } i,j;\\ &(\mathbf{c},i)<(\mathbf{c},j) &&\text{for any } i<j;\\ &(\mathbf{i},i)<(\mathbf{i},j) &&\text{for any } i<j. \end{aligned}$$

This will be the index set for lower and upper numbering of new ramification subgroups.

Let $G=\operatorname{Gal}(L/K)$. We put $G_{-1}=G$, and denote by G_0 the usual inertia subgroup in G.

[5]i.e. such that $[\overline{K}:\overline{K}^p]=p$

To introduce subgroups $G_{(\mathbf{c},i)} = G_{\mathbf{c},i}$, we consider first the case when L_c/K is constant and contains no unramified subextension. Then $L_c = lK$, and we have a natural projection

$$p: \operatorname{Gal}(L/K) \to \operatorname{Gal}(L_c/K) = \operatorname{Gal}(l/k) = \operatorname{Gal}(l/k)_0.$$

Then we put $G_{\mathbf{c},i} = p^{-1}(\operatorname{Gal}(l/k)_i)$. In the general case take an unramified extension K'/K such that $K'L/K'$ contains no unramified subextension, and the maximal almost constant subextension in $K'L/K'$ (i.e., $K'L_c/K'$) is constant. We put $G_{\mathbf{c},i} = \operatorname{Gal}(K'L/K')_{\mathbf{c},i}$. Next,

$$G_{\mathbf{c},\infty} = \operatorname{Gal}(L/L_c) = G_{\mathbf{c},m}.$$

for m big enough.

Assume that L_c is standard and L/L_c is ferocious. Let $t \in \mathcal{O}_L$, $\bar{t} \notin \overline{L}^p$. We define

$$G_{\mathbf{c},i} = \{g \in \operatorname{Gal}(L/L_c) | v_K(g(t) - t) \geq i\} \tag{15}$$

for all $i > 0$.

In the general case choose a finite extension l'/l such that $l'L_c$ is standard and $e(l'L/l'L_c) = 1$; this is possible by Epp's theorem. Then $\operatorname{Gal}(l'L/l'L_c) = \operatorname{Gal}(L/L_c)$, and $l'L/l'L_c$ is ferocious. We define

$$G_{\mathbf{c},i} = \operatorname{Gal}(l'L/l'L_c)_{\mathbf{c},i} = \operatorname{Gal}(l'L/l'K)_{\mathbf{c},i}$$

for all $i > 0$; these groups are independent of the choice of l' since we used v_K (and not v_L) in (15).

This gives a well-defined lower ramification filtration on G indexed by $\mathbb{I}$; one can define Hasse-Herbrand functions from $\mathbb{I}$ to $\mathbb{I}$ with usual properties and, consequently, construct the upper filtration. The compatibility with subgroups and factor groups mimics that of the classical case, and a ramification filtration for infinite Galois extensions is defined.

One can note also that we obtained filtration (on finite Galois groups) which factors G_i/G_{i+} are abelian for $i \geq 0$ (even elementary abelian for $i > 0$). This would not be true if we did not consider the contribution of **c**-part. For a 2-dimensional local field, one could also define a refined $\mathbb{I}_2$-filtration using rank 2 valuations in the **i**-part ([Z03, §4]).

7.4. **Further properties.** There exists also a partial result on compatibility with the higher class field theory. Namely, for an equal characteristic 2-dimensional local field K with finite residue field, one can define explicitly an $\mathbb{I}_2$-filtration on $K_2^{\mathrm{top}}K$ which coincides with the inverse image of the ramification filtration on $\operatorname{Gal}(K^{\mathrm{ab}}/K)$ with respect to the reciprocity map $\Theta_K : K_2^{\mathrm{top}}K \to \operatorname{Gal}(K^{\mathrm{ab}}/K)$, see [Z03, §6].

It is not so easy to do the same in the mixed characteristic case because of the more complicated $\operatorname{Gal}(K^{\mathrm{ab}}/K)$ and the presence of p-torsion elements in $K_2^{\mathrm{top}}K$. In particular, the following question is of interest.

7.4.1. **Question.** What is $C_K = \Theta_K^{-1}(\operatorname{Gal}(K^{\mathrm{ab}}/K_c^{\mathrm{ab}}))$?

By the results of Miki [Mik74], any extension of K with the Galois group $\mathbb{Z}_p$ is almost constant. This means that K^Γ/K, the compositum of all $\mathbb{Z}_p$-extensions, is a subextension of K_c^{ab}/K. On the other hand, $K_c^{\mathrm{ab}} = k^{\mathrm{ab}}K^{\mathrm{ab,ur}} = k^{\mathrm{ab}}K^{\mathrm{ab,tr}}$, and it is easy to see that $K^\Gamma K^{\mathrm{ab,tr}} = k^\Gamma K^{\mathrm{ab,tr}}$, where $K^{\mathrm{ab,ur}}/K$ (resp. $K^{\mathrm{ab,tr}}/K$) is the maximal abelian unramified (resp. tamely ramified) extension of K. Therefore,

$$\begin{aligned}\mathrm{Gal}(K_c^{\mathrm{ab}}/K^\Gamma K^{\mathrm{ab,tr}}) &= \mathrm{Gal}(k^{\mathrm{ab}}K^{\mathrm{ab,tr}}/k^\Gamma K^{\mathrm{ab,tr}})\\ &\simeq \mathrm{Gal}(k^{\mathrm{ab}}/k^\Gamma k^{\mathrm{ab,tr}})\\ &\simeq \text{ torsion subgroup in } U_{1,k}\end{aligned}$$

by usual local class field theory.

Let T_K be the topological closure of the p-torsion subgroup in $K_2^{\mathrm{top}}K$. Since there is no p-torsion elements in $K^\Gamma K^{\mathrm{ab,tr}}/K$, we have $\Theta_K(T_K) \subset \mathrm{Gal}(K^{\mathrm{ab}}/K^\Gamma K^{\mathrm{ab,tr}})$. From the explicit description of generators of $K_2^{\mathrm{top}}K/T_K$ (see [Z97], [I08]), it is clear that even $\Theta_K(T_K) = \mathrm{Gal}(K^{\mathrm{ab}}/K^\Gamma K^{\mathrm{ab,tr}})$. This means that C_K should be a subgroup of index p^m in T_K, where p^m is the order of p-torsion subgroup in k^* (or in K^*). However, what are the generators of C_K?

The above described ramification filtration gives a way of generalizing the "anabelian yoga" (see Subsection 3.9) to higher local fields. Abrashkin [Abr02] generalized the above construction from 2-dimensional case to n-dimensional local fields, introducing ramification theory that depends on the choice of i-dimensional subfields K_i $(1 \le i \le n-1)$ in the given n-dimensional local field, and proved a complete analog of his 1-dimensional result (announced in [Abr02], full proof in the equal characteristic 2-dimensional case in [Abr03]).

Next, Abrashkin used his generalized ramification theory to develop an analogous functor of field of norms for higher dimensional local fields, see [Abr07]. Note that there exists further generalization of the field of norms functor to the case of arbitrary imperfect residue field with finite p-basis by Scholl [Sch]; his construction does not use any kind of higher ramification theory.

Despite these nice properties, the $\mathbb{I}$-ramification theory is quite far from being a "Traumverzweigungstheorie". In particular, even for an extension of prime degree its $\mathbb{I}$-ramification break does not determine its depth of ramification and even its genome ("W" or "F"). For example, let $K = F((t))((\pi))$ and $k = F((\pi))$, F being a finite field. Assume that L/K corresponds to the Artin-Schreier equation $x^p - x = \pi^{-n} + t\pi^{-pm}$, where m, n are positive integers. Then the $\mathbb{I}$-break of L/K is m for any n, whereas $d_K(L/K) = \frac{p-1}{p}\max\{n, pm\}$, and L/K is wild if and only if $n \ge pm$.

However, in the equal characteristic case one can vary the constant subfield k of K thus collecting more information on ramification. For example, if L/K is wild of degree p with the Swan number s_0, then, for some choices of k, the $\mathbb{I}$-break of L/K is $(\mathbf{c}, s)$ and necessarily $s = s_0$. In this example m is not an invariant of L/K. However, if in the example of §4 we consider only such k that the $\mathbb{I}$-break of K_2/K is some $(\mathbf{c}, s)$ (clearly, $s = N$), then the $\mathbb{I}$-break of K_1/K will be $(\mathbf{i}, m/p)$. Therefore, the knowledge of $\mathbb{I}$-breaks of K_1/K and K_2/K *for all choices of* k determines the ramification of K_1K_2/K.

7.4.2. **Question.** Can we construct a powerful ramification theory for equal characteristic 2-dimensional fields by varying the constant subfield?

7.4.3. **Question.** Can we use this approach even in the mixed characteristic case using truncations from [De84]?

8. Semi-global modeling

Now we describe one more approach to description of ramification in the imperfect residue field case. This approach goes back to Deligne who sketched a proof of a Grothendieck-Ogg-Shafarevich formula for surfaces in his famous letter to Illusie [De76].

8.1. **Background.** We recall some starting points of Deligne's program. Let $\mathcal{F}$ be a locally constant étale $\mathbb{F}_l$-sheaf of finite rank on U, where U is the complement to some divisor D on a smooth projective surface S over an algebraically closed field of prime characteristic $p \neq l$. In order to understand the ramification data associated with $\mathcal{F}$ at the generic point of a component D_0 of D, Deligne considers various regular arcs $\mathcal{C}$ transversal to D_0 and studies the restrictions of $\mathcal{F}$ to these arcs. It is expected that the Swan conductor of $\mathcal{F}|_{\mathcal{C}}$ (at the point where $\mathcal{C}$ meets D_0) depends only on the jet of $\mathcal{C}$ of certain order r. Thus, we can consider the Swan conductor as a function on the space $T_{1,r}$ of r-jets of regular arcs transversal to D_0; this space has a natural structure of a vector bundle over D_0. Next, this function is expected to be lower semi-continuous; in particular, it should take its maximal value over certain Zariski open subset W of $T_{1,r}$. The next claim is that the complement of W has pure codimension one in $T_{1,r}$, i.e., is a union of several irreducible hypersurfaces. The further work is based on geometry of these hypersurfaces including intersection theory.

Some of these facts were proved in [La] under assumption of "absence of ferocious ramification". This means that the locally constant sheaf $\mathcal{F}$ is trivialized in some finite extension of $k(S)$ such that all extensions of residue fields are separable. In particular, the semi-continuity of Swan conductor has been proved under this assumption.

Brylinski in [Br] considers a cyclic p-extension of the function field of s surface S over a field of characteristic p given by the Witt vector $x = (x_0, \dots, x_{r-1})$. He assumes that the branch locus D_0 is smooth at a certain regular point P of S and the valuations of all x_i at the generic point of D_0 are either positive or prime to p. (This condition implies absence of ferocious ramification if $r = 1$ but not in general.) Under this assumption he proves that, for all curves C transversal to D_0 at P, the Swan conductors of corresponding extensions of $k(C)$ are equal, and their common value is Kato-Swan conductor of the extension of the 2-dimensional local field $k(S)_{D_0,P}$ corresponding to x.

Consider a cyclic extension L of degree p of $k(S)$ as above such that the branch locus D_0 is smooth with one component, and the ramification at this component is wild. We see from the papers of Laumon and Brylinski that in this case for all curves C transversal to D at a fixed point, the corresponding ramification numbers will be the same (and equal to the ramification number of $L/k(S)$). However, in order to approach a more comprehensive description of ramification in the sense of Subsection 7.1, it appeared useful to consider curves which are *tangent* to D_0 of certain fixed order (and *smooth*).

8.1.1. **Example.** Let k be algebraically closed, $\operatorname{char} k = 2$, $S = \mathbb{A}^2_k$ with coordinates t, u, S' the normalization of S in the Artin-Schreier extension $L_\alpha/k(t,u)$ given by

$$x^2 - x = t^{-2n+1}(1 + \alpha u),$$

where $\alpha \in k$. Introducing $t_1 = t^n x$, we see that t_1 is integral over $k[t,u]$ and $S'_0 = \operatorname{Spec} k[t,u,t_1]$ is regular, whence $S' = S'_0$. Let O' be the closed point of S' above the origin O. (It is unique since O belongs to the branch locus of normalization morphism.) Replacing S and S' with the spectra of completed local rings at O and O' respectively, and introducing $t_0 = t(1+\alpha u)$, we arrive at the homomorphism $\varphi : k[[t_0,u]] \to k[[t_1,u]]$ given by

$$\varphi(t_0) = t_1^2 + t_1^{2n+1} + \text{ terms of higher order.}$$

Notice that the branch locus of φ is determined by the prime ideal (t_0) of $k[[t_0,u]]$. Consider a family of curves C_λ on $\operatorname{Spec} k[[t_0,u]]$ with the equations

$$t_0 = u^2 + \lambda u^3 + u^5, \quad \lambda \in k,$$

and denote by C'_λ their pullbacks in $\operatorname{Spec} k[[t_1,u]]$. It is not difficult to calculate that

$$s(k(C'_\lambda)/k(C_\lambda)) = \begin{cases} 4n-3, & \lambda \neq 0, \\ 4n-5, & \lambda = 0, \end{cases}$$

(assuming $n \geq 2$). Moreover, let C be an arbitrary regular curve on $\operatorname{Spec} k[[t_0,u]]$ which is simply tangent to the branch divisor, i.e., with an equation

$$t_0 = \lambda_2 u^2 + \lambda_3 u^3 + \dots,$$

where $\lambda_2 \neq 0$, and let C' be its pullback. Then C' is irreducible; $s(k(C')/k(C)) = 4n-3$ if $\lambda_3 \neq 0$, and $s(k(C')/k(C)) < 4n-3$ if $\lambda_3 = 0$ ("exceptional hypersurface"). Note that if C is determined by an equation

$$t = \mu_2 u^2 + \mu_3 u^3 + \dots$$

in the original coordinates t,u, then $\mu_3 = \lambda_3 + \alpha\mu_2$. This means that the equation of the "exceptional hypersurface" H_α is $\mu_3 = \alpha\mu_2$, and thus H_α "detects the α".

8.2. **Semi-global models.** Deligne's program is intended to compute Euler-Poincaré characteristic of an étale sheaf on a surface or, more generally, to describe ramification of a finite morphism of algebraic or arithmetic surfaces. However, we can try to use this approach as a source of rich information about ramification of extensions of 2-dimensional local fields by constructing geometric "models" for given extensions.

Namely, let $h : A \to B$ be a finite k-homomorphism of 2-dimensional regular local rings with perfect coefficient subfield k of characteristic $p > 0$. Let $\mathfrak{p}$ be a prime ideal of height 1 in B such that $B/\mathfrak{p}$ and $A/h^{-1}(\mathfrak{p})$ are regular. We shall say that $(h, \mathfrak{p})$ is a *model* for a finite extension of 2-dimensional local fields L/K, if there exists an isomorphism i of 2-dimensional local fields $\widehat{Q(B)}_{\mathfrak{p}} \simeq L$ mapping $\widehat{Q(A)}_{h^{-1}(\mathfrak{p})}$ onto K.

We suggest to study ramification in L/K by considering various regular curves on $\operatorname{Spec} A$ and their pullbacks in $\operatorname{Spec} B$. For each such curve C and a component of its pullback C', the field extension $k(C')/k(C)$ is a finite extension of 1-dimensional local fields inheriting information on L/K.

Of course, since we are interested only in "ramification in codimension 1", we have a huge freedom in choosing models for given L/K. (We can make blow-ups preserving L/K etc.) We hope to describe a class of morphisms h having as simple structure as possible to make the study of $k(C')/k(C)$ easy but still providing models for all L/K of interest.

For example, in [Z10] we proposed to study pairs $(h, \mathfrak{p})$ such that for some choice of regular local parameters t, u in A and x, y in B with $\mathfrak{p} = (x)$ and $h^{-1}(\mathfrak{p}) = (t)$ the following conditions are satisfied:

(i) $h(t) = \delta \cdot x^{e_x}$,

(ii) $h(u) \equiv \varepsilon \cdot y^{e_y} \mod x$,

(iii) $J(t,u) = \begin{vmatrix} \frac{\partial h(t)}{\partial x} & \frac{\partial h(t)}{\partial y} \\ \frac{\partial h(u)}{\partial x} & \frac{\partial h(u)}{\partial y} \end{vmatrix} = \gamma \cdot x^M$,

where e_x, e_y are positive integers, e_y being a nonnegative power of p, M is a nonnegative integer; $\delta, \varepsilon, \gamma \in B^*$.

Such morphisms appeared in [CuP] in the context of resolution of a finite morphism between regular algebraic surfaces over a field of characteristic $p > 0$.

It was proved in [Z10, Prop. 2.4] that an extension of 2-dimensional local fields L/K has a model with properties (i), (ii), if the following 2 conditions are satisfied.

(1) $f_s(L/K) = 1$.

(2) Let $(e_{ij})_{i,j=1,2}$ be the matrix ramification index for some choice of rank 2 valuations $\mathbf{v}_L$ and $\mathbf{v}_K$, i.e., $\mathbf{v}_L|_K = \mathbf{v}_K \cdot (e_{ij})$. Then $\gcd(e_{11}, e_{22})|e_{12}$.

Moreover, in this case we have $e_w(L/K) = e_x$ and $f_i(L/K) = e_y$, see [Z10, Prop. 2.2].

8.3. **Initial questions.** Let $(h, \mathfrak{p})$ be as in Subsection 8.2; denote by D_0 the prime divisor of $X = \operatorname{Spec} A$ corresponding to $h^{-1}(\mathfrak{p})$. Fix a positive integer r and consider the set T_r of all regular arcs C on X such that $(C.D_0) = r$. Assume the above condition (iii); then C is not a component of the branch divisor, and $h^*C = C_1' + \dots + C_n'$, where $C_1', \dots, C_n'$ are distinct prime divisors of $\operatorname{Spec} B$, and $n = n(C)$ is a positive integer. For each i $(1 \le i \le n)$, we have an extension of complete discrete valuation fields with perfect residue fields $k(C_i')/k(C)$. Our plan is to study the ramification invariants of the extensions $k(C_i')/k(C)$ as functions on the set T_r.

First of all, we have to check that $n(C)$ and all the ramification invariants depend only on the jet of C of certain order $R = R(r)$. Having this proved, we can consider $n(C)$ and ramification invariants as functions on the set $T_{r,R}$ of R-jets of arcs from T_r.

Each $T_{r,R}$ has a structure of an affine variety over k. Indeed, let t, u be local parameters of A such that $(t) = h^{-1}(\mathfrak{p})$. Then, in view of Weierstraß preparation theorem, each curve from T_r has a unique equation of the form

$$f = \begin{cases} -u + \alpha_1 t + \alpha_2 t^2 + \dots, & r = 1, \\ -t + \beta_r u^r + \beta_{r+1} u^{r+1} + \dots, & r > 1, \end{cases}$$

where α_i and β_i are any elements of k with an only restriction $\beta_r \neq 0$. If $r > 1$, $T_{r,R}$ can be identified with $\{(\beta_r, \dots, \beta_R) \in \mathbb{A}_k^{R-r+1} | \beta_r \neq 0\}$; if $r = 1$, $T_{r,R}$ can be identified with $\mathbb{A}_k^R$; see more details in [Z02a].

Next, we would like to check that certain functions of these ramification invariants are semi-continuous on $T_{r,R}$ with respect to corresponding Zariski topology. (These functions are reduced to conductors or the order of different if $s = 1$, and the precise definitions in the general case are still to be understood.)

Some results in this direction are included into the next section.

9. Some results on semi-continuity

9.1. **Artin-Schreier extensions.** The paper [Z02a] is devoted to the study of questions raised in Subsection 8.3 in the case of Artin-Schreier coverings of the spectrum of a complete 2-dimensional regular local ring (of characteristic $p > 0$). Such coverings can serve as semi-global models of Artin-Schreier extensions of 2-dimensional local fields. However, the setting in this work is somewhat more general: the morphisms with 2 (transversal) components in the branch locus are also included into consideration.

More precisely, let A be a regular two-dimensional local ring (not necessarily complete), $\operatorname{char} A = p > 0$, $K = Q(A)$, $\mathfrak{m}$ the maximal ideal of A, and k the residue field which is assumed to be algebraically closed. For a prime ideal $\mathfrak{p}$ of height 1, denote by $F_{\mathfrak{p}}$ the corresponding prime divisor of $\operatorname{Spec} A$. For any two distinct prime divisors $F_{\mathfrak{p}}$, $F_{\mathfrak{p}'}$ we define their intersection number as

$$(F_{\mathfrak{p}}.F_{\mathfrak{p}'}) = \dim_k A/(\mathfrak{p} + \mathfrak{p}');$$

by linearity this definition can be extended to any two divisors C, D with no common components.

Let L/K be a cyclic extension of degree p, and let B be the integral closure of A in L. For the sake of simplicity of statements we assume here that the branch divisor of B/A consists of one smooth component $F_{\mathfrak{p}_1}$; for the case of two transversal components, see [Z02a]. Denote by U_A the set of prime ideals of height 1 of A other than $\mathfrak{p}_1$. For $\mathfrak{p} \in U_A$, denote by $\mathfrak{q}$ any prime ideal of B over $\mathfrak{p}$. Denote

$$s_{\mathfrak{p}}(L/K) = \begin{cases} s(L(\mathfrak{q})/K(\mathfrak{p})), & e(L(\mathfrak{q})/K(\mathfrak{p})) = p, \\ 0, & \text{otherwise}, \end{cases}$$

where $K(\mathfrak{p})$ is the fraction field of $A/\mathfrak{p}$, and $L(\mathfrak{q})$ is the fraction field of $B/\mathfrak{q}$.

Introduce T_r and $T_{r,n}$ as in Subsection 8.3 and identify $\mathfrak{p}$ with the arc $F_{\mathfrak{p}}$.

9.1.1. **Proposition.** (existence of a uniform sufficient jet order, [Z02a, Theorem 2.1]) *For any $r \geq 1$ there exists R such that if $\mathfrak{p}, \mathfrak{p}' \in T_r$ and $(F_{\mathfrak{p}}.F_{\mathfrak{p}'}) \geq R+1$, then $s_{\mathfrak{p}}(L/K) = s_{\mathfrak{p}'}(L/K)$. Let $\mathrm{su}_{1,r}(L/K)$ be the minimal such R. Then there exists $N \geq 1$ such that $\mathrm{su}_{1,r}(L/K) < Nr$ for any r.*

9.1.2. **Remark.** There was a mistake in the proof of "sufficient jet order conjecture" in [Z02b]. The correct part of this preprint on the bounded growth of curve singularity invariants along certain tame and wild morphisms of surfaces was published later as [Z06].

Next, introduce Zariski topology in all $T_{r,n}$ as in Subsection 8.3. Then the following statements hold.

9.1.3. **Proposition.** (semi-continuity of a break, [Z02a, Theorems 2.2–2.4])

1. Let $n \geq \mathrm{su}_{1,r}(L/K)$. Denote by $J_n(\mathfrak{p})$ the n-jet of the arc $F_{\mathfrak{p}}$. Then for any $s \geq 0$ the set

$$\{J_n(\mathfrak{p}) | \mathfrak{p} \in T_r;\ s_{\mathfrak{p}}(L/K) \leq s\}$$

is a closed subset in $T_{r,n}$.

2. The supremum

$$s_r(L/K) = \sup\{s_{\mathfrak{p}}(L/K) | \mathfrak{p} \in T_r\}$$

is finite.

3. Assume in addition that A is a G-ring. Then the sequence $(s_r(L/K)/r)_r$ is convergent.

9.2. **Extensions of prime degree.** The paper [Fa] is devoted to morphisms $h : A \to B$ of Subsection 8.3 with properties (i), (ii) and (iii) without assumption that B is a Galois algebra over A.

Let T_r, $T_{r,R}$, C, $n(C)$, C'_i have the same meaning as in Subsection 8.3. Under the assumption $n(C) = 1$, denote by s_C the only ramification break of $k(C'_1)/k(C)$ as defined at the very end of §1. Then we have ([Fa, Theorem 4]):

9.2.1. **Proposition.** (existence of a uniform sufficient jet order) *For any $r \geq 1$ there exists R such that if $C, \tilde{C} \in T_r$ and $(C.\tilde{C}) \geq R+1$, then $s_C = s_{\tilde{C}}$. Let $\mathrm{su}_{1,r}(h)$ be the minimal such R. Then there exists $N \geq 1$ such that $\mathrm{su}_{1,r}(h) < Nr$ for any r.*

Next, Faizov proved the following semi-continuity statement ([Fa, Theorems 5 and 6]).

9.2.2. **Proposition.** (semi-continuity of a break) *1. Let $n \geq \mathrm{su}_{1,r}(h)$. Then for any rational $s \geq 0$ the set*

$$\{J_n(C) | C \in T_r;\ s_C \leq s\}$$

is a closed subset in $T_{r,n}$.

2. The supremum

$$s_r(h) = \sup\{s_C | C \in T_r\}$$

is finite.

The proofs are based on careful work with Hamburger-Noether algorithm for curve C_1 yielding an explicit form of a uinformizing element of $k(C)$.

9.3. **Relation to singularity invariants.** In the context of Subsection 8.3, we considered regular arcs on $\operatorname{Spec} A$; however, the arcs C'_i on $\operatorname{Spec} B$ are in general singular, and the complexity of singularity can reflect the ramification data of the morphism h; this phenomenon was first observed in [Z06]. In [CZ] we relate the semi-continuity property of ramification invariants with the semi-continuity of δ-invariant in families of singular arcs.

Let A, B be complete 2-dimensional regular local rings with algebraically closed coefficient subfield k. A finite k-homomorphism $h : A \to B$ will be called *unmixed* if $h(\mathfrak{m}_A) \subset \mathfrak{m}_B$ and $h(\mathfrak{m}_A) \not\subset \mathfrak{m}_B^2$. In particular, a homomorphism with properties (i) and (ii) is unmixed if in its definition either $e_x = 1$ or $e_y = 1$.

A *decomposable* homomorphism is by definition a composition of several unmixed homomorphisms.

The following statement is proved in [CZ].

9.3.1. **Proposition.** *Let $h : A \to B$ be a decomposable homomorphism of degree m, and $\mathcal{B}$ its branch divisor in $\operatorname{Spec} A$. Let C be a reduced curve on $\operatorname{Spec} A$ having no common components with $\mathcal{B}$; $C' = h^*C$. Let $C'_1, \ldots, C'_r$ be all components of C'; $C_i = h_*C'_i$, $i = 1, \ldots, r$; d_i the order of different in the extension of discrete valuation fields $k(C'_i)/k(C_i)$. Then we have*

$$2\delta(C') - 2m\delta(C) = (C.\mathcal{B}) - \sum_{i=1}^{r} d_i. \tag{16}$$

This immediately implies

9.3.2. **Corollary.** *Let $h : A \to B$ be a decomposable homomorphism, and $\mathcal{B}$ its branch divisor in* Spec A*. Let C be a regular curve on* Spec A *which is not a component of $\mathcal{B}$; $C'_1, \ldots, C'_r$ all components of $C' = h^*C$, $i = 1, \ldots, l$; d_i the order of different in the extension of discrete valuation fields $k(C'_i)/k(C)$. Then*

1. $\sum_{i=1}^r d_i \le (C.\mathcal{B})$.
2. $\delta(C') \le \frac{1}{2}(C.\mathcal{B})$.

Consider a decomposable homomorphism $h : A \to B$ and assume that the branch divisor of h is of the form $\mathcal{B} = bD_0$, where D_0 is a regular reduced irreducible curve on Spec A and b is a positive integer. (It is always so when h has properties (i)–(iii) from Subsection 8.2.)

9.3.3. **Lemma.** *Let Δ be a positive integer. Let A be a complete 2-dimensional regular local ring having a coefficient subfield. Consider two curves $C, \tilde{C}$ on* Spec A *such that $\delta(C) \le \Delta$, and $C, \tilde{C}$ have the same 2Δ-jet. Let $C_1, \ldots, C_r$ be all irreducible components of C. Then $\tilde{C}$ also has r irreducible components $\tilde{C}_1, \ldots, \tilde{C}_r$ with $\delta(\tilde{C}_i) = \delta(C_i)$ and $(\tilde{C}_i.\tilde{C}_j) = (C_i.C_j)$ for all i, j.*

9.3.4. **Question.** Is it possible to estimate Milnor and Tjurina numbers $\mu(C)$ or $\tau(C)$ in terms of $\delta(C)$? Maybe, one could apply formulas for $\mu(C)$ from [BGM], [MHW]. If yes, this would enable us to estimate finite determinacy of C.

Next, let $T_{r,R}$, $n(C)$, C'_i have the same meaning as in Subsection 8.3.

9.3.5. **Proposition.** *Let C be a regular curve on* Spec A *with $(C.D_0) = r < \infty$. Then, for the curve h^*C, the number of components, their δ-invariants and intersection numbers depend only on the jet of C in $T_{r,br}$.*

Proof. Let C and $\tilde{C}$ have the same br-jet. Then obviously h^*C and $h^*\tilde{C}$ also have the same br-jet. In view of Corollary 9.3.2, $\delta(h^*C) \le br/2$. It remains to apply Lemma 9.3.3 with $\Delta = [br/2]$. □

9.3.6. **Corollary.** *For C as in the above proposition, let d_i be the order of different in the extension of discrete valuation fields $k(C'_i)/k(C)$, $i = 1, \ldots, n(C)$. Then $\sum_{i=1}^r d_i$ depends only on the br-jet of C.*

Proof. It follows from Proposition 9.3.5 and formula (16). □

Let us make the following Assumption S_δ on the semi-continuity of the δ-invariant.

Let A be a complete 2-dimensional regular local ring with algebraically closed coefficient subfield k, and let U be an open subset of $\mathbb{A}_k^N$ for some positive integer N. Let $f \in A[X_1, \ldots, X_N]$ be such that for any closed point $(a_1, \ldots, a_N) \in U$ the curve $C(a_1, \ldots, a_N) =$ Spec $A/(f, X_1 - a_1, \ldots, X_N - a_N)$ is reduced. Assume that there exists a positive integer Δ such that $\delta(C(a_1, \ldots, a_N)) \le \Delta$ for all $(a_1, \ldots, a_N) \in U$. Then $\delta(C(a_1, \ldots, a_N))$ is an upper semi-continuous function on U.

9.3.7. **Proposition.** *If Assumption S_δ is satisfied, then for any $r \ge 1$, $\delta(h^*C)$ determines an upper semi-continuous function on $T_{r,br}$.*

Proof. It follows immediately from Corollary 9.3.2. □

9.3.8. **Question.** Is it true that $n(C)$ (the number of components of h^*C) determines a lower semi-continuous function on $T_{r,br}$? What can be said about the generic value of $n(C)$?

9.3.9. **Corollary.** *For a regular curve C on* $\operatorname{Spec} A$ *with* $(C.D_0) = r$*, let* $C'_1, \dots, C'_n$ *be all components of* h^*C*,* $n = n(C)$*, and* d_i *the order of different in the extension of discrete valuation fields* $k(C'_i)/k(C)$*. Then* $\sum_{i=1}^n d_i$ *determines a lower semi-continuous function on* $T_{r,br}$*, if the Assumption* S_δ *is satisfied.*

Proof. It follows immediately from Prop.9.3.7 and 9.3.1, since $(C.\mathcal{B}) = br$. □

9.3.10. **Question.** We suggest to say that a lower semi-continuous integer-valued function h on a variety S is *purely lower semi-continuous* if for every N each component of the closed subset

$$S_N = \{P \in S | h(P) < N\}$$

has codimension ≤ 1 in the respective component of S_{N+1}.

Is it true that $\sum_{i=1}^n d_i$ determines a purely lower semi-continuous function on $T_{r,br}$? Equivalently, is $\delta(\beta_r, \dots, \beta_{pr})$ purely upper semi-continuous on $T_{r,br}$? (Pure upper semi-continuity is defined similarly.)

This is related to Deligne's conjecture that the loci of exceptional values of ramification invariants are always hypersurfaces.

10. Algebraic-geometric consequences of Abbes-Saito filtration

The theory of Abbes-Saito ramification filtrations has deep applications in algebraic geometry, including Grothendieck-Ogg-Shafarevich type formulas for Euler characteristic of étale sheaves. A survey of these geometric applications is also given in T. Saito's ICM talk [Sa10]. Here we prefer to discuss at the same time the global version of three analogous objects: lisse $\mathbb{Q}_l$-sheaves, overconvergent F-isocrystals, and locally free coherent sheaves with integrable connections; this way, we can compare their similarities as well as differences.

10.0.1. **Question.** This section can lead the reader to the following question: could some of the results in this section find an application to the geometric Langlands program? The authors are very interested in such potential relations.

10.1. **Setup.** Let k be a field. For a smooth variety X over k, let $D = \cup_{i=1}^r D_i$ be a divisor on X with strict simple normal crossings, where D_i are irreducible components. Let $U = X \backslash D$ denote the complement. Suppose that we are in one of the following situations.

(a) $\mathcal{F}$ is a lisse $\mathbb{Q}_l$-sheaf on U, where l is a prime number different from char k;
(b) $\mathcal{F}$ is an F-isocrystal on U overconvergent along D, while char $k = p > 0$;
(c) $\mathcal{F}$ is a locally free coherent sheaf on U with an integrable connection, while char $k = 0$.

At the generic point η_i of an irreducible component D_i of the divisor D, one can talk about

(a) the Swan conductor $\mathrm{Sw}(\mathcal{F}; D_i)$, obtained by considering the representation $G_{k(X)^{\eta_i}} \to \pi_1(U) \to \mathrm{GL}(V_{\mathcal{F}})$, where the latter homomorphism is the representation associated to the lisse sheaf $\mathcal{F}$; or
(b) the (differential) Swan conductor $\mathrm{Sw}(\mathcal{F}; D_i)$, obtained by passing to the generic point in the sense of Subsection 6.5; or
(c) the irregularity $\mathrm{Irr}(\mathcal{F}; D_i)$ in the sense of Subsection 6.9 by base changing to the completion at η_i; we rename it as the Swan conductor $\mathrm{Sw}(\mathcal{F}; D_i)$.

We define the Euler characteristic to be $\chi(U,\mathcal{F}) = \sum_j (-1)^j \dim H^j_?(X,\mathcal{F})$, where ? is the étale cohomology (after base change to k^{alg}) in case (a), is the rigid cohomology in case (b), and is the de Rham cohomology in case (c). When $\mathcal{F}$ is the trivial object, we write $\chi(U)$ for $\chi(U,\mathcal{F})$.

We list these three cases together because most of the results on ramification theory hold in a similar fashion.

10.2. **Results of variation of Swan conductors.** The approach we will take is local-to-global; building on the study of variation of Swan conductors locally on X, we expect a global result from the local data at the end.

We explain the main results of [KeX, Ke11a, Ke10b] on the variation properties of Swan conductors by means of an example. Historically, the same result in the rank one case was already known to Kato, as explained implicitly in his foundational work [Ka94]. We take $X = \mathbb{A}^2 = \operatorname{Spec} k[x,y]$, $D_0 = Z(y)$ and $D_1 = Z(x)$. Let $\mathcal{F}$ be as in either case considered in Subsection 10.1 over $U = X \backslash (D_0 \cup D_1)$ as above. We can consider the Swan conductors $\mathrm{Sw}(\mathcal{F}; D_0)$ and $\mathrm{Sw}(\mathcal{F}; D_1)$.

We may blowup X at the origin $P = D_0 \cap D_1$ to get $X' = \mathrm{Bl}_P X$; let $D_{1/2}$ denote the exceptional divisor. Since $\mathcal{F}$ is defined on U, we can talk about the Swan conductor $\mathrm{Sw}(\mathcal{F}; D_{1/2})$ of the sheaf $\mathcal{F}$ along $D_{1/2}$ as in Subsection 10.1(b). Carrying on this idea, we can continue to blow up X' along the intersections of $D_{1/2}$ with the proper transforms of D_0 and D_1. We use $D_{1/3}$ and $D_{2/3}$ to denote the two exceptional divisors for this blowup. Similarly, the Swan conductors $\mathrm{Sw}(\mathcal{F}; D_{1/3})$ and $\mathrm{Sw}(\mathcal{F}; D_{2/3})$ are then well-defined. We can iterate this process to blow up intersections of these divisors and then consider the Swan conductors along all the exceptional divisors. We label the exceptional divisors as follows: for each pair of coprime integers $(m,n) \in \mathbb{N}^2$, there is exactly one exceptional divisor $D_{n/m+n}$ such that, for the valuation v corresponding to $D_{n/m+n}$, we have $v(x) = n$ and $v(y) = m$. Along this divisor, a Swan conductor $\mathrm{Sw}(\mathcal{F}; D_{n/m+n})$ can be defined as in Subsection 10.1(b).

10.2.1. **Proposition.** *The function*

$$\frac{n}{n+m} \longmapsto \frac{1}{n+m}\mathrm{Sw}(\mathcal{F}; D_{n/m+n})$$

extends by continuity to a convex piecewise linear function on $[0,1]$ *with integral slopes.*

This proposition is a special case of the results proved in [KeX, Ke11a, Ke10b] for a higher dimensional variety X and for an intersection point of simple normal crossing divisors. (The essential part of the proof is in [KeX]; the statements appear in [Ke11a] for cases (a) and (b) and in [Ke10b] for case (c).) Moreover, the slopes of the piecewise linear function are related to the refined Swan conductor homomorphism defined in Subsection 6.2; see [X12b] for details.

10.2.2. **Remark.** We point out a caveat: there is no analogous result of Proposition 10.2.1 for Artin conductors, because blowing up is log-smooth but not smooth. So Swan conductors are better adapted to this type of variation questions.

10.3. **Approach to ramification theory using cutting-by-curves.** It would be interesting to clarify the relation between the Abbes-Saito filtration at generic points (as discussed above) and the ramification data from cutting-by-curves (as discussed in details in Section 9).

We first explain the "cut-by-curve" Swan conductors. Let D_i be an irreducible divisor of X, then one can define a new Swan conductor by taking

$$\mathrm{Sw}_{\mathrm{curve}}(\mathcal{F}; D_i) := \sup_C \Big(\frac{\mathrm{Sw}(\mathcal{F}|_C; C \cap D_i)}{(C.D_i)} \Big),$$

where $(C.D_i)$ is the intersection number of C with D_i and the supremum is taken over all curves C that intersects with D_i (not necessarily transversely). A suggestion to study $\mathrm{Sw}_{\mathrm{curve}}$ appeared (in 2-dimensional case) in [Z02b, Remark 2.5.3]; a computation in the Artin-Schreier case was done in [Z02a] (see above Prop. 9.1.3).

The natural question to ask is whether $\mathrm{Sw}_{\mathrm{curve}}(\mathcal{F}; D_i)$ is the same as $\mathrm{Sw}(\mathcal{F}; D_i)$ which is defined using the Abbes-Saito ramification filtration (as in Subsection 10.1). This question is addressed by Barrientos [Ba] in case (a) when the sheaf has rank one, which generalizes an idea of Deligne-Esnault-Kerz [EK]. It would be interesting to generalize this to all cases in Subsection 10.1 for arbitrary rank objects. We also emphasize that using curves that are not transversal to the divisor is essential in this theory, as shown in the following example.

10.3.1. **Example.** Let $X = \mathbb{A}^2$ be the xy-plane over a field k of characteristic p and let D be the divisor $Z(y)$. Consider the Artin-Schreier sheaf $\mathcal{F}$ over $U = X - D$ given by the equation $z^p - z = x/y^p$, that is the lisse sheaf associated to a nontrivial character of the Galois group $\mathbb{Z}/p\mathbb{Z}$ of the cover of U given by this equation.

Using Example 6.1.1, we see that $\mathrm{Sw}(\mathcal{F}; D) = p$, as x is *not* a pth power in the residue field $k(x)$. When restricted to each line $C_a : x = a$ for $a \in k^{\mathrm{alg}}$, the Artin-Schreier equation becomes $z^p - z = a/y^p$ which is the same as $z'^p - z' = a^{1/p}/y$ for $z' = z - a^{1/p}/y$. So $\mathrm{Sw}(\mathcal{F}|_{C_a}; D \cap C_a) = 1$. In other words, the generic Swan conductor (using Abbes-Saito's filtration) is not equal to the Swan conductor restricted to *any* such curve C_a.

If instead we consider the curve $C_{a,m} : y = (x-a)^m$ for $a \in k^{\mathrm{alg}}$ and $m \gg 0$, the Artin-Schreier equation becomes $z^p - z = x/(x-a)^{pm}$. Since the intersection point is $x = a$, we use change of variable $x' = x - a$; the equation becomes $z^p - z = (x' + a)x'^{-pm}$. If we substitute z' for $z - a^{1/p}x'^{-m}$, we get $z'^p - z' = x'^{-pm+1} + a^{1/p}x'^{-m}$. It is now clear that $\mathrm{Sw}(\mathcal{F}|_{C_{a,m}}; D \cap C_{a,m}) = pm - 1$. Thus,

$$\limsup_m \frac{\mathrm{Sw}(\mathcal{F}|_{C_{a,m}}; D \cap C_{a,m})}{(D.C_{a,m})} = p.$$

We also point out that when $m = 1$, the curve $y = x - a$ is still transversal to D, but $\mathrm{Sw}(\mathcal{F}|_{C_{a,1}}; D \cap C_{a,1}) = p - 1$, which is different from $\mathrm{Sw}(\mathcal{F}|_{C_a}; D \cap C_a) = 1$; thus restricting to different transversal curves may give different Swan conductors. The largest Swan conductor obtained by restricting to transversal curves is $p - 1$, which is still smaller than the "correct answer" p, as seen at the "generic point". This is why we need to consider curves non-transversal to the divisor.

10.3.2. **Question.** Using the results on variation properties of Abbes-Saito Swan conductors (Prop. 10.2.1) and the information of refined Swan conductors, can we say something along the line of semi-continuity type statement proposed by Deligne [De76] (and proved in [La] in case of absence of ferocious ramification)?

10.4. **Towards generalized Grothendieck-Ogg-Shafarevich formulas.** One of the goals of Abbes and T. Saito's project is to generalize the Euler characteristic formula for l-adic sheaves. In fact, this should be applicable to all three cases we discussed above. We will refer to such formulas as *Grothendieck-Ogg-Shafarevich* *type formulas* (*GOS type formulas* for short). Under a *cleanliness condition* which we explain later, a GOS type formula is expected to take the following form (when $\operatorname{rank}\mathcal{F} = 1$ and $\dim X = 2$)

$$\chi(\mathcal{F}) = \chi(U) - \sum_{j=1}^{r} \mathrm{Sw}_j \cdot \chi(D_j^\circ) + \sum_{j_1, j_2=1}^{r} \mathrm{Sw}_{j_1}\mathrm{Sw}_{j_2} \cdot (D_{j_1}.D_{j_2}), \tag{17}$$

where Sw_j is the Swan conductor of $\mathcal{F}$ along D_j as in Subsection 10.1 and $D_j^\circ = D_j - (\cup_{j' \neq j} D_{j'})$. (Compare this with the classical Grothendieck-Ogg-Shafarevich formula in Subsection 3.11.) The expression of the formula becomes more complicated when $\operatorname{rank}\mathcal{F} > 1$.

GOS type formulas are known when X is a curve. Case (a) is discussed in Subsection 3.11. Case (b) is due to Christol, Crew, Matsuda, Mebkhout, and Tsuzuki; a complete reference with a proof is given in [Ke06, Theorem 4.3.1]. Case (c) is due to Deligne and Gabber; one can find a proof in [Katz, Theorem 2.9.9].

In [Ka94], Kato studied the GOS type formulas for higher dimensional varieties and for $\mathcal{F}$ of rank one. There have been some recent generalizations of Kato's work to the case when both X and $\mathcal{F}$ are general. A GOS type formula for case (a) is conjectured in [AS11, Sa10] under the cleanliness condition, and is proved under additional assumptions in [Sa09]. In case (c), a GOS type formula under the cleanliness condition plus a very mild assumption is proved in [X12+], which follows the idea of [Ka94, §1].

We now explain the key points that enter the proof of these GOS type formulas.

First, it appears to be impossible to obtain an unconditional formula that takes the form of (17). This is because the ramification data at the generic points of the divisors do not determine the ramification at the closed points. One has to impose a *cleanliness condition* on the object $\mathcal{F}$, which roughly says that the ramification at all closed points on D is determined by the ramification data at generic points of D. The cleanliness condition is discussed in [AS11] for case (a), but also note the subtlety of different versions of cleanliness, as discussed in [X12+] for case (c).

10.4.1. **Question.** We often encounter situations when $\mathcal{F}$ is not clean on X. But we expect that there exists a birational proper morphism

$$f : (X', D') \to (X, D)$$

such that $f^*\mathcal{F}$ satisfies the aforementioned cleanliness condition. In case (c), this expectation is known as the Sabbah Conjecture, proved by Kedlaya [Ke10b, Ke11b] and Mochizuki [Mo-T] independently. It would be very interesting to prove this expectation in cases (a) and (b). This may be thought of as a version of desingulization, except that we are resolving the "singularities of a sheaf".

Second, let us assume the cleanliness condition from now on. Along the way of proving GOS type formulas, we expect that the ramification data (namely the Swan conductors and the refined Swan conductors) also provides information about the log-characteristic cycle of $\mathcal{F}$ (as a cycle in the log-cotangent space of X). There are a good surprise and a bad surprise when one tries to realize such philosophy. The good surprise is that, unlike in the usual (non-logarithmic) characteristic cycle for an algebraic D-module, where all irreducible components are conormal bundles of some closed subvarieties of X (see, e.g., [HTT]), the log-characteristic cycle can contain arbitrary subbundles of the log-cotangent space over some subvarieties of

X.[6] The expectation is that the coefficients from the refined Swan conductors define the aforementioned subbundles which constitute the log-characteristic cycle. We also point out that the Euler characteristic is only sensitive to the multiplicities of these subbundles but not to how they are embedded in the log-tangent space of X.

The bad surprise is that the definition of log-characteristic cycles is a big mystery! On one hand, it seems that there has not been a successful theory of (log-)characteristic cycles for l-adic sheaves; on the other hand, even in the cases (b) and (c), where a standard theory of characteristic cycles is available (see [HTT] for case (c) and [Be] for case (b)), it is not entirely clear how to make an analogous logarithmic theory. Two major difficulties are the lack of appropriate log-holonomicity theorem for $\mathcal{F}$ (which may not even be finitely generated over $\mathcal{D}_X^{\log}$) and absence of Bernstein inequality. (We refer to [X12+] for more discussion on pathological examples.) In case (c), the first author [X12+] developed a theory tailored for the application to the GOS type formulas. He does not know how to make analogous construction in case (b).

Third, the Euler characteristic and the log-characteristic cycles are expected to be related. In the standard theory of algebraic D-modules and overconvergent F-isocrystals, the intersection number of the characteristic cycle with the zero section of the cotangent space gives the Euler characteristic of $\mathcal{F}$; this formula is known as the Kashiwara-Dubson formula. See [HTT] for the algebraic D-module case and [Be] for the overconvergent F-isocrystal case. One may hope to use a log-version of such a formula to deduce GOS type formulas by computing explicitly the log-characteristic cycles, at least in the case of F-isocrystals and algebraic D-modules. Unfortunately, this comes back to the bad surprise mentioned earlier: we do not have a satisfactory theory of log-characteristic cycles for general $\mathcal{F}$, except in case (c) where a GOS type formula is proved in [X12+] under a mild hypothesis.

10.4.2. **Remark.** A very important application for an appropriate definition of log-characteristic cycles for overconvergent F-isocrystals would be the following. Kedlaya develops a trick in [Ke11a, Section 5] that can "transfer" the ramification data of a lisse l-adic sheaf to a (virtual) overconvergent F-isocrystal. Then we would get a natural definition of log-characteristic cycles for lisse l-adic sheaves for free. To our knowledge, a general construction of the (log-)characteristic cycles is not known for lisse l-adic sheaves. (Under the cleanliness condition, Abbes and T. Saito [AS11] give a definition using the refined Swan conductors, but it is unclear how to remove the cleanliness hypothesis.)

10.5. **A global approach by Kato-Saito.** In the end, we briefly mention an approach of Kato and T. Saito, in which they interpret the ramification information of a lisse $\mathbb{Q}_l$-sheaf $\mathcal{F}$ as cycle classes supported on the boundary divisor D. The method is global and hence is different from the view point we took in previous subsections. We will only summarize the gist of the idea but refer to [KS08] for details.

Roughly speaking, one first chooses a $\mathbb{Z}_l$-lattice $\mathcal{F}_0$ of $\mathcal{F}$ and consider $\bar{\mathcal{F}} = \mathcal{F}_0/l\mathcal{F}_0$ instead. It turns out that the (wild) ramification information is completely contained in the reduction $\bar{\mathcal{F}}$. Then there exists a finite Galois étale cover V of U over which $\bar{\mathcal{F}}$ is trivial. Put $G = \mathrm{Gal}(V/U)$; the sheaf $\bar{\mathcal{F}}$ corresponds to an $\mathbb{F}_l$-representation $\rho_{\bar{\mathcal{F}}}$ of G. Suppose that V admits a compactification Y such that

[6] This is related to the fact that the Poisson structure on the log-cotangent space is degenerate.

$E = Y \backslash V$ is a divisor with simple normal crossings. Let f denote the natural morphism $f : Y\backslash V \to X \backslash U$. One can consider the intersection of the diagonal Y and the graph of $g \in G$ in certain log-product $Y \times_X Y$, viewed as a cycle $s_{V/U}(g)$ on $Y\backslash V$. One defines $s_{V/U}(\mathrm{id})$ so that $\sum_{g\in G} s_{V/U}(g) = 0$. Kato and T. Saito [KS08] then formally define the *Swan class* to be

$$\mathrm{Sw}(\mathcal{F}) := \sum_{g\in G} f_*(s_{V/U}(g))\mathrm{Tr}^{Br}(g; \rho_{\bar{\mathcal{F}}}) \in CH_0(D),$$

where Tr^{Br} is the Brauer trace. (Compare this to the definition of Swan character in Subsection 3.7.) The upshot of [KS08] is that, even if V does not admit a good compactification as above, one can use alteration to reproduce the construction (at the expense of passing to Chow group with rational coefficients). Moreover, the Swan class does not depend on the choice of the lattice $\mathcal{F}_0$. Essentially by construction, the degree of the Swan class measures the difference $\chi(U, \mathcal{F}) - \chi(U, \mathbb{Q}_l) \cdot \mathrm{rank}\mathcal{F}$.

10.5.1. **Question.** Can one prove an analogous result of Kato and T. Saito in the cases (b) and (c) of Subsection 10.1?

We also mention that Abbes and T. Saito construct certain cohomology classes for lisse l-adic sheaves (under a mild hypothesis) using purely cohomological method; they check that their construction is consistent with the work of Kato and T. Saito above. Recently, Kato and T. Saito [KS13] extended their work to varieties over $\mathbb{Q}_p$; in this case, the focus is no longer the Euler characteristic of $\mathcal{F}$, but the Swan conductor of the cohomology of $\mathcal{F}$ as a representation of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{alg}}/\mathbb{Q}_p)$.

10.5.2. **Question.** It would be interesting to know if one can reproduce some of the results in this subsection by working Zariski locally on X. Also, can we relate this to the local approaches we discussed earlier?

11. Miscellaneous questions

Here are some questions which are of interest for us but do not fit into other sections.

11.1. **Ramification numbers and structure of Galois groups.** There exists a number of results relating the structure of Galois groups with the possible values of ramification invariants. Hasse-Arf theorem gives an example; another example is the following Hyodo inequality ([Hy, Lemma (4-1)] or, without class field theory, [Z95, §1]).

11.1.1. **Proposition.** *Let M/K be a cyclic extension p^2, L the intermediate subfield. Then*

$$d_K(M/L) \geq \min\big((p-1+p^{-1})d_K(L/K), e_K - p^{-1}e_K + p^{-1}d_K(L/K)\big). \tag{18}$$

11.1.2. **Question.** Given a complete discrete valuation field K, a word $T = T_1 \dots T_n$ in the alphabet $\{W, F\}$ and an n-tuple of integers $(i_1, \dots, i_n)$, does there exist a cyclic extension L/K with genome T and lower breaks $(i_1, \dots, i_n)$?

The answer is known only in 2 cases.

(1) The classical case: we only give the reference [Mik81] for the mixed characteristic case. For equal characteristic case, a related work is [Th].

(2) Ferocious extensions of 2-dimensional fields ([We]).

In general, we cannot even answer the following question.

11.1.3. **Question.** Given a complete discrete valuation field K and a word $T = T_1 \dots T_n$ in the alphabet $\{W, F\}$, does there exist a cyclic extension L/K with genome T?

If $\operatorname{char} K = p$, the answer is expected to be positive for any T; however, it cannot be so if $\operatorname{char} K = 0$. Indeed, according to [Kur], in this case any complete discrete valuation field belongs to one of two types; the fields of type I (resp. of type II) do not have arbitrarily big cyclic ferocious (resp. wild) extensions. It would be interesting to try to answer Question 11.1.3 in terms of refinement of Kurihara's classification by Ivanova [I12a, I12b].

One more aspect of this topic is the following phenomenon in the mixed characteristic case: the assumption that the minimal ramification break of L/K takes its almost maximal value, namely, $h \geq \frac{pe_K}{p-1} - 1$, has strong implications for the whole ramification filtration; see [PVZ] for a number of results in this direction.

11.2. **Small ramification numbers and embedding problem.** In this subsection, we assume $\operatorname{char} K = 0$.

By a result of Miki [Mik74], if L/K is a cyclic extension of degree p, it can be embedded into a cyclic extension of degree p^n if and only if $L(\zeta_p) = K_1(x)$, where $x^p \in N_{K_n/K_1} K_n^*$, and K_n denotes $K(\zeta_{p^n})$. The following statement is an easy consequence ([VZ, §2]).

11.2.1. **Proposition.** *Let L/K is a cyclic extension of degree p with $d_K(L/K) < \frac{e_K}{p-1}$. Then L/K can be embedded into a cyclic extension of degree p^2.*

We are interested in generalization of this observation to any Galois groups.

11.2.2. **Question.** Let $f : G' \to G$ be an epimorphism of finite groups. Does there exist an $\varepsilon_f > 0$ such that, for any Galois extension L/K of mixed characteristic complete discrete valuation fields with $\operatorname{Gal}(L/K) \simeq G$ and $d_K(L/K) < \varepsilon_f e_K$, the embedding problem $(L/K, f)$ has a solution?

11.3. **Ramification and higher adèles.** It would be interesting to understand what kind of ramification data are needed in adelic theory of arithmetic surfaces. For example, the non-wild part of the conductor of the curve appears in [Fe10, Subsection 3.4]; can we allow wild ramification here?

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E-mail address: `lxiao@math.uchicago.edu, zhukov@math.spbu.ru`