7 ☐ TECHNIQUES OF INTEGRATION

7.1 Integration by Parts

1. Let $u = x$, $dv = e^{2x}\,dx \quad\Rightarrow\quad du = dx$, $v = \frac{1}{2}e^{2x}$. Then by Equation 2,

$\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C.$

2. Let $u = \ln x$, $dv = \sqrt{x}\,dx \quad\Rightarrow\quad du = \dfrac{1}{x}\,dx$, $v = \frac{2}{3}x^{3/2}$. Then by Equation 2,

$$\int \sqrt{x}\ln x\,dx = \tfrac{2}{3}x^{3/2}\ln x - \int \tfrac{2}{3}x^{3/2}\cdot\frac{1}{x}\,dx = \tfrac{2}{3}x^{3/2}\ln x - \int \tfrac{2}{3}x^{1/2}\,dx = \tfrac{2}{3}x^{3/2}\ln x - \tfrac{4}{9}x^{3/2} + C.$$

Note: A mnemonic device which is helpful for selecting u when using integration by parts is the LIATE principle of precedence for u:

Logarithmic
Inverse trigonometric
Algebraic
Trigonometric
Exponential

If the integrand has several factors, then we try to choose among them a u which appears as high as possible on the list. For example, in $\int xe^{2x}\,dx$ the integrand is xe^{2x}, which is the product of an algebraic function (x) and an exponential function (e^{2x}). Since Algebraic appears before Exponential, we choose $u = x$. Sometimes the integration turns out to be similar regardless of the selection of u and dv, but it is advisable to refer to LIATE when in doubt.

3. Let $u = x$, $dv = \cos 5x\,dx \quad\Rightarrow\quad du = dx$, $v = \frac{1}{5}\sin 5x$. Then by Equation 2,

$\int x\cos 5x\,dx = \frac{1}{5}x\sin 5x - \int \frac{1}{5}\sin 5x\,dx = \frac{1}{5}x\sin 5x + \frac{1}{25}\cos 5x + C.$

4. Let $u = y$, $dv = e^{0.2y}dy \quad\Rightarrow\quad du = dy$, $v = \frac{1}{0.2}e^{0.2y}$. Then by Equation 2,

$\int ye^{0.2y}dy = 5ye^{0.2y} - \int 5e^{0.2y}dy = 5ye^{0.2y} - 25e^{0.2y} + C.$

5. Let $u = t$, $dv = e^{-3t}dt \quad\Rightarrow\quad du = dt$, $v = -\frac{1}{3}e^{-3t}$. Then by Equation 2,

$\int te^{-3t}dt = -\frac{1}{3}te^{-3t} - \int -\frac{1}{3}e^{-3t}dt = -\frac{1}{3}te^{-3t} + \frac{1}{3}\int e^{-3t}dt = -\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t} + C.$

6. Let $u = x - 1$, $dv = \sin \pi x\,dx \quad\Rightarrow\quad du = dx$, $v = -\frac{1}{\pi}\cos \pi x$. Then by Equation 2,

$$\begin{aligned}\int (x-1)\sin \pi x\,dx &= -\frac{1}{\pi}(x-1)\cos \pi x - \int -\frac{1}{\pi}\cos \pi x\,dx = -\frac{1}{\pi}(x-1)\cos \pi x + \frac{1}{\pi}\int \cos \pi x\,dx \\ &= -\frac{1}{\pi}(x-1)\cos \pi x + \frac{1}{\pi^2}\sin \pi x + C\end{aligned}$$

7. First let $u = x^2 + 2x$, $dv = \cos x\,dx \quad\Rightarrow\quad du = (2x+2)\,dx$, $v = \sin x$. Then by Equation 2,

$I = \int (x^2+2x)\cos x\,dx = (x^2+2x)\sin x - \int (2x+2)\sin x\,dx$. Next let $U = 2x+2$, $dV = \sin x\,dx \quad\Rightarrow\quad dU = 2\,dx$, $V = -\cos x$, so $\int (2x+2)\sin x\,dx = -(2x+2)\cos x - \int -2\cos x\,dx = -(2x+2)\cos x + 2\sin x$. Thus,

$I = (x^2+2x)\sin x + (2x+2)\cos x - 2\sin x + C.$

8. First let $u = t^2$, $dv = \sin\beta t\,dt \;\Rightarrow\; du = 2t\,dt$, $v = -\frac{1}{\beta}\cos\beta t$. Then by Equation 2,

$I = \int t^2 \sin\beta t\,dt = -\frac{1}{\beta}t^2\cos\beta t - \int -\frac{2}{\beta}t\cos\beta t\,dt$. Next let $U = t$, $dV = \cos\beta t\,dt \;\Rightarrow\; dU = dt$,

$V = \dfrac{1}{\beta}\sin\beta t$, so $\int t\cos\beta t\,dt = \dfrac{1}{\beta}t\sin\beta t - \displaystyle\int \frac{1}{\beta}\sin\beta t\,dt = \frac{1}{\beta}t\sin\beta t + \frac{1}{\beta^2}\cos\beta t$. Thus,

$$I = -\frac{1}{\beta}t^2\cos\beta t + \frac{2}{\beta}\left(\frac{1}{\beta}t\sin\beta t + \frac{1}{\beta^2}\cos\beta t\right) + C = -\frac{1}{\beta}t^2\cos\beta t + \frac{2}{\beta^2}t\sin\beta t + \frac{2}{\beta^3}\cos\beta t + C.$$

9. Let $u = \cos^{-1}x$, $dv = dx \;\Rightarrow\; du = \dfrac{-1}{\sqrt{1-x^2}}\,dx$, $v = x$. Then by Equation 2,

$$\int \cos^{-1}x\,dx = x\cos^{-1}x - \int \frac{-x}{\sqrt{1-x^2}}\,dx = x\cos^{-1}x - \int \frac{1}{\sqrt{t}}\left(\frac{1}{2}\,dt\right) \qquad \begin{bmatrix} t = 1-x^2, \\ dt = -2x\,dx \end{bmatrix}$$
$$= x\cos^{-1}x - \tfrac{1}{2}\cdot 2t^{1/2} + C = x\cos^{-1}x - \sqrt{1-x^2} + C$$

10. Let $u = \ln\sqrt{x}$, $dv = dx \;\Rightarrow\; du = \dfrac{1}{\sqrt{x}}\cdot\dfrac{1}{2\sqrt{x}}\,dx = \dfrac{1}{2x}\,dx$, $v = x$. Then by Equation 2,

$$\int \ln\sqrt{x}\,dx = x\ln\sqrt{x} - \int x\cdot\frac{1}{2x}\,dx = x\ln\sqrt{x} - \int \frac{1}{2}\,dx = x\ln\sqrt{x} - \frac{1}{2}x + C.$$

Note: We could start by using $\ln\sqrt{x} = \frac{1}{2}\ln x$.

11. Let $u = \ln t$, $dv = t^4\,dt \;\Rightarrow\; du = \dfrac{1}{t}\,dt$, $v = \dfrac{1}{5}t^5$. Then by Equation 2,

$$\int t^4\ln t\,dt = \frac{1}{5}t^5\ln t - \int \frac{1}{5}t^5\cdot\frac{1}{t}\,dt = \frac{1}{5}t^5\ln t - \int \frac{1}{5}t^4\,dt = \frac{1}{5}t^5\ln t - \frac{1}{25}t^5 + C.$$

12. Let $u = \tan^{-1}2y$, $dv = dy \;\Rightarrow\; du = \dfrac{2}{1+4y^2}\,dy$, $v = y$. Then by Equation 2,

$$\int \tan^{-1}2y\,dy = y\tan^{-1}2y - \int \frac{2y}{1+4y^2}\,dy = y\tan^{-1}2y - \int \frac{1}{t}\left(\frac{1}{4}\,dt\right) \qquad \begin{bmatrix} t = 1+4y^2, \\ dt = 8y\,dy \end{bmatrix}$$
$$= y\tan^{-1}2y - \tfrac{1}{4}\ln|t| + C = y\tan^{-1}2y - \tfrac{1}{4}\ln(1+4y^2) + C$$

13. Let $u = t$, $dv = \csc^2 t\,dt \;\Rightarrow\; du = dt$, $v = -\cot t$. Then by Equation 2,

$$\int t\csc^2 t\,dt = -t\cot t - \int -\cot t\,dt = -t\cot t + \int \frac{\cos t}{\sin t}\,dt = -t\cot t + \int \frac{1}{z}\,dz \qquad \begin{bmatrix} z = \sin t, \\ dz = \cos t\,dt \end{bmatrix}$$
$$= -t\cot t + \ln|z| + C = -t\cot t + \ln|\sin t| + C$$

14. Let $u = x$, $dv = \cosh ax\,dx \;\Rightarrow\; du = dx$, $v = \dfrac{1}{a}\sinh ax$. Then by Equation 2,

$$\int x\cosh ax\,dx = \frac{1}{a}x\sinh ax - \int \frac{1}{a}\sinh ax\,dx = \frac{1}{a}x\sinh ax - \frac{1}{a^2}\cosh ax + C.$$

15. First let $u = (\ln x)^2$, $dv = dx \;\Rightarrow\; du = 2\ln x\cdot\frac{1}{x}\,dx$, $v = x$. Then by Equation 2,

$I = \int (\ln x)^2\,dx = x(\ln x)^2 - 2\int x\ln x\cdot\frac{1}{x}\,dx = x(\ln x)^2 - 2\int \ln x\,dx$. Next let $U = \ln x$, $dV = dx \;\Rightarrow$ $dU = 1/x\,dx$, $V = x$ to get $\int \ln x\,dx = x\ln x - \int x\cdot(1/x)\,dx = x\ln x - \int dx = x\ln x - x + C_1$. Thus,

$I = x(\ln x)^2 - 2(x\ln x - x + C_1) = x(\ln x)^2 - 2x\ln x + 2x + C$, where $C = -2C_1$.

16. $\int \frac{z}{10^z}\,dz = \int z\,10^{-z}\,dz$. Let $u = z$, $dv = 10^{-z}\,dz \quad\Rightarrow\quad du = dz$, $v = \frac{-10^{-z}}{\ln 10}$. Then by Equation 2,

$$\int z\,10^{-z}\,dz = \frac{-z\,10^{-z}}{\ln 10} - \int \frac{-10^{-z}}{\ln 10}\,dz = \frac{-z}{10^z \ln 10} - \frac{10^{-z}}{(\ln 10)(\ln 10)} + C = -\frac{z}{10^z \ln 10} - \frac{1}{10^z(\ln 10)^2} + C.$$

17. First let $u = \sin 3\theta$, $dv = e^{2\theta}\,d\theta \quad\Rightarrow\quad du = 3\cos 3\theta\,d\theta$, $v = \frac{1}{2}e^{2\theta}$. Then

$I = \int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{2}\int e^{2\theta}\cos 3\theta\,d\theta$. Next let $U = \cos 3\theta$, $dV = e^{2\theta}\,d\theta \quad\Rightarrow\quad dU = -3\sin 3\theta\,d\theta$,

$V = \frac{1}{2}e^{2\theta}$ to get $\int e^{2\theta}\cos 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\cos 3\theta + \frac{3}{2}\int e^{2\theta}\sin 3\theta\,d\theta$. Substituting in the previous formula gives

$I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}\int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}I \quad\Rightarrow$

$\frac{13}{4}I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta + C_1$. Hence, $I = \frac{1}{13}e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$, where $C = \frac{4}{13}C_1$.

18. First let $u = e^{-\theta}$, $dv = \cos 2\theta\,d\theta \quad\Rightarrow\quad du = -e^{-\theta}\,d\theta$, $v = \frac{1}{2}\sin 2\theta$. Then

$I = \int e^{-\theta}\cos 2\theta\,d\theta = \frac{1}{2}e^{-\theta}\sin 2\theta - \int \frac{1}{2}\sin 2\theta\left(-e^{-\theta}\,d\theta\right) = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\int e^{-\theta}\sin 2\theta\,d\theta$.

Next let $U = e^{-\theta}$, $dV = \sin 2\theta\,d\theta \quad\Rightarrow\quad dU = -e^{-\theta}\,d\theta$, $V = -\frac{1}{2}\cos 2\theta$, so

$\int e^{-\theta}\sin 2\theta\,d\theta = -\frac{1}{2}e^{-\theta}\cos 2\theta - \int\left(-\frac{1}{2}\right)\cos 2\theta\left(-e^{-\theta}\,d\theta\right) = -\frac{1}{2}e^{-\theta}\cos 2\theta - \frac{1}{2}\int e^{-\theta}\cos 2\theta\,d\theta$.

So $I = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\left[\left(-\frac{1}{2}e^{-\theta}\cos 2\theta\right) - \frac{1}{2}I\right] = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta - \frac{1}{4}I \quad\Rightarrow$

$\frac{5}{4}I = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1 \quad\Rightarrow\quad I = \frac{4}{5}\left(\frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1\right) = \frac{2}{5}e^{-\theta}\sin 2\theta - \frac{1}{5}e^{-\theta}\cos 2\theta + C$.

19. First let $u = z^3$, $dv = e^z dz \quad\Rightarrow\quad du = 3z^2 dz$, $v = e^z$. Then $I_1 = \int z^3 e^z dz = z^3 e^z - 3\int z^2 e^z dz$. Next let $u_1 = z^2$,

$dv_1 = e^z dz \quad\Rightarrow\quad du_1 = 2z\,dz$, $v_1 = e^z$. Then $I_2 = z^2 e^z - 2\int z e^z dz$. Finally, let $u_2 = z$, $dv_2 = e^z dz \quad\Rightarrow\quad du_2 = dz$,

$v_2 = e^z$. Then $\int z e^z dz = z e^z - \int e^z dz = z e^z - e^z + C_1$. Substituting in the expression for I_2, we get

$I_2 = z^2 e^z - 2(z e^z - e^z + C_1) = z^2 e^z - 2z e^z + 2e^z - 2C_1$. Substituting the last expression for I_2 into I_1 gives

$I_1 = z^3 e^z - 3(z^2 e^z - 2z e^z + 2e^z - 2C_1) = z^3 e^z - 3z^2 e^z + 6z e^z - 6e^z + C$, where $C = 6C_1$.

20. $\int x\tan^2 x\,dx = \int x(\sec^2 x - 1)\,dx = \int x\sec^2 x\,dx - \int x\,dx$. Let $u = x$, $dv = \sec^2 x\,dx \quad\Rightarrow\quad du = dx$, $v = \tan x$.

Then by Equation 2, $\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x - \ln|\sec x|$, and thus,

$\int x\tan^2 x\,dx = x\tan x - \ln|\sec x| - \frac{1}{2}x^2 + C$.

21. Let $u = xe^{2x}$, $dv = \frac{1}{(1+2x)^2}\,dx \quad\Rightarrow\quad du = (x\cdot 2e^{2x} + e^{2x}\cdot 1)\,dx = e^{2x}(2x+1)\,dx$, $v = -\frac{1}{2(1+2x)}$.

Then by Equation 2,

$$\int \frac{xe^{2x}}{(1+2x)^2}\,dx = -\frac{xe^{2x}}{2(1+2x)} + \frac{1}{2}\int \frac{e^{2x}(2x+1)}{1+2x}\,dx = -\frac{xe^{2x}}{2(1+2x)} + \frac{1}{2}\int e^{2x}dx = -\frac{xe^{2x}}{2(1+2x)} + \frac{1}{4}e^{2x} + C.$$

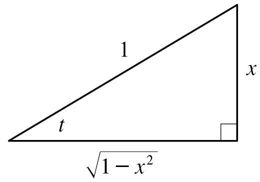
The answer could be written as $\frac{e^{2x}}{4(2x+1)} + C$.

22. First let $u = (\arcsin x)^2$, $dv = dx \quad\Rightarrow\quad du = 2\arcsin x\cdot\frac{1}{\sqrt{1-x^2}}\,dx$, $v = x$. Then

$I = \int (\arcsin x)^2 dx = x(\arcsin x)^2 - 2\int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx$. To simplify the last integral, let $t = \arcsin x$ $[x = \sin t]$, so

$dt = \dfrac{1}{\sqrt{1-x^2}}\,dx$, and $\displaystyle\int \frac{x \arcsin x}{\sqrt{1-x^2}}\,dx = \int t \sin t\,dt$. To evaluate just the last integral, now let $U = t$, $dV = \sin t\,dt \Rightarrow$ $dU = dt$, $V = -\cos t$. Thus,

$$\int t \sin t\,dt = -t\cos t + \int \cos t\,dt = -t\cos t + \sin t + C$$
$$= -\arcsin x \cdot \frac{\sqrt{1-x^2}}{1} + x + C_1 \quad \text{[refer to the figure]}$$



Returning to I, we get $I = x(\arcsin x)^2 + 2\sqrt{1-x^2}\,\arcsin x - 2x + C$, where $C = -2C_1$.

23. Let $u = x$, $dv = \cos \pi x\,dx \Rightarrow du = dx$, $v = \frac{1}{\pi}\sin \pi x$. By (6),

$$\int_0^{1/2} x\cos \pi x\,dx = \left[\frac{1}{\pi}x \sin \pi x\right]_0^{1/2} - \int_0^{1/2} \frac{1}{\pi}\sin \pi x\,dx = \frac{1}{2\pi} - 0 - \frac{1}{\pi}\left[-\frac{1}{\pi}\cos \pi x\right]_0^{1/2}$$
$$= \frac{1}{2\pi} + \frac{1}{\pi^2}(0-1) = \frac{1}{2\pi} - \frac{1}{\pi^2} \text{ or } \frac{\pi-2}{2\pi^2}$$

24. First let $u = x^2+1$, $dv = e^{-x}\,dx \Rightarrow du = 2x\,dx$, $v = -e^{-x}$. By (6),

$\int_0^1 (x^2+1)e^{-x}\,dx = \left[-(x^2+1)e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}\,dx = -2e^{-1} + 1 + 2\int_0^1 xe^{-x}\,dx$.

Next let $U = x$, $dV = e^{-x}\,dx \Rightarrow dU = dx$, $V = -e^{-x}$. By (6) again,

$\int_0^1 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx = -e^{-1} + \left[-e^{-x}\right]_0^1 = -e^{-1} - e^{-1} + 1 = -2e^{-1} + 1$. So

$\int_0^1 (x^2+1)e^{-x}\,dx = -2e^{-1} + 1 + 2(-2e^{-1}+1) = -2e^{-1} + 1 - 4e^{-1} + 2 = -6e^{-1} + 3$.

25. Let $u = y$, $dv = \sinh y\,dy \Rightarrow du = dy$, $v = \cosh y$. By (6),

$\int_0^2 y \sinh y\,dy = \Big[y\cosh y\Big]_0^2 - \int_0^2 \cosh y\,dy = 2\cosh 2 - 0 - \Big[\sinh y\Big]_0^2 = 2\cosh 2 - \sinh 2$.

26. Let $u = \ln w$, $dv = w^2\,dw \Rightarrow du = \dfrac{1}{w}\,dw$, $v = \frac{1}{3}w^3$. By (6),

$\int_1^2 w^2 \ln w\,dw = \left[\frac{1}{3}w^3 \ln w\right]_1^2 - \int_1^2 \frac{1}{3}w^2\,dw = \frac{8}{3}\ln 2 - 0 - \left[\frac{1}{9}w^3\right]_1^2 = \frac{8}{3}\ln 2 - \left(\frac{8}{9} - \frac{1}{9}\right) = \frac{8}{3}\ln 2 - \frac{7}{9}$.

27. Let $u = \ln R$, $dv = \dfrac{1}{R^2}\,dR \Rightarrow du = \dfrac{1}{R}\,dR$, $v = -\dfrac{1}{R}$. By (6),

$$\int_1^5 \frac{\ln R}{R^2}\,dR = \left[-\frac{1}{R}\ln R\right]_1^5 - \int_1^5 -\frac{1}{R^2}\,dR = -\tfrac{1}{5}\ln 5 - 0 - \left[\frac{1}{R}\right]_1^5 = -\tfrac{1}{5}\ln 5 - \left(\tfrac{1}{5} - 1\right) = \tfrac{4}{5} - \tfrac{1}{5}\ln 5.$$

28. First let $u = t^2$, $dv = \sin 2t\,dt \Rightarrow du = 2t\,dt$, $v = -\frac{1}{2}\cos 2t$. By (6),

$\int_0^{2\pi} t^2 \sin 2t\,dt = \left[-\frac{1}{2}t^2 \cos 2t\right]_0^{2\pi} + \int_0^{2\pi} t\cos 2t\,dt = -2\pi^2 + \int_0^{2\pi} t\cos 2t\,dt$. Next let $U = t$, $dV = \cos 2t\,dt \Rightarrow$ $dU = dt$, $V = \frac{1}{2}\sin 2t$. By (6) again,

$\int_0^{2\pi} t\cos 2t\,dt = \left[\frac{1}{2}t\sin 2t\right]_0^{2\pi} - \int_0^{2\pi} \frac{1}{2}\sin 2t\,dt = 0 - \left[-\frac{1}{4}\cos 2t\right]_0^{2\pi} = \frac{1}{4} - \frac{1}{4} = 0$. Thus, $\int_0^{2\pi} t^2 \sin 2t\,dt = -2\pi^2$.

29. $\sin 2x = 2\sin x\,\cos x$, so $\int_0^{\pi} x \sin x\,\cos x\,dx = \frac{1}{2}\int_0^{\pi} x\,\sin 2x\,dx$. Let $u = x$, $dv = \sin 2x\,dx \Rightarrow du = dx$, $v = -\frac{1}{2}\cos 2x$. By (6), $\frac{1}{2}\int_0^{\pi} x\,\sin 2x\,dx = \frac{1}{2}\left[-\frac{1}{2}x\cos 2x\right]_0^{\pi} - \frac{1}{2}\int_0^{\pi} -\frac{1}{2}\cos 2x\,dx = -\frac{1}{4}\pi - 0 + \frac{1}{4}\left[\frac{1}{2}\sin 2x\right]_0^{\pi} = -\dfrac{\pi}{4}$.

30. Let $u = \arctan(1/x)$, $dv = dx \quad \Rightarrow \quad du = \dfrac{1}{1+(1/x)^2} \cdot \dfrac{-1}{x^2}\,dx = \dfrac{-dx}{x^2+1}$, $v = x$. By (6),

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) dx = \left[x \arctan\left(\frac{1}{x}\right)\right]_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x\,dx}{x^2+1} = \sqrt{3}\,\frac{\pi}{6} - 1 \cdot \frac{\pi}{4} + \frac{1}{2}\Big[\ln(x^2+1)\Big]_1^{\sqrt{3}}$$
$$= \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}(\ln 4 - \ln 2) = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}\ln\frac{4}{2} = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}\ln 2$$

31. Let $u = M$, $dv = e^{-M}\,dM \quad \Rightarrow \quad du = dM$, $v = -e^{-M}$. By (6),

$$\int_1^5 \frac{M}{e^M}\,dM = \int_1^5 Me^{-M}\,dM = \Big[-Me^{-M}\Big]_1^5 - \int_1^5 -e^{-M}\,dM = -5e^{-5} + e^{-1} - \Big[e^{-M}\Big]_1^5$$
$$= -5e^{-5} + e^{-1} - (e^{-5} - e^{-1}) = 2e^{-1} - 6e^{-5}$$

32. Let $u = (\ln x)^2$, $dv = x^{-3}\,dx \quad \Rightarrow \quad du = \dfrac{2\ln x}{x}\,dx$, $v = -\tfrac{1}{2}x^{-2}$. By (6),

$I = \displaystyle\int_1^2 \frac{(\ln x)^2}{x^3}\,dx = \left[-\frac{(\ln x)^2}{2x^2}\right]_1^2 + \int_1^2 \frac{\ln x}{x^3}\,dx$. Now let $U = \ln x$, $dV = x^{-3}\,dx \quad \Rightarrow \quad dU = \dfrac{1}{x}\,dx$, $V = -\tfrac{1}{2}x^{-2}$.

Then

$$\int_1^2 \frac{\ln x}{x^3}\,dx = \left[-\frac{\ln x}{2x^2}\right]_1^2 + \tfrac{1}{2}\int_1^2 x^{-3}dx = -\tfrac{1}{8}\ln 2 + 0 + \tfrac{1}{2}\left[-\tfrac{1}{2x^2}\right]_1^2 = -\tfrac{1}{8}\ln 2 + \tfrac{1}{2}\left(-\tfrac{1}{8} + \tfrac{1}{2}\right) = \tfrac{3}{16} - \tfrac{1}{8}\ln 2.$$

Thus $I = \left(-\tfrac{1}{8}(\ln 2)^2 + 0\right) + \left(\tfrac{3}{16} - \tfrac{1}{8}\ln 2\right) = -\tfrac{1}{8}(\ln 2)^2 - \tfrac{1}{8}\ln 2 + \tfrac{3}{16}$.

33. Let $u = \ln(\cos x)$, $dv = \sin x\,dx \quad \Rightarrow \quad du = \dfrac{1}{\cos x}(-\sin x)\,dx$, $v = -\cos x$. By (6),

$$\textstyle\int_0^{\pi/3} \sin x\,\ln(\cos x)\,dx = \Big[-\cos x\,\ln(\cos x)\Big]_0^{\pi/3} - \int_0^{\pi/3} \sin x\,dx = -\tfrac{1}{2}\ln\tfrac{1}{2} - 0 - \Big[-\cos x\Big]_0^{\pi/3}$$
$$= -\tfrac{1}{2}\ln\tfrac{1}{2} + \left(\tfrac{1}{2} - 1\right) = \tfrac{1}{2}\ln 2 - \tfrac{1}{2}$$

34. Let $u = r^2$, $dv = \dfrac{r}{\sqrt{4+r^2}}\,dr \quad \Rightarrow \quad du = 2r\,dr$, $v = \sqrt{4+r^2}$. By (6),

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr = \Big[r^2\sqrt{4+r^2}\Big]_0^1 - 2\int_0^1 r\sqrt{4+r^2}\,dr = \sqrt{5} - \tfrac{2}{3}\Big[(4+r^2)^{3/2}\Big]_0^1$$
$$= \sqrt{5} - \tfrac{2}{3}(5)^{3/2} + \tfrac{2}{3}(8) = \sqrt{5}\left(1 - \tfrac{10}{3}\right) + \tfrac{16}{3} = \tfrac{16}{3} - \tfrac{7}{3}\sqrt{5}$$

35. Let $u = (\ln x)^2$, $dv = x^4\,dx \quad \Rightarrow \quad du = 2\,\dfrac{\ln x}{x}\,dx$, $v = \dfrac{x^5}{5}$. By (6),

$$\int_1^2 x^4(\ln x)^2\,dx = \left[\frac{x^5}{5}(\ln x)^2\right]_1^2 - 2\int_1^2 \frac{x^4}{5}\ln x\,dx = \tfrac{32}{5}(\ln 2)^2 - 0 - 2\int_1^2 \frac{x^4}{5}\ln x\,dx.$$

Let $U = \ln x$, $dV = \dfrac{x^4}{5}\,dx \quad \Rightarrow \quad dU = \dfrac{1}{x}\,dx$, $V = \dfrac{x^5}{25}$.

Then $\displaystyle\int_1^2 \frac{x^4}{5}\ln x\,dx = \left[\frac{x^5}{25}\ln x\right]_1^2 - \int_1^2 \frac{x^4}{25}\,dx = \tfrac{32}{25}\ln 2 - 0 - \left[\frac{x^5}{125}\right]_1^2 = \tfrac{32}{25}\ln 2 - \left(\tfrac{32}{125} - \tfrac{1}{125}\right)$.

So $\int_1^2 x^4(\ln x)^2\,dx = \tfrac{32}{5}(\ln 2)^2 - 2\left(\tfrac{32}{25}\ln 2 - \tfrac{31}{125}\right) = \tfrac{32}{5}(\ln 2)^2 - \tfrac{64}{25}\ln 2 + \tfrac{62}{125}$.

36. Let $u = \sin(t-s)$, $dv = e^s\,ds \;\Rightarrow\; du = -\cos(t-s)\,ds$, $v = e^s$. Then

$I = \int_0^t e^s \sin(t-s)\,ds = \Big[e^s \sin(t-s)\Big]_0^t + \int_0^t e^s \cos(t-s)\,ds = e^t \sin 0 - e^0 \sin t + I_1$. For I_1, let $U = \cos(t-s)$,

$dV = e^s\,ds \;\Rightarrow\; dU = \sin(t-s)\,ds$, $V = e^s$. So $I_1 = \big[e^s \cos(t-s)\big]_0^t - \int_0^t e^s \sin(t-s)\,ds = e^t \cos 0 - e^0 \cos t - I$.

Thus, $I = -\sin t + e^t - \cos t - I \;\Rightarrow\; 2I = e^t - \cos t - \sin t \;\Rightarrow\; I = \frac{1}{2}(e^t - \cos t - \sin t)$.

37. Let $t = \sqrt{x}$, so that $t^2 = x$ and $2t\,dt = dx$. Thus, $\int e^{\sqrt{x}}\,dx = \int e^t(2t)\,dt$. Now use parts with $u = t$, $dv = e^t\,dt$, $du = dt$, and $v = e^t$ to get $2\int te^t\,dt = 2t\,e^t - 2\int e^t\,dt = 2t\,e^t - 2e^t + C = 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$.

38. Let $t = \ln x$, so that $e^t = x$ and $e^t\,dt = dx$. Thus, $\int \cos(\ln x)\,dx = \int \cos t \cdot e^t\,dt = I$. Now use parts with $u = \cos t$, $dv = e^t\,dt$, $du = -\sin t\,dt$, and $v = e^t$ to get $\int e^t \cos t\,dt = e^t \cos t - \int -e^t \sin t\,dt = e^t \cos t + \int e^t \sin t\,dt$. Now use parts with $U = \sin t$, $dV = e^t\,dt$, $dU = \cos t\,dt$, and $V = e^t$ to get

$\int e^t \sin t\,dt = e^t \sin t - \int e^t \cos t\,dt$. Thus, $I = e^t \cos t + e^t \sin t - I \;\Rightarrow\; 2I = e^t \cos t + e^t \sin t \;\Rightarrow$

$I = \frac{1}{2}e^t \cos t + \frac{1}{2}e^t \sin t + C = \frac{1}{2}x\cos(\ln x) + \frac{1}{2}x\sin(\ln x) + C$.

39. Let $x = \theta^2$, so that $dx = 2\theta\,d\theta$. Thus, $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\;d\theta = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^2 \cos(\theta^2) \cdot \tfrac{1}{2}(2\theta\,d\theta) = \tfrac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx$. Now use parts with $u = x$, $dv = \cos x\,dx$, $du = dx$, $v = \sin x$ to get

$$\begin{aligned}\tfrac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx &= \tfrac{1}{2}\left(\big[x\sin x\big]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \sin x\,dx\right) = \tfrac{1}{2}\,\big[x\sin x + \cos x\big]_{\pi/2}^{\pi} \\ &= \tfrac{1}{2}(\pi\sin\pi + \cos\pi) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\sin\tfrac{\pi}{2} + \cos\tfrac{\pi}{2}\right) = \tfrac{1}{2}(\pi\cdot 0 - 1) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\cdot 1 + 0\right) = -\tfrac{1}{2} - \tfrac{\pi}{4}\end{aligned}$$

40. Let $x = \cos t$, so that $dx = -\sin t\,dt$. Thus,

$\int_0^{\pi} e^{\cos t} \sin 2t\,dt = \int_0^{\pi} e^{\cos t}(2\sin t\,\cos t)\,dt = \int_1^{-1} e^x \cdot 2x\,(-dx) = 2\int_{-1}^{1} xe^x\,dx$. Now use parts with $u = x$, $dv = e^x\,dx$, $du = dx$, $v = e^x$ to get

$2\int_{-1}^{1} xe^x\,dx = 2\left(\big[xe^x\big]_{-1}^{1} - \int_{-1}^{1} e^x\,dx\right) = 2\left(e^1 + e^{-1} - \big[e^x\big]_{-1}^{1}\right) = 2(e + e^{-1} - [e^1 - e^{-1}]) = 2(2e^{-1}) = 4/e$.

41. Let $y = 1 + x$, so that $dy = dx$. Thus, $\int x\ln(1+x)\,dx = \int (y-1)\ln y\,dy$. Now use parts with $u = \ln y$, $dv = (y-1)\,dy$, $du = \frac{1}{y}\,dy$, $v = \frac{1}{2}y^2 - y$ to get

$$\begin{aligned}\int (y-1)\ln y\,dy &= \left(\tfrac{1}{2}y^2 - y\right)\ln y - \int\left(\tfrac{1}{2}y - 1\right)dy = \tfrac{1}{2}y(y-2)\ln y - \tfrac{1}{4}y^2 + y + C \\ &= \tfrac{1}{2}(1+x)(x-1)\ln(1+x) - \tfrac{1}{4}(1+x)^2 + 1 + x + C,\end{aligned}$$

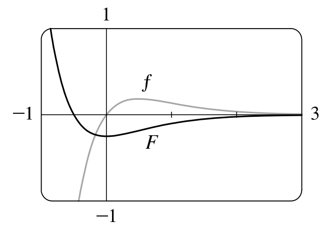
which can be written as $\frac{1}{2}(x^2-1)\ln(1+x) - \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{4} + C$.

42. Let $y = \ln x$, so that $dy = \dfrac{1}{x}\,dx$. Thus, $\displaystyle\int \frac{\arcsin(\ln x)}{x}\,dx = \int \arcsin y\,dy$. Now use parts with $u = \arcsin y$, $dv = dy$, $du = \dfrac{1}{\sqrt{1-y^2}}\,dy$, and $v = y$ to get

$$\int \arcsin y\,dy = y\arcsin y - \int \frac{y}{\sqrt{1-y^2}}\,dy = y\arcsin y + \sqrt{1-y^2} + C = (\ln x)\arcsin(\ln x) + \sqrt{1-(\ln x)^2} + C.$$

43. Let $u = x$, $dv = e^{-2x}\,dx \quad\Rightarrow\quad du = dx$, $v = -\frac{1}{2}e^{-2x}$. Then

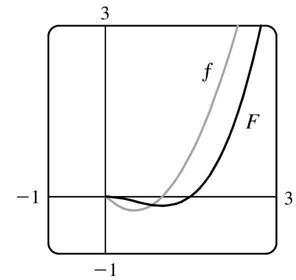
$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$. We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive. Also, F increases where f is positive and F decreases where f is negative.



44. Let $u = \ln x$, $dv = x^{3/2}\,dx \quad\Rightarrow\quad du = \frac{1}{x}\,dx$, $v = \frac{2}{5}x^{5/2}$. Then

$$\int x^{3/2}\ln x\,dx = \tfrac{2}{5}x^{5/2}\ln x - \tfrac{2}{5}\int x^{3/2}\,dx = \tfrac{2}{5}x^{5/2}\ln x - \left(\tfrac{2}{5}\right)^2 x^{5/2} + C$$
$$= \tfrac{2}{5}x^{5/2}\ln x - \tfrac{4}{25}x^{5/2} + C$$

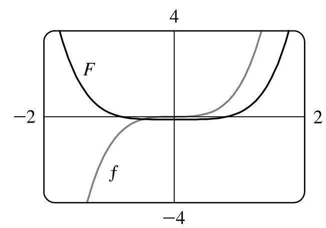
We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



45. Let $u = \frac{1}{2}x^2$, $dv = 2x\sqrt{1+x^2}\,dx \quad\Rightarrow\quad du = x\,dx$, $v = \frac{2}{3}(1+x^2)^{3/2}$.

Then

$$\int x^3\sqrt{1+x^2}\,dx = \tfrac{1}{2}x^2\left[\tfrac{2}{3}(1+x^2)^{3/2}\right] - \tfrac{2}{3}\int x(1+x^2)^{3/2}dx$$
$$= \tfrac{1}{3}x^2(1+x^2)^{3/2} - \tfrac{2}{3}\cdot\tfrac{2}{5}\cdot\tfrac{1}{2}(1+x^2)^{5/2} + C$$
$$= \tfrac{1}{3}x^2(1+x^2)^{3/2} - \tfrac{2}{15}(1+x^2)^{5/2} + C$$



We see from the graph that this is reasonable, since F increases where f is positive and F decreases where f is negative. Note also that f is an odd function and F is an even function.

Another method: Use substitution with $u = 1 + x^2$ to get $\frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C$.

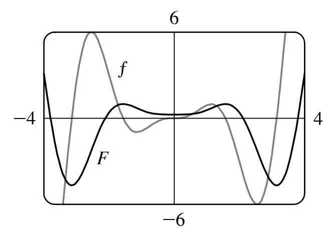
46. First let $u = x^2$, $dv = \sin 2x\,dx \quad\Rightarrow\quad du = 2x\,dx$, $v = -\frac{1}{2}\cos 2x$.

Then $I = \int x^2\sin 2x\,dx = -\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx$.

Next let $U = x$, $dV = \cos 2x\,dx \quad\Rightarrow\quad dU = dx$, $V = \frac{1}{2}\sin 2x$, so

$\int x\cos 2x\,dx = \frac{1}{2}x\sin 2x - \int\frac{1}{2}\sin 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$.

Thus, $I = -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$.



We see from the graph that this is reasonable, since F increases where f is positive and F decreases where f is negative. Note also that f is an odd function and F is an even function.

47. (a) Take $n = 2$ in Example 6 to get $\displaystyle\int \sin^2 x\,dx = -\frac{1}{2}\cos x\sin x + \frac{1}{2}\int 1\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$.

(b) $\int \sin^4 x\,dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{4}\int\sin^2 x\,dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{8}x - \frac{3}{16}\sin 2x + C$.

48. (a) Let $u = \cos^{n-1}x$, $dv = \cos x\,dx \quad\Rightarrow\quad du = -(n-1)\cos^{n-2}x\,\sin x\,dx$, $v = \sin x$ in (2):

$$\int\cos^n x\,dx = \cos^{n-1}x\,\sin x + (n-1)\int\cos^{n-2}x\,\sin^2 x\,dx$$
$$= \cos^{n-1}x\,\sin x + (n-1)\int\cos^{n-2}x\,(1-\cos^2 x)dx$$
$$= \cos^{n-1}x\,\sin x + (n-1)\int\cos^{n-2}x\,dx - (n-1)\int\cos^n x\,dx$$

[continued]

Rearranging terms gives $n\int \cos^n x\,dx = \cos^{n-1} x\,\sin x + (n-1)\int \cos^{n-2} x\,dx$ or

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\,\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

(b) Take $n = 2$ in part (a) to get $\int \cos^2 x\,dx = \frac{1}{2}\cos x\,\sin x + \frac{1}{2}\int 1\,dx = \dfrac{x}{2} + \dfrac{\sin 2x}{4} + C.$

(c) $\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x\,\sin x + \frac{3}{4}\int \cos^2 x\,dx = \frac{1}{4}\cos^3 x\,\sin x + \frac{3}{8}x + \frac{3}{16}\sin 2x + C$

49. (a) From Example 6, $\displaystyle\int \sin^n x\,dx = -\frac{1}{n}\cos x\,\sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$. Using (6),

$$\int_0^{\pi/2} \sin^n x\,dx = \left[-\frac{\cos x\,\sin^{n-1} x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$
$$= (0-0) + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

(b) Using $n = 3$ in part (a), we have $\int_0^{\pi/2} \sin^3 x\,dx = \frac{2}{3}\int_0^{\pi/2} \sin x\,dx = \left[-\frac{2}{3}\cos x\right]_0^{\pi/2} = \frac{2}{3}$.

Using $n = 5$ in part (a), we have $\int_0^{\pi/2} \sin^5 x\,dx = \frac{4}{5}\int_0^{\pi/2} \sin^3 x\,dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$.

(c) The formula holds for $n = 1$ (that is, $2n + 1 = 3$) by (b). Assume it holds for some $k \geq 1$. Then

$\displaystyle\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)}$. By Example 6,

$$\int_0^{\pi/2} \sin^{2k+3} x\,dx = \frac{2k+2}{2k+3}\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2k+2}{2k+3}\cdot\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)}$$
$$= \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)[2\,(k+1)]}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)[2\,(k+1)+1]},$$

so the formula holds for $n = k + 1$. By induction, the formula holds for all $n \geq 1$.

50. Using Exercise 49(a), we see that the formula holds for $n = 1$, because $\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2} 1\,dx = \frac{1}{2}\left[x\right]_0^{\pi/2} = \frac{1}{2}\cdot\frac{\pi}{2}$.

Now assume it holds for some $k \geq 1$. Then $\displaystyle\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}\frac{\pi}{2}$. By Exercise 49(a),

$$\int_0^{\pi/2} \sin^{2(k+1)} x\,dx = \frac{2k+1}{2k+2}\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{2k+1}{2k+2}\cdot\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}\frac{\pi}{2}$$
$$= \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)(2k+1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)(2k+2)}\cdot\frac{\pi}{2},$$

so the formula holds for $n = k + 1$. By induction, the formula holds for all $n \geq 1$.

51. Let $u = (\ln x)^n$, $dv = dx \quad\Rightarrow\quad du = n(\ln x)^{n-1}(dx/x)$, $v = x$. By Equation 2,

$\int (\ln x)^n\,dx = x(\ln x)^n - \int nx(\ln x)^{n-1}(dx/x) = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$.

52. Let $u = x^n$, $dv = e^x\,dx \quad\Rightarrow\quad du = nx^{n-1}\,dx$, $v = e^x$. By Equation 2, $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$.

53. $\int \tan^n x\,dx = \int \tan^{n-2} x\,\tan^2 x\,dx = \int \tan^{n-2} x\,(\sec^2 x - 1)\,dx = \int \tan^{n-2} x\,\sec^2 x\,dx - \int \tan^{n-2} x\,dx$
$= I - \int \tan^{n-2} x\,dx$.

Let $u = \tan^{n-2} x$, $dv = \sec^2 x\,dx \;\Rightarrow\; du = (n-2)\,\tan^{n-3} x\,\sec^2 x\,dx$, $v = \tan x$. Then, by Equation 2,

$$\begin{aligned} I &= \tan^{n-1} x - (n-2) \int \tan^{n-2} x\,\sec^2 x\,dx \\ 1I &= \tan^{n-1} x - (n-2)I \\ (n-1)I &= \tan^{n-1} x \\ I &= \frac{\tan^{n-1} x}{n-1} \end{aligned}$$

Returning to the original integral, $\int \tan^n x\,dx = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$.

54. Let $u = \sec^{n-2} x$, $dv = \sec^2 x\,dx \;\Rightarrow\; du = (n-2)\,\sec^{n-3} x\,\sec x\,\tan x\,dx$, $v = \tan x$. Then, by Equation 2,

$$\begin{aligned} \int \sec^n x\,dx &= \tan x\,\sec^{n-2} x - (n-2) \int \sec^{n-2} x\,\tan^2 x\,dx \\ &= \tan x\,\sec^{n-2} x - (n-2) \int \sec^{n-2} x\,(\sec^2 x - 1)\,dx \\ &= \tan x\,\sec^{n-2} x - (n-2) \int \sec^n x\,dx + (n-2) \int \sec^{n-2} x\,dx \end{aligned}$$

so $(n-1) \int \sec^n x\,dx = \tan x\,\sec^{n-2} x + (n-2) \int \sec^{n-2} x\,dx$. If $n - 1 \neq 0$, then

$$\int \sec^n x\,dx = \frac{\tan x\,\sec^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\,dx.$$

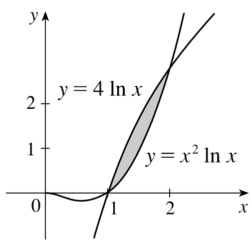
55. By repeated applications of the reduction formula in Exercise 51,

$$\begin{aligned} \int (\ln x)^3\,dx &= x\,(\ln x)^3 - 3 \int (\ln x)^2\,dx = x(\ln x)^3 - 3\left[x(\ln x)^2 - 2 \int (\ln x)^1\,dx\right] \\ &= x\,(\ln x)^3 - 3x(\ln x)^2 + 6\left[x(\ln x)^1 - 1 \int (\ln x)^0\,dx\right] \\ &= x\,(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6 \int 1\,dx = x\,(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x + C \end{aligned}$$

56. By repeated applications of the reduction formula in Exercise 52,

$$\begin{aligned} \int x^4 e^x\,dx &= x^4 e^x - 4 \int x^3 e^x\,dx = x^4 e^x - 4\left(x^3 e^x - 3 \int x^2 e^x\,dx\right) \\ &= x^4 e^x - 4x^3 e^x + 12\left(x^2 e^x - 2 \int x^1 e^x\,dx\right) = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24\left(x^1 e^x - \int x^0 e^x\,dx\right) \\ &= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + C \quad \left[\text{or } e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + C\right] \end{aligned}$$

57. The curves $y = x^2 \ln x$ and $y = 4 \ln x$ intersect when $x^2 \ln x = 4 \ln x \;\Leftrightarrow\; x^2 \ln x - 4 \ln x = 0 \;\Leftrightarrow\; (x^2 - 4) \ln x = 0 \;\Leftrightarrow\; x = 1$ or 2 [since $x > 0$]. For $1 < x < 2$, $4 \ln x > x^2 \ln x$. Thus, area $= \int_1^2 (4 \ln x - x^2 \ln x)\,dx = \int_1^2 [(4 - x^2) \ln x]\,dx$. Let $u = \ln x$, $dv = (4 - x^2)\,dx \;\Rightarrow\; du = \frac{1}{x}\,dx$, $v = 4x - \frac{1}{3}x^3$. Then



$$\begin{aligned} \text{area} &= \left[(\ln x)\left(4x - \tfrac{1}{3}x^3\right)\right]_1^2 - \int_1^2 \left[\left(4x - \tfrac{1}{3}x^3\right)\frac{1}{x}\right] dx = (\ln 2)\left(\tfrac{16}{3}\right) - 0 - \int_1^2 \left(4 - \tfrac{1}{3}x^2\right) dx \\ &= \tfrac{16}{3} \ln 2 - \left[4x - \tfrac{1}{9}x^3\right]_1^2 = \tfrac{16}{3} \ln 2 - \left(\tfrac{64}{9} - \tfrac{35}{9}\right) = \tfrac{16}{3} \ln 2 - \tfrac{29}{9} \end{aligned}$$

58. The curves $y = x^2 e^{-x}$ and $y = xe^{-x}$ intersect when $x^2 e^{-x} = xe^{-x} \quad \Leftrightarrow$ $x^2 - x = 0 \quad \Leftrightarrow \quad x(x-1) = 0 \quad \Leftrightarrow \quad x = 0$ or 1.

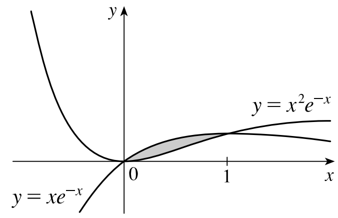
For $0 < x < 1$, $xe^{-x} > x^2 e^{-x}$. Thus,

area $= \int_0^1 (xe^{-x} - x^2 e^{-x})\, dx = \int_0^1 (x - x^2) e^{-x}\, dx$. Let $u = x - x^2$, $dv = e^{-x} dx \quad \Rightarrow \quad du = (1 - 2x)\, dx$, $v = -e^{-x}$. Then

area $= \left[(x - x^2)(-e^{-x})\right]_0^1 - \int_0^1 [-e^{-x}(1 - 2x)]\, dx = 0 + \int_0^1 (1 - 2x) e^{-x}\, dx$.

Now let $U = 1 - 2x$, $dV = e^{-x} dx \quad \Rightarrow \quad dU = -2\, dx$, $V = -e^{-x}$. Now

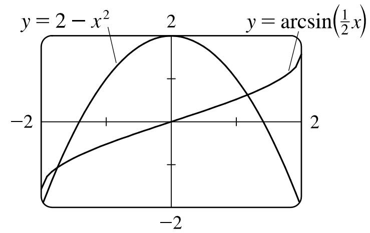
area $= \left[(1 - 2x)(-e^{-x})\right]_0^1 - \int_0^1 2e^{-x}\, dx = e^{-1} + 1 - \left[-2e^{-x}\right]_0^1 = e^{-1} + 1 + 2(e^{-1} - 1) = 3e^{-1} - 1$.



59. The curves $y = \arcsin\left(\frac{1}{2}x\right)$ and $y = 2 - x^2$ intersect at $x = a \approx -1.75119$ and $x = b \approx 1.17210$. From the figure, the area bounded by the curves is given by

$A = \int_a^b [(2 - x^2) - \arcsin\left(\frac{1}{2}x\right)]\, dx = \left[2x - \frac{1}{3}x^3\right]_a^b - \int_a^b \arcsin\left(\frac{1}{2}x\right) dx$.

Let $u = \arcsin\left(\frac{1}{2}x\right)$, $dv = dx \quad \Rightarrow \quad du = \dfrac{1}{\sqrt{1 - \left(\frac{1}{2}x\right)^2}} \cdot \dfrac{1}{2}\, dx$, $v = x$.



Then

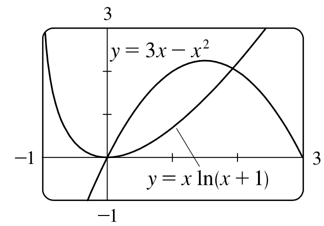
$$A = \left[2x - \frac{1}{3}x^3\right]_a^b - \left\{\left[x \arcsin\left(\frac{1}{2}x\right)\right]_a^b - \int_a^b \frac{x}{2\sqrt{1 - \frac{1}{4}x^2}}\, dx\right\}$$

$$= \left[2x - \tfrac{1}{3}x^3 - x\arcsin\left(\tfrac{1}{2}x\right) - 2\sqrt{1 - \tfrac{1}{4}x^2}\,\right]_a^b \approx 3.99926$$

60. The curves $y = x\ln(x + 1)$ and $y = 3x - x^2$ intersect at $x = 0$ and $x = a \approx 1.92627$. From the figure, the area bounded by the curves is given by

$A = \int_0^a [(3x - x^2) - x\ln(x+1)]\, dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^a - \int_0^a x\ln(x+1)\, dx$.

Let $u = \ln(x + 1)$, $dv = x\, dx \quad \Rightarrow \quad du = \dfrac{1}{x+1}\, dx$, $v = \frac{1}{2}x^2$. Then



$$A = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^a - \left\{\left[\frac{1}{2}x^2 \ln(x+1)\right]_0^a - \frac{1}{2}\int_0^a \frac{x^2}{x+1}\, dx\right\}$$

$$= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^a - \left[\frac{1}{2}x^2\ln(x+1)\right]_0^a + \frac{1}{2}\int_0^a \left(x - 1 + \frac{1}{x+1}\right) dx$$

$$= \left[\tfrac{3}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\ln(x+1) + \tfrac{1}{4}x^2 - \tfrac{1}{2}x + \tfrac{1}{2}\ln|x+1|\right]_0^a \approx 1.69260$$

61. Volume $= \int_0^1 2\pi x \cos(\pi x/2)\, dx$. Let $u = x$, $dv = \cos(\pi x/2)\, dx \quad \Rightarrow \quad du = dx$, $v = \frac{2}{\pi}\sin(\pi x/2)$.

$$V = 2\pi\left[\frac{2}{\pi}x\sin\left(\frac{\pi x}{2}\right)\right]_0^1 - 2\pi \cdot \frac{2}{\pi}\int_0^1 \sin\left(\frac{\pi x}{2}\right) dx = 2\pi\left(\frac{2}{\pi} - 0\right) - 4\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_0^1 = 4 + \frac{8}{\pi}(0 - 1) = 4 - \frac{8}{\pi}.$$

62. Volume $= \int_0^1 2\pi x(e^x - e^{-x})\, dx = 2\pi\int_0^1 (xe^x - xe^{-x})\, dx = 2\pi\left[\int_0^1 xe^x\, dx - \int_0^1 xe^{-x}\, dx\right]$ [both integrals by parts]

$= 2\pi\left[(xe^x - e^x) - (-xe^{-x} - e^{-x})\right]_0^1 = 2\pi[2/e - 0] = 4\pi/e$

63. Volume $= \int_{-1}^{0} 2\pi(1-x)e^{-x}\,dx$. Let $u = 1-x$, $dv = e^{-x}\,dx \;\Rightarrow\; du = -\,dx$, $v = -e^{-x}$.

$V = 2\pi\big[(1-x)(-e^{-x})\big]_{-1}^{0} - 2\pi\int_{-1}^{0} e^{-x}\,dx = 2\pi\big[(x-1)(e^{-x}) + e^{-x}\big]_{-1}^{0} = 2\pi\big[xe^{-x}\big]_{-1}^{0} = 2\pi(0+e) = 2\pi e.$

64. $y = e^x \;\Leftrightarrow\; x = \ln y$. Volume $= \int_1^3 2\pi y\,\ln y\,dy$. Let $u = \ln y$, $dv = y\,dy \;\Rightarrow\; du = \dfrac{1}{y}\,dy$, $v = \frac{1}{2}y^2$.

$$V = 2\pi\left[\tfrac{1}{2}y^2\ln y\right]_1^3 - 2\pi\int_1^3 \tfrac{1}{2}y\,dy = 2\pi\left[\tfrac{1}{2}y^2\ln y - \tfrac{1}{4}y^2\right]_1^3$$
$$= 2\pi\left[\left(\tfrac{9}{2}\ln 3 - \tfrac{9}{4}\right) - \left(0 - \tfrac{1}{4}\right)\right] = 2\pi\left(\tfrac{9}{2}\ln 3 - 2\right) = (9\ln 3 - 4)\,\pi$$

65. (a) Use shells about the y-axis:

$$V = \int_1^2 2\pi x\ln x\,dx \qquad \begin{bmatrix} u = \ln x, & dv = x\,dx \\ du = \frac{1}{x}\,dx, & v = \frac{1}{2}x^2 \end{bmatrix}$$
$$= 2\pi\left\{\left[\tfrac{1}{2}x^2\ln x\right]_1^2 - \int_1^2 \tfrac{1}{2}x\,dx\right\} = 2\pi\left\{(2\ln 2 - 0) - \left[\tfrac{1}{4}x^2\right]_1^2\right\} = 2\pi\left(2\ln 2 - \tfrac{3}{4}\right)$$

(b) Use disks about the x-axis:

$$V = \int_1^2 \pi(\ln x)^2\,dx \qquad \begin{bmatrix} u = (\ln x)^2, & dv = dx \\ du = 2\ln x\cdot\frac{1}{x}\,dx, & v = x \end{bmatrix}$$
$$= \pi\left\{\left[x(\ln x)^2\right]_1^2 - \int_1^2 2\ln x\,dx\right\} \qquad \begin{bmatrix} u = \ln x, & dv = dx \\ du = \frac{1}{x}\,dx, & v = x \end{bmatrix}$$
$$= \pi\left\{2(\ln 2)^2 - 2\left(\Big[x\ln x\Big]_1^2 - \int_1^2 dx\right)\right\} = \pi\left\{2(\ln 2)^2 - 4\ln 2 + 2\Big[x\Big]_1^2\right\}$$
$$= \pi[2(\ln 2)^2 - 4\ln 2 + 2] = 2\pi[(\ln 2)^2 - 2\ln 2 + 1]$$

66. $$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\pi/4 - 0}\int_0^{\pi/4} x\sec^2 x\,dx \qquad \begin{bmatrix} u = x, & dv = \sec^2 x\,dx \\ du = dx, & v = \tan x \end{bmatrix}$$
$$= \frac{4}{\pi}\left\{\Big[x\tan x\Big]_0^{\pi/4} - \int_0^{\pi/4}\tan x\,dx\right\} = \frac{4}{\pi}\left\{\frac{\pi}{4} - \Big[\ln|\sec x|\Big]_0^{\pi/4}\right\} = \frac{4}{\pi}\left(\frac{\pi}{4} - \ln\sqrt{2}\right)$$
$$= 1 - \tfrac{4}{\pi}\ln\sqrt{2} \;\text{ or }\; 1 - \tfrac{2}{\pi}\ln 2$$

67. $S(x) = \displaystyle\int_0^x \sin\left(\tfrac{1}{2}\pi t^2\right)dt \;\Rightarrow\; \int S(x)\,dx = \int\left[\int_0^x \sin\left(\tfrac{1}{2}\pi t^2\right)dt\right]dx.$

Let $u = \displaystyle\int_0^x \sin\left(\tfrac{1}{2}\pi t^2\right)dt = S(x)$, $dv = dx \;\Rightarrow\; du = \sin\left(\tfrac{1}{2}\pi x^2\right)dx$, $v = x$. Thus,

$$\int S(x)\,dx = xS(x) - \int x\sin\left(\tfrac{1}{2}\pi x^2\right)dx = xS(x) - \int \sin y\left(\tfrac{1}{\pi}\,dy\right) \qquad \begin{bmatrix} u = \frac{1}{2}\pi x^2, \\ du = \pi x\,dx \end{bmatrix}$$
$$= xS(x) + \tfrac{1}{\pi}\cos y + C = xS(x) + \tfrac{1}{\pi}\cos\left(\tfrac{1}{2}\pi x^2\right) + C$$

68. The rocket will have height $H = \int_0^{60} v(t)\,dt$ after 60 seconds.

$$H = \int_0^{60}\left[-gt - v_e\ln\left(\frac{m-rt}{m}\right)\right]dt = -g\left[\tfrac{1}{2}t^2\right]_0^{60} - v_e\left[\int_0^{60}\ln(m-rt)\,dt - \int_0^{60}\ln m\,dt\right]$$
$$= -g(1800) + v_e(\ln m)(60) - v_e\int_0^{60}\ln(m-rt)\,dt$$

Let $u = \ln(m-rt)$, $dv = dt \;\Rightarrow\; du = \dfrac{1}{m-rt}(-r)\,dt$, $v = t$. Then

$$\int_0^{60} \ln(m-rt)\,dt = \Big[t\ln(m-rt)\Big]_0^{60} + \int_0^{60} \frac{rt}{m-rt}\,dt = 60\ln(m-60r) + \int_0^{60}\left(-1+\frac{m}{m-rt}\right)dt$$

$$= 60\ln(m-60r) + \left[-t - \frac{m}{r}\ln(m-rt)\right]_0^{60} = 60\ln(m-60r) - 60 - \frac{m}{r}\ln(m-60r) + \frac{m}{r}\ln m$$

So $H = -1800g + 60v_e \ln m - 60v_e \ln(m-60r) + 60v_e + \dfrac{m}{r}v_e\ln(m-60r) - \dfrac{m}{r}v_e \ln m$. Substituting $g = 9.8$, $m = 30{,}000$, $r = 160$, and $v_e = 3000$ gives us $H \approx 14{,}844$ m.

69. Since $v(t) > 0$ for all t, the desired distance is $s(t) = \int_0^t v(w)\,dw = \int_0^t w^2 e^{-w}\,dw$.

First let $u = w^2$, $dv = e^{-w}\,dw \quad\Rightarrow\quad du = 2w\,dw$, $v = -e^{-w}$. Then $s(t) = \left[-w^2e^{-w}\right]_0^t + 2\int_0^t we^{-w}\,dw$.

Next let $U = w$, $dV = e^{-w}\,dw \quad\Rightarrow\quad dU = dw$, $V = -e^{-w}$. Then

$$s(t) = -t^2e^{-t} + 2\left(\left[-we^{-w}\right]_0^t + \int_0^t e^{-w}\,dw\right) = -t^2e^{-t} + 2\left(-te^{-t} + 0 + \left[-e^{-w}\right]_0^t\right)$$

$$= -t^2e^{-t} + 2(-te^{-t} - e^{-t} + 1) = -t^2e^{-t} - 2te^{-t} - 2e^{-t} + 2 = 2 - e^{-t}(t^2+2t+2) \text{ meters}$$

70. Suppose $f(0) = g(0) = 0$ and let $u = f(x)$, $dv = g''(x)\,dx \quad\Rightarrow\quad du = f'(x)\,dx$, $v = g'(x)$.

Then $\int_0^a f(x)\,g''(x)\,dx = \Big[f(x)\,g'(x)\Big]_0^a - \int_0^a f'(x)\,g'(x)\,dx = f(a)\,g'(a) - \int_0^a f'(x)\,g'(x)\,dx$.

Now let $U = f'(x)$, $dV = g'(x)\,dx \quad\Rightarrow\quad dU = f''(x)\,dx$ and $V = g(x)$, so

$\int_0^a f'(x)\,g'(x)\,dx = \Big[f'(x)\,g(x)\Big]_0^a - \int_0^a f''(x)\,g(x)\,dx = f'(a)\,g(a) - \int_0^a f''(x)\,g(x)\,dx$.

Combining the two results, we get $\int_0^a f(x)\,g''(x)\,dx = f(a)\,g'(a) - f'(a)\,g(a) + \int_0^a f''(x)\,g(x)\,dx$.

71. For $I = \int_1^4 xf''(x)\,dx$, let $u = x$, $dv = f''(x)\,dx \quad\Rightarrow\quad du = dx$, $v = f'(x)$. Then

$I = \left[xf'(x)\right]_1^4 - \int_1^4 f'(x)\,dx = 4f'(4) - 1\cdot f'(1) - [f(4) - f(1)] = 4\cdot 3 - 1\cdot 5 - (7-2) = 12 - 5 - 5 = 2$.

We used the fact that f'' is continuous to guarantee that I exists.

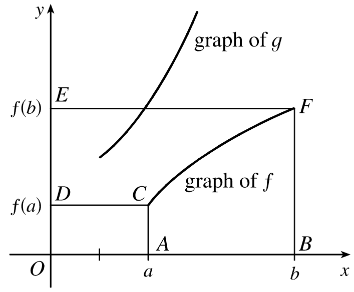
72. (a) Take $g(x) = x$ and $g'(x) = 1$ in Equation 1.

(b) By part (a), $\int_a^b f(x)\,dx = bf(b) - a\,f(a) - \int_a^b x\,f'(x)\,dx$. Now let $y = f(x)$, so that $x = g(y)$ and $dy = f'(x)\,dx$. Then $\int_a^b x\,f'(x)\,dx = \int_{f(a)}^{f(b)} g(y)\,dy$. The result follows.

(c) Part (b) says that the area of region $ABFC$ is

$$= \quad bf(b) \quad - \quad af(a) \quad - \quad \int_{f(a)}^{f(b)} g(y)\,dy$$

$= \text{(area of rectangle } OBFE) - \text{(area of rectangle } OACD) - \text{(area of region } DCFE)$



(d) We have $f(x) = \ln x$, so $f^{-1}(x) = e^x$, and since $g = f^{-1}$, we have $g(y) = e^y$. By part (b),

$$\int_1^e \ln x\,dx = e\ln e - 1\ln 1 - \int_{\ln 1}^{\ln e} e^y\,dy = e - \int_0^1 e^y\,dy = e - \left[\,e^y\,\right]_0^1 = e - (e-1) = 1.$$

73. Using the formula for volumes of rotation and the figure, we see that

Volume $= \int_0^d \pi b^2\,dy - \int_0^c \pi a^2\,dy - \int_c^d \pi[g(y)]^2\,dy = \pi b^2 d - \pi a^2 c - \int_c^d \pi[g(y)]^2\,dy$. Let $y = f(x)$,

which gives $dy = f'(x)\,dx$ and $g(y) = x$, so that $V = \pi b^2 d - \pi a^2 c - \pi\int_a^b x^2 f'(x)\,dx$.

Now integrate by parts with $u = x^2$, and $dv = f'(x)\,dx \quad\Rightarrow\quad du = 2x\,dx$, $v = f(x)$, and

$\int_a^b x^2\,f'(x)\,dx = \left[x^2\,f(x)\right]_a^b - \int_a^b 2x\,f(x)\,dx = b^2\,f(b) - a^2\,f(a) - \int_a^b 2x\,f(x)\,dx$, but $f(a) = c$ and $f(b) = d \quad\Rightarrow$

$V = \pi b^2 d - \pi a^2 c - \pi\left[b^2 d - a^2 c - \int_a^b 2x\,f(x)\,dx\right] = \int_a^b 2\pi x\,f(x)\,dx$.

74. (a) We note that for $0 \le x \le \frac{\pi}{2}$, $0 \le \sin x \le 1$, so $\sin^{2n+2} x \le \sin^{2n+1} x \le \sin^{2n} x$. So by the second Comparison Property of the Integral, $I_{2n+2} \le I_{2n+1} \le I_{2n}$.

(b) Substituting directly into the result from Exercise 50, we get

$$\frac{I_{2n+2}}{I_{2n}} = \frac{\dfrac{1\cdot3\cdot5\cdot\cdots\cdot[2(n+1)-1]}{2\cdot4\cdot6\cdot\cdots\cdot[2(n+1)]}\,\dfrac{\pi}{2}}{\dfrac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{2\cdot4\cdot6\cdot\cdots\cdot(2n)}\,\dfrac{\pi}{2}} = \frac{2(n+1)-1}{2(n+1)} = \frac{2n+1}{2n+2}$$

(c) We divide the result from part (a) by I_{2n}. The inequalities are preserved since I_{2n} is positive: $\dfrac{I_{2n+2}}{I_{2n}} \le \dfrac{I_{2n+1}}{I_{2n}} \le \dfrac{I_{2n}}{I_{2n}}$.

Now from part (b), the left term is equal to $\dfrac{2n+1}{2n+2}$, so the expression becomes $\dfrac{2n+1}{2n+2} \le \dfrac{I_{2n+1}}{I_{2n}} \le 1$. Now

$\lim\limits_{n\to\infty} \dfrac{2n+1}{2n+2} = \lim\limits_{n\to\infty} 1 = 1$, so by the Squeeze Theorem, $\lim\limits_{n\to\infty} \dfrac{I_{2n+1}}{I_{2n}} = 1$.

(d) We substitute the results from Exercises 49 and 50 into the result from part (c):

$$1 = \lim_{n\to\infty}\frac{I_{2n+1}}{I_{2n}} = \lim_{n\to\infty}\frac{\dfrac{2\cdot4\cdot6\cdot\cdots\cdot(2n)}{3\cdot5\cdot7\cdot\cdots\cdot(2n+1)}}{\dfrac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{2\cdot4\cdot6\cdot\cdots\cdot(2n)}\,\dfrac{\pi}{2}} = \lim_{n\to\infty}\left[\frac{2\cdot4\cdot6\cdot\cdots\cdot(2n)}{3\cdot5\cdot7\cdot\cdots\cdot(2n+1)}\right]\left[\frac{2\cdot4\cdot6\cdot\cdots\cdot(2n)}{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}\left(\frac{2}{\pi}\right)\right]$$

$$= \lim_{n\to\infty}\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\frac{2}{\pi} \qquad \text{[rearrange terms]}$$

Multiplying both sides by $\frac{\pi}{2}$ gives us the *Wallis product*:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots$$

(e) The area of the kth rectangle is k. At the $2n$th step, the area is increased from $2n-1$ to $2n$ by multiplying the width by $\dfrac{2n}{2n-1}$, and at the $(2n+1)$th step, the area is increased from $2n$ to $2n+1$ by multiplying the height by $\dfrac{2n+1}{2n}$. These two steps multiply the ratio of width to height by $\dfrac{2n}{2n-1}$ and $\dfrac{1}{(2n+1)/(2n)} = \dfrac{2n}{2n+1}$ respectively. So, by part (d), the limiting ratio is $\dfrac{2}{1}\cdot\dfrac{2}{3}\cdot\dfrac{4}{3}\cdot\dfrac{4}{5}\cdot\dfrac{6}{5}\cdot\dfrac{6}{7}\cdot\cdots = \dfrac{\pi}{2}$.

7.2 Trigonometric Integrals

The symbols $\overset{s}{=}$ and $\overset{c}{=}$ indicate the use of the substitutions $\{u = \sin x, du = \cos x\,dx\}$ and $\{u = \cos x, du = -\sin x\,dx\}$, respectively.

1. $\int \sin^2 x \cos^3 x\,dx = \int \sin^2 x \cos^2 x \cos x\,dx = \int \sin^2 x\,(1 - \sin^2 x) \cos x\,dx$

$$\overset{s}{=} \int u^2(1-u^2)\,du = \int (u^2 - u^4)\,du = \tfrac{1}{3}u^3 - \tfrac{1}{5}u^5 + C = \tfrac{1}{3}\sin^3 x - \tfrac{1}{5}\sin^5 x + C$$

2. $\int \sin^3\theta \cos^4\theta\,d\theta = \int \sin^2\theta \cos^4\theta \sin\theta\,d\theta = \int (1 - \cos^2\theta) \cos^4\theta \sin\theta\,d\theta$

$$\overset{c}{=} \int (1-u^2)u^4(-du) = \int (u^6 - u^4)\,du = \tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + C = \tfrac{1}{7}\cos^7\theta - \tfrac{1}{5}\cos^5\theta + C$$

3. $\int_0^{\pi/2} \sin^7\theta \cos^5\theta\,d\theta = \int_0^{\pi/2} \sin^7\theta \cos^4\theta \cos\theta\,d\theta = \int_0^{\pi/2} \sin^7\theta\,(1-\sin^2\theta)^2 \cos\theta\,d\theta$

$$\overset{s}{=} \int_0^1 u^7(1-u^2)^2\,du = \int_0^1 u^7(1 - 2u^2 + u^4)\,du = \int_0^1 (u^7 - 2u^9 + u^{11})\,du$$

$$= \left[\frac{1}{8}u^8 - \frac{1}{5}u^{10} + \frac{1}{12}u^{12}\right]_0^1 = \left(\frac{1}{8} - \frac{1}{5} + \frac{1}{12}\right) - 0 = \frac{15 - 24 + 10}{120} = \frac{1}{120}$$

4. $\int_0^{\pi/2} \sin^5 x\,dx = \int_0^{\pi/2} \sin^4 x \sin x\,dx = \int_0^{\pi/2} (1 - \cos^2 x)^2 \sin x\,dx \overset{c}{=} \int_1^0 (1-u^2)^2(-du)$

$$= \int_0^1 (1 - 2u^2 + u^4)\,du = \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1 = \left(1 - \frac{2}{3} + \frac{1}{5}\right) - 0 = \frac{15 - 10 + 3}{15} = \frac{8}{15}$$

5. $\int \sin^5(2t)\cos^2(2t)\,dt = \int \sin^4(2t)\cos^2(2t)\sin(2t)\,dt = \int [1 - \cos^2(2t)]^2 \cos^2(2t)\sin(2t)\,dt$

$$= \int (1-u^2)^2 u^2 \left(-\tfrac{1}{2}\,du\right) \qquad [u = \cos(2t),\ du = -2\sin(2t)\,dt]$$

$$= -\tfrac{1}{2}\int (u^4 - 2u^2 + 1)u^2\,du = -\tfrac{1}{2}\int (u^6 - 2u^4 + u^2)\,du$$

$$= -\tfrac{1}{2}\left(\tfrac{1}{7}u^7 - \tfrac{2}{5}u^5 + \tfrac{1}{3}u^3\right) + C = -\tfrac{1}{14}\cos^7(2t) + \tfrac{1}{5}\cos^5(2t) - \tfrac{1}{6}\cos^3(2t) + C$$

6. $\int t\cos^5(t^2)\,dt = \int t\cos^4(t^2)\cos(t^2)\,dt = \int t[1 - \sin^2(t^2)]^2\cos(t^2)\,dt$

$$= \int \tfrac{1}{2}(1-u^2)^2\,du \qquad [u = \sin(t^2),\ du = 2t\cos(t^2)\,dt]$$

$$= \tfrac{1}{2}\int (u^4 - 2u^2 + 1)\,du = \tfrac{1}{2}\left(\tfrac{1}{5}u^5 - \tfrac{2}{3}u^3 + u\right) + C = \tfrac{1}{10}\sin^5(t^2) - \tfrac{1}{3}\sin^3(t^2) + \tfrac{1}{2}\sin(t^2) + C$$

7. $\int_0^{\pi/2} \cos^2\theta\,d\theta = \int_0^{\pi/2} \tfrac{1}{2}(1 + \cos 2\theta)\,d\theta$ [half-angle identity]

$$= \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{1}{2}\left[\left(\tfrac{\pi}{2} + 0\right) - (0 + 0)\right] = \tfrac{\pi}{4}$$

8. $\int_0^{2\pi} \sin^2\left(\tfrac{1}{3}\theta\right)d\theta = \int_0^{2\pi} \tfrac{1}{2}\left[1 - \cos\left(2 \cdot \tfrac{1}{3}\theta\right)\right]d\theta$ [half-angle identity]

$$= \frac{1}{2}\left[\theta - \frac{3}{2}\sin\left(\frac{2}{3}\theta\right)\right]_0^{2\pi} = \frac{1}{2}\left[\left(2\pi - \frac{3}{2}\left(-\frac{\sqrt{3}}{2}\right)\right) - 0\right] = \pi + \frac{3}{8}\sqrt{3}$$

9. $\int_0^{\pi} \cos^4(2t)\,dt = \int_0^{\pi} [\cos^2(2t)]^2\,dt = \int_0^{\pi} \left[\tfrac{1}{2}(1 + \cos(2\cdot 2t))\right]^2 dt$ [half-angle identity]

$$= \tfrac{1}{4}\int_0^{\pi} [1 + 2\cos 4t + \cos^2(4t)]\,dt = \tfrac{1}{4}\int_0^{\pi} [1 + 2\cos 4t + \tfrac{1}{2}(1 + \cos 8t)]\,dt$$

$$= \tfrac{1}{4}\int_0^{\pi} \left(\tfrac{3}{2} + 2\cos 4t + \tfrac{1}{2}\cos 8t\right)dt = \tfrac{1}{4}\left[\tfrac{3}{2}t + \tfrac{1}{2}\sin 4t + \tfrac{1}{16}\sin 8t\right]_0^{\pi} = \tfrac{1}{4}\left[\left(\tfrac{3}{2}\pi + 0 + 0\right) - 0\right] = \tfrac{3}{8}\pi$$

10. $\int_0^{\pi} \sin^2 t \cos^4 t\,dt = \tfrac{1}{4}\int_0^{\pi} (4\sin^2 t \cos^2 t)\cos^2 t\,dt = \tfrac{1}{4}\int_0^{\pi} (2\sin t \cos t)^2\, \tfrac{1}{2}(1 + \cos 2t)\,dt$

$$= \tfrac{1}{8}\int_0^{\pi} (\sin 2t)^2(1 + \cos 2t)\,dt = \tfrac{1}{8}\int_0^{\pi} (\sin^2 2t + \sin^2 2t \cos 2t)\,dt$$

$$= \tfrac{1}{8}\int_0^{\pi} \sin^2 2t\,dt + \tfrac{1}{8}\int_0^{\pi} \sin^2 2t \cos 2t\,dt = \tfrac{1}{8}\int_0^{\pi} \tfrac{1}{2}(1 - \cos 4t)\,dt + \tfrac{1}{8}\left[\tfrac{1}{3}\cdot\tfrac{1}{2}\sin^3 2t\right]_0^{\pi}$$

$$= \tfrac{1}{16}\left[t - \tfrac{1}{4}\sin 4t\right]_0^{\pi} + \tfrac{1}{8}(0 - 0) = \tfrac{1}{16}[(\pi - 0) - 0] = \tfrac{\pi}{16}$$

11. $\int_0^{\pi/2} \sin^2 x \, \cos^2 x \, dx = \int_0^{\pi/2} \frac{1}{4}(4\sin^2 x \, \cos^2 x) \, dx = \int_0^{\pi/2} \frac{1}{4}(2\sin x \, \cos x)^2 dx = \frac{1}{4}\int_0^{\pi/2} \sin^2 2x \, dx$

$= \frac{1}{4}\int_0^{\pi/2} \frac{1}{2}(1 - \cos 4x) \, dx = \frac{1}{8}\int_0^{\pi/2}(1 - \cos 4x) \, dx = \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/2} = \frac{1}{8}\left(\frac{\pi}{2}\right) = \frac{\pi}{16}$

12. $\int_0^{\pi/2}(2 - \sin\theta)^2 \, d\theta = \int_0^{\pi/2}(4 - 4\sin\theta + \sin^2\theta) \, d\theta = \int_0^{\pi/2}\left[4 - 4\sin\theta + \frac{1}{2}(1 - \cos 2\theta)\right] d\theta$

$= \int_0^{\pi/2}\left(\frac{9}{2} - 4\sin\theta - \frac{1}{2}\cos 2\theta\right) d\theta = \left[\frac{9}{2}\theta + 4\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2}$

$= \left(\frac{9\pi}{4} + 0 - 0\right) - (0 + 4 - 0) = \frac{9}{4}\pi - 4$

13. $\int \sqrt{\cos\theta}\,\sin^3\theta \, d\theta = \int \sqrt{\cos\theta}\,\sin^2\theta \, \sin\theta \, d\theta = \int (\cos\theta)^{1/2}(1 - \cos^2\theta)\sin\theta \, d\theta$

$\overset{c}{=} \int u^{1/2}(1 - u^2)\,(-du) = \int (u^{5/2} - u^{1/2}) \, du$

$= \frac{2}{7}u^{7/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{7}(\cos\theta)^{7/2} - \frac{2}{3}(\cos\theta)^{3/2} + C$

14. $\displaystyle\int \frac{\sin^2(1/t)}{t^2} \, dt = \int \sin^2 u \, (-du) \qquad \left[u = \frac{1}{t},\, du = -\frac{1}{t^2}\, dt\right]$

$\displaystyle= -\int \frac{1}{2}(1 - \cos 2u) \, du = -\frac{1}{2}\left(u - \frac{1}{2}\sin 2u\right) + C = -\frac{1}{2t} + \frac{1}{4}\sin\left(\frac{2}{t}\right) + C$

15. $\displaystyle\int \cot x \, \cos^2 x \, dx = \int \frac{\cos x}{\sin x}\,(1 - \sin^2 x) \, dx$

$\displaystyle\overset{s}{=} \int \frac{1 - u^2}{u} \, du = \int \left(\frac{1}{u} - u\right) du = \ln|u| - \frac{1}{2}u^2 + C = \ln|\sin x| - \frac{1}{2}\sin^2 x + C$

16. $\displaystyle\int \tan^2 x \, \cos^3 x \, dx = \int \frac{\sin^2 x}{\cos^2 x}\cos^3 x \, dx = \int \sin^2 x \, \cos x \, dx \overset{s}{=} \int u^2 \, du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C$

17. $\int \sin^2 x \, \sin 2x \, dx = \int \sin^2 x \,(2\sin x \, \cos x) \, dx \overset{s}{=} \int 2u^3 \, du = \frac{1}{2}u^4 + C = \frac{1}{2}\sin^4 x + C$

18. $\int \sin x \, \cos\left(\frac{1}{2}x\right) dx = \int \sin\left(2 \cdot \frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right) dx = \int 2\sin\left(\frac{1}{2}x\right)\cos^2\left(\frac{1}{2}x\right) dx$

$= \int 2u^2\,(-2\,du) \qquad \left[u = \cos\left(\frac{1}{2}x\right),\, du = -\frac{1}{2}\sin\left(\frac{1}{2}x\right) dx\right]$

$= -\frac{4}{3}u^3 + C = -\frac{4}{3}\cos^3\left(\frac{1}{2}x\right) + C$

19. $\int t\sin^2 t \, dt = \int t\left[\frac{1}{2}(1 - \cos 2t)\right] dt = \frac{1}{2}\int (t - t\cos 2t) \, dt = \frac{1}{2}\int t \, dt - \frac{1}{2}\int t\cos 2t \, dt$

$= \frac{1}{2}\left(\frac{1}{2}t^2\right) - \frac{1}{2}\left(\frac{1}{2}t\sin 2t - \int \frac{1}{2}\sin 2t \, dt\right) \qquad \begin{bmatrix} u = t, & dv = \cos 2t \, dt \\ du = dt, & v = \frac{1}{2}\sin 2t \end{bmatrix}$

$= \frac{1}{4}t^2 - \frac{1}{4}t\sin 2t + \frac{1}{2}\left(-\frac{1}{4}\cos 2t\right) + C = \frac{1}{4}t^2 - \frac{1}{4}t\sin 2t - \frac{1}{8}\cos 2t + C$

20. $I = \int x\sin^3 x \, dx$. First, evaluate

$\int \sin^3 x \, dx = \int (1 - \cos^2 x)\, \sin x \, dx \overset{c}{=} \int (1 - u^2)(-du) = \int (u^2 - 1) \, du = \frac{1}{3}u^3 - u + C_1 = \frac{1}{3}\cos^3 x - \cos x + C_1$.

Now for I, let $u = x$, $dv = \sin^3 x \;\Rightarrow\; du = dx$, $v = \frac{1}{3}\cos^3 x - \cos x$, so

$I = \frac{1}{3}x\cos^3 x - x\cos x - \int\left(\frac{1}{3}\cos^3 x - \cos x\right) dx = \frac{1}{3}x\cos^3 x - x\cos x - \frac{1}{3}\int \cos^3 x \, dx + \sin x$

$= \frac{1}{3}x\cos^3 x - x\cos x - \frac{1}{3}(\sin x - \frac{1}{3}\sin^3 x) + \sin x + C$ [by Example 1]

$= \frac{1}{3}x\cos^3 x - x\cos x + \frac{2}{3}\sin x + \frac{1}{9}\sin^3 x + C$

21. $\int \tan x \, \sec^3 x \, dx = \int \tan x \, \sec x \, \sec^2 x \, dx = \int u^2 \, du \qquad [u = \sec x,\, du = \sec x \, \tan x \, dx]$

$= \frac{1}{3}u^3 + C = \frac{1}{3}\sec^3 x + C$

22. $\int \tan^2\theta\,\sec^4\theta\,d\theta = \int \tan^2\theta\,\sec^2\theta\,\sec^2\theta\,d\theta = \int \tan^2\theta\,(\tan^2\theta+1)\,\sec^2\theta\,d\theta$

$= \int u^2(u^2+1)\,du \qquad [u=\tan\theta,\ du=\sec^2\theta\,d\theta]$

$= \int (u^4+u^2)\,du = \frac{1}{5}u^5+\frac{1}{3}u^3+C = \frac{1}{5}\tan^5\theta+\frac{1}{3}\tan^3\theta+C$

23. $\int \tan^2 x\,dx = \int(\sec^2 x-1)\,dx = \tan x - x + C$

24. $\int(\tan^2 x+\tan^4 x)\,dx = \int \tan^2 x\,(1+\tan^2 x)\,dx = \int \tan^2 x\,\sec^2 x\,dx = \int u^2\,du \qquad [u=\tan x,\ du=\sec^2 x\,dx]$

$= \frac{1}{3}u^3+C = \frac{1}{3}\tan^3 x + C$

25. Let $u=\tan x$. Then $du=\sec^2 x\,dx$, so

$$\begin{aligned}\int \tan^4 x\sec^6 x\,dx &= \int \tan^4 x\sec^4 x\,(\sec^2 x\,dx) = \int \tan^4 x(1+\tan^2 x)^2\,(\sec^2 x\,dx)\\ &= \int u^4(1+u^2)^2\,du = \int(u^8+2u^6+u^4)\,du\\ &= \tfrac{1}{9}u^9+\tfrac{2}{7}u^7+\tfrac{1}{5}u^5+C = \tfrac{1}{9}\tan^9 x+\tfrac{2}{7}\tan^7 x+\tfrac{1}{5}\tan^5 x+C\end{aligned}$$

26. $\int_0^{\pi/4}\sec^6\theta\,\tan^6\theta\,d\theta = \int_0^{\pi/4}\tan^6\theta\,\sec^4\theta\,\sec^2\theta\,d\theta = \int_0^{\pi/4}\tan^6\theta(1+\tan^2\theta)^2\sec^2\theta\,d\theta$

$= \int_0^1 u^6(1+u^2)^2\,du \qquad \left[\begin{array}{c}u=\tan\theta,\\ du=\sec^2\theta\,d\theta\end{array}\right]$

$= \int_0^1 u^6(u^4+2u^2+1)\,du = \int_0^1(u^{10}+2u^8+u^6)\,du$

$= \left[\frac{1}{11}u^{11}+\frac{2}{9}u^9+\frac{1}{7}u^7\right]_0^1 = \frac{1}{11}+\frac{2}{9}+\frac{1}{7} = \frac{63+154+99}{693} = \frac{316}{693}$

27. $\int \tan^3 x\,\sec x\,dx = \int \tan^2 x\,\sec x\,\tan x\,dx = \int(\sec^2 x-1)\,\sec x\,\tan x\,dx$

$= \int(u^2-1)\,du \quad [u=\sec x,\ du=\sec x\,\tan x\,dx] \quad = \frac{1}{3}u^3-u+C = \frac{1}{3}\sec^3 x-\sec x+C$

28. Let $u=\sec x$, so $du=\sec x\tan x\,dx$. Thus,

$$\begin{aligned}\int \tan^5 x\sec^3 x\,dx &= \int \tan^4 x\sec^2 x\,(\sec x\tan x)\,dx = \int(\sec^2 x-1)^2\sec^2 x\,(\sec x\tan x\,dx)\\ &= \int(u^2-1)^2u^2\,du = \int(u^6-2u^4+u^2)\,du\\ &= \tfrac{1}{7}u^7-\tfrac{2}{5}u^5+\tfrac{1}{3}u^3+C = \tfrac{1}{7}\sec^7 x-\tfrac{2}{5}\sec^5 x+\tfrac{1}{3}\sec^3 x+C\end{aligned}$$

29. $\int \tan^3 x\,\sec^6 x\,dx = \int \tan^3 x\,\sec^4 x\,\sec^2 x\,dx = \int \tan^3 x\,(1+\tan^2 x)^2\sec^2 x\,dx$

$= \int u^3(1+u^2)^2\,du \qquad \left[\begin{array}{c}u=\tan x,\\ du=\sec^2 x\,dx\end{array}\right]$

$= \int u^3(u^4+2u^2+1)\,du = \int(u^7+2u^5+u^3)\,du$

$= \frac{1}{8}u^8+\frac{1}{3}u^6+\frac{1}{4}u^4+C = \frac{1}{8}\tan^8 x+\frac{1}{3}\tan^6 x+\frac{1}{4}\tan^4 x+C$

30. $\int_0^{\pi/4}\tan^4 t\,dt = \int_0^{\pi/4}\tan^2 t\,(\sec^2 t-1)\,dt = \int_0^{\pi/4}\tan^2 t\,\sec^2 t\,dt - \int_0^{\pi/4}\tan^2 t\,dt$

$= \int_0^1 u^2\,du \ [u=\tan t] \ - \int_0^{\pi/4}(\sec^2 t-1)\,dt = \left[\frac{1}{3}u^3\right]_0^1 - \left[\tan t - t\right]_0^{\pi/4}$

$= \frac{1}{3} - \left[\left(1-\frac{\pi}{4}\right)-0\right] = \frac{\pi}{4}-\frac{2}{3}$

31. $\int \tan^5 x\,dx = \int(\sec^2 x-1)^2\,\tan x\,dx = \int \sec^4 x\,\tan x\,dx - 2\int \sec^2 x\,\tan x\,dx + \int \tan x\,dx$

$= \int \sec^3 x\,\sec x\,\tan x\,dx - 2\int \tan x\,\sec^2 x\,dx + \int \tan x\,dx$

$= \frac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C \quad \left[\text{or } \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C\right]$

32. $\int \tan^2 x \sec x\,dx = \int (\sec^2 x - 1)\sec x\,dx = \int \sec^3 x\,dx - \int \sec x\,dx$

$= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) - \ln|\sec x + \tan x| + C$ [by Example 8 and (1)]

$= \frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x|) + C$

33. Let $u = x, dv = \sec x \tan x\,dx \Rightarrow du = dx, v = \sec x$. Then

$\int x \sec x \tan x\,dx = x\sec x - \int \sec x\,dx = x\sec x - \ln|\sec x + \tan x| + C$.

34. $\displaystyle\int \frac{\sin\phi}{\cos^3\phi}\,d\phi = \int \frac{\sin\phi}{\cos\phi}\cdot\frac{1}{\cos^2\phi}\,d\phi = \int \tan\phi \sec^2\phi\,d\phi = \int u\,du$ $\quad[u = \tan\phi,\ du = \sec^2\phi\,d\phi]$

$= \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2\phi + C$

Alternate solution: Let $u = \cos\phi$ to get $\frac{1}{2}\sec^2\phi + C$.

35. $\int_{\pi/6}^{\pi/2} \cot^2 x\,dx = \int_{\pi/6}^{\pi/2} (\csc^2 x - 1)\,dx = \left[-\cot x - x\right]_{\pi/6}^{\pi/2} = \left(0 - \frac{\pi}{2}\right) - \left(-\sqrt{3} - \frac{\pi}{6}\right) = \sqrt{3} - \frac{\pi}{3}$

36. $\displaystyle\int_{\pi/4}^{\pi/2} \cot^3 x\,dx = \int_{\pi/4}^{\pi/2} \cot x\,(\csc^2 x - 1)\,dx = \int_{\pi/4}^{\pi/2} \cot x \csc^2 x\,dx - \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}\,dx$

$= \left[-\frac{1}{2}\cot^2 x - \ln|\sin x|\right]_{\pi/4}^{\pi/2} = (0 - \ln 1) - \left[-\frac{1}{2} - \ln\frac{1}{\sqrt{2}}\right] = \frac{1}{2} + \ln\frac{1}{\sqrt{2}} = \frac{1}{2}(1 - \ln 2)$

37. $\int_{\pi/4}^{\pi/2} \cot^5\phi \csc^3\phi\,d\phi = \int_{\pi/4}^{\pi/2} \cot^4\phi \csc^2\phi \csc\phi \cot\phi\,d\phi = \int_{\pi/4}^{\pi/2} (\csc^2\phi - 1)^2 \csc^2\phi \csc\phi \cot\phi\,d\phi$

$\displaystyle= \int_{\sqrt{2}}^{1} (u^2 - 1)^2 u^2\,(-du)$ $\quad[u = \csc\phi,\ du = -\csc\phi \cot\phi\,d\phi]$

$\displaystyle= \int_{1}^{\sqrt{2}} (u^6 - 2u^4 + u^2)\,du = \left[\frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3\right]_1^{\sqrt{2}} = \left(\frac{8}{7}\sqrt{2} - \frac{8}{5}\sqrt{2} + \frac{2}{3}\sqrt{2}\right) - \left(\frac{1}{7} - \frac{2}{5} + \frac{1}{3}\right)$

$\displaystyle= \frac{120 - 168 + 70}{105}\sqrt{2} - \frac{15 - 42 + 35}{105} = \frac{22}{105}\sqrt{2} - \frac{8}{105}$

38. $\int_{\pi/4}^{\pi/2} \csc^4\theta \cot^4\theta\,d\theta = \int_{\pi/4}^{\pi/2} \cot^4\theta \csc^2\theta \csc^2\theta\,d\theta = \int_{\pi/4}^{\pi/2} \cot^4\theta\,(\cot^2\theta + 1)\csc^2\theta\,d\theta$

$= \int_1^0 u^4(u^2 + 1)\,(-du)$ $\quad\begin{bmatrix} u = \cot\theta, \\ du = -\csc^2\theta\,d\theta \end{bmatrix}$

$= \int_0^1 (u^6 + u^4)\,du$

$= \left[\frac{1}{7}u^7 + \frac{1}{5}u^5\right]_0^1 = \frac{1}{7} + \frac{1}{5} = \frac{12}{35}$

39. $\displaystyle I = \int \csc x\,dx = \int \frac{\csc x\,(\csc x - \cot x)}{\csc x - \cot x}\,dx = \int \frac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x}\,dx$. Let $u = \csc x - \cot x \Rightarrow$

$du = (-\csc x \cot x + \csc^2 x)\,dx$. Then $I = \int du/u = \ln|u| = \ln|\csc x - \cot x| + C$.

40. Let $u = \csc x, dv = \csc^2 x\,dx$. Then $du = -\csc x \cot x\,dx, v = -\cot x \Rightarrow$

$\int \csc^3 x\,dx = -\csc x \cot x - \int \csc x \cot^2 x\,dx = -\csc x \cot x - \int \csc x\,(\csc^2 x - 1)\,dx$

$= -\csc x \cot x + \int \csc x\,dx - \int \csc^3 x\,dx$

Solving for $\int \csc^3 x\,dx$ and using Exercise 39, we get

$\int \csc^3 x\,dx = -\frac{1}{2}\csc x\,\cot x + \frac{1}{2}\int \csc x\,dx = -\frac{1}{2}\csc x\,\cot x + \frac{1}{2}\ln|\csc x - \cot x| + C$. Thus,

$$\begin{aligned}\int_{\pi/6}^{\pi/3} \csc^3 x\,dx &= \left[-\tfrac{1}{2}\csc x\,\cot x + \tfrac{1}{2}\ln|\csc x - \cot x|\right]_{\pi/6}^{\pi/3} \\ &= -\tfrac{1}{2}\cdot\tfrac{2}{\sqrt{3}}\cdot\tfrac{1}{\sqrt{3}} + \tfrac{1}{2}\ln\left|\tfrac{2}{\sqrt{3}} - \tfrac{1}{\sqrt{3}}\right| + \tfrac{1}{2}\cdot 2\cdot\sqrt{3} - \tfrac{1}{2}\ln\left|2-\sqrt{3}\right| \\ &= -\tfrac{1}{3} + \sqrt{3} + \tfrac{1}{2}\ln\tfrac{1}{\sqrt{3}} - \tfrac{1}{2}\ln\left(2-\sqrt{3}\right) \approx 1.7825\end{aligned}$$

41. $$\begin{aligned}\int \sin 8x\,\cos 5x\,dx &\overset{2a}{=} \int \tfrac{1}{2}[\sin(8x-5x)+\sin(8x+5x)]\,dx = \tfrac{1}{2}\int(\sin 3x + \sin 13x)\,dx \\ &= \tfrac{1}{2}(-\tfrac{1}{3}\cos 3x - \tfrac{1}{13}\cos 13x) + C = -\tfrac{1}{6}\cos 3x - \tfrac{1}{26}\cos 13x + C\end{aligned}$$

42. $$\begin{aligned}\int \sin 2\theta\,\sin 6\theta\,d\theta &\overset{2b}{=} \int \tfrac{1}{2}[\cos(2\theta-6\theta)-\cos(2\theta+6\theta)]\,d\theta \\ &= \tfrac{1}{2}\int[\cos(-4\theta)-\cos 8\theta]\,d\theta = \tfrac{1}{2}\int(\cos 4\theta - \cos 8\theta)\,d\theta \\ &= \tfrac{1}{2}\left(\tfrac{1}{4}\sin 4\theta - \tfrac{1}{8}\sin 8\theta\right) + C = \tfrac{1}{8}\sin 4\theta - \tfrac{1}{16}\sin 8\theta + C\end{aligned}$$

43. $$\begin{aligned}\int_0^{\pi/2} \cos 5t\,\cos 10t\,dt &\overset{2c}{=} \int_0^{\pi/2} \tfrac{1}{2}[\cos(5t-10t)+\cos(5t+10t)]\,dt \\ &= \tfrac{1}{2}\int_0^{\pi/2}[\cos(-5t)+\cos 15t]\,dt = \tfrac{1}{2}\int_0^{\pi/2}(\cos 5t + \cos 15t)\,dt \\ &= \tfrac{1}{2}\left[\tfrac{1}{5}\sin 5t + \tfrac{1}{15}\sin 15t\right]_0^{\pi/2} = \tfrac{1}{2}\left(\tfrac{1}{5}-\tfrac{1}{15}\right) = \tfrac{1}{15}\end{aligned}$$

44. $$\int \sin x\,\sec^5 x\,dx = \int \frac{\sin x}{\cos^5 x}\,dx \overset{c}{=} \int \frac{1}{u^5}\,(-du) = \frac{1}{4u^4} + C = \frac{1}{4\cos^4 x} + C = \tfrac{1}{4}\sec^4 x + C$$

45. $$\begin{aligned}\int_0^{\pi/6} \sqrt{1+\cos 2x}\,dx &= \int_0^{\pi/6} \sqrt{1+(2\cos^2 x - 1)}\,dx = \int_0^{\pi/6} \sqrt{2\cos^2 x}\,dx = \sqrt{2}\int_0^{\pi/6}\sqrt{\cos^2 x}\,dx \\ &= \sqrt{2}\int_0^{\pi/6}|\cos x|\,dx = \sqrt{2}\int_0^{\pi/6}\cos x\,dx \qquad [\text{since } \cos x > 0 \text{ for } 0 \le x \le \pi/6] \\ &= \sqrt{2}\Big[\sin x\Big]_0^{\pi/6} = \sqrt{2}\left(\tfrac{1}{2}-0\right) = \tfrac{1}{2}\sqrt{2}\end{aligned}$$

46. $$\begin{aligned}\int_0^{\pi/4} \sqrt{1-\cos 4\theta}\,d\theta &= \int_0^{\pi/4}\sqrt{1-(1-2\sin^2(2\theta))}\,d\theta = \int_0^{\pi/4}\sqrt{2\sin^2(2\theta)}\,d\theta = \sqrt{2}\int_0^{\pi/4}\sqrt{\sin^2(2\theta)}\,d\theta \\ &= \sqrt{2}\int_0^{\pi/4}|\sin 2\theta|\,d\theta = \sqrt{2}\int_0^{\pi/4}\sin 2\theta\,d\theta \qquad [\text{since } \sin 2\theta \ge 0 \text{ for } 0 \le \theta \le \pi/4] \\ &= \sqrt{2}\left[-\tfrac{1}{2}\cos 2\theta\right]_0^{\pi/4} = -\tfrac{1}{2}\sqrt{2}\,(0-1) = \tfrac{1}{2}\sqrt{2}\end{aligned}$$

47. $$\int \frac{1-\tan^2 x}{\sec^2 x}\,dx = \int\left(\cos^2 x - \sin^2 x\right)dx = \int \cos 2x\,dx = \tfrac{1}{2}\sin 2x + C$$

48. $$\begin{aligned}\int \frac{dx}{\cos x - 1} &= \int \frac{1}{\cos x - 1}\cdot\frac{\cos x + 1}{\cos x + 1}\,dx = \int \frac{\cos x + 1}{\cos^2 x - 1}\,dx = \int \frac{\cos x + 1}{-\sin^2 x}\,dx \\ &= \int\left(-\cot x\,\csc x - \csc^2 x\right)dx = \csc x + \cot x + C\end{aligned}$$

49. $$\begin{aligned}\int x\tan^2 x\,dx &= \int x(\sec^2 x - 1)\,dx = \int x\sec^2 x\,dx - \int x\,dx \\ &= x\tan x - \int \tan x\,dx - \tfrac{1}{2}x^2 \qquad \begin{bmatrix} u = x, & dv = \sec^2 x\,dx \\ du = dx, & v = \tan x\end{bmatrix} \\ &= x\tan x - \ln|\sec x| - \tfrac{1}{2}x^2 + C\end{aligned}$$

50. Let $u = \tan^7 x$, $dv = \sec x \, \tan x \, dx \quad \Rightarrow \quad du = 7\tan^6 x \, \sec^2 x \, dx$, $v = \sec x$. Then

$$\begin{aligned} \int \tan^8 x \, \sec x \, dx = \int \tan^7 x \cdot \sec x \, \tan x \, dx &= \tan^7 x \, \sec x - \int 7\tan^6 x \sec^2 x \sec x \, dx \\ &= \tan^7 x \, \sec x - 7 \int \tan^6 x \, (\tan^2 x + 1) \, \sec x \, dx \\ &= \tan^7 x \, \sec x - 7 \int \tan^8 x \sec x \, dx - 7 \int \tan^6 x \sec x \, dx. \end{aligned}$$

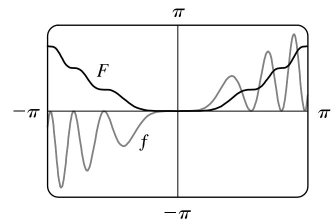
Thus, $8 \int \tan^8 x \sec x \, dx = \tan^7 x \, \sec x - 7 \int \tan^6 x \sec x \, dx$ and

$$\int_0^{\pi/4} \tan^8 x \sec x \, dx = \frac{1}{8}\big[\tan^7 x \, \sec x\big]_0^{\pi/4} - \frac{7}{8}\int_0^{\pi/4} \tan^6 x \, \sec x \, dx = \frac{\sqrt{2}}{8} - \frac{7}{8}I.$$

In Exercises 51–54, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

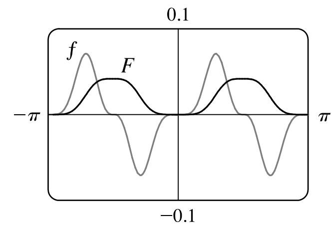
51. Let $u = x^2$, so that $du = 2x \, dx$. Then

$$\begin{aligned} \int x \sin^2(x^2) \, dx = \int \sin^2 u \left(\tfrac{1}{2} \, du\right) &= \tfrac{1}{2} \int \tfrac{1}{2}(1 - \cos 2u) \, du \\ &= \tfrac{1}{4}\left(u - \tfrac{1}{2}\sin 2u\right) + C = \tfrac{1}{4}u - \tfrac{1}{4}\left(\tfrac{1}{2} \cdot 2 \sin u \, \cos u\right) + C \\ &= \tfrac{1}{4}x^2 - \tfrac{1}{4}\sin(x^2) \, \cos(x^2) + C \end{aligned}$$



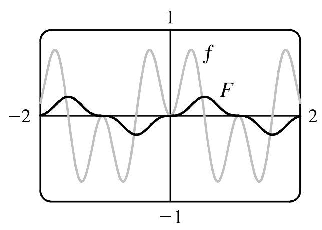
We see from the graph that this is reasonable, since F increases where f is positive and F decreases where f is negative. Note also that f is an odd function and F is an even function.

52. $$\begin{aligned} \int \sin^5 x \, \cos^3 x \, dx &= \int \sin^5 x \, \cos^2 x \, \cos x \, dx \\ &= \int \sin^5 x \, (1 - \sin^2 x) \, \cos x \, dx \\ &\overset{s}{=} \int u^5(1 - u^2) \, du = \int (u^5 - u^7) \, du \\ &= \tfrac{1}{6}\sin^6 x - \tfrac{1}{8}\sin^8 x + C \end{aligned}$$



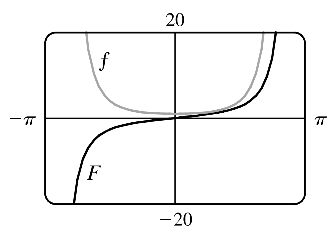
We see from the graph that this is reasonable, since F increases where f is positive and F decreases where f is negative. Note also that f is an odd function and F is an even function.

53. $$\begin{aligned} \int \sin 3x \, \sin 6x \, dx &= \int \tfrac{1}{2}[\cos(3x - 6x) - \cos(3x + 6x)] \, dx \\ &= \tfrac{1}{2}\int (\cos 3x - \cos 9x) \, dx \\ &= \tfrac{1}{6}\sin 3x - \tfrac{1}{18}\sin 9x + C \end{aligned}$$



Notice that $f(x) = 0$ whenever F has a horizontal tangent.

54. $$\begin{aligned} \int \sec^4\left(\tfrac{1}{2}x\right) \, dx &= \int \left(\tan^2 \tfrac{x}{2} + 1\right) \sec^2 \tfrac{x}{2} \, dx \\ &= \int (u^2 + 1) \, 2 \, du \quad \left[u = \tan \tfrac{x}{2}, \, du = \tfrac{1}{2}\sec^2 \tfrac{x}{2} \, dx\right] \\ &= \tfrac{2}{3}u^3 + 2u + C = \tfrac{2}{3}\tan^3 \tfrac{x}{2} + 2\tan \tfrac{x}{2} + C \end{aligned}$$



Notice that F is increasing and f is positive on the intervals on which they are defined. Also, F has no horizontal tangent and f is never zero.

55. $f_{\text{ave}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 x\, \cos^3 x\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 x\,(1-\sin^2 x)\,\cos x\, dx$

$= \frac{1}{2\pi}\int_0^0 u^2(1-u^2)\,du$ [where $u = \sin x$] $= 0$

56. (a) Let $u = \cos x$. Then $du = -\sin x\,dx \;\Rightarrow\; \int \sin x\,\cos x\,dx = \int u(-du) = -\frac{1}{2}u^2 + C = -\frac{1}{2}\cos^2 x + C_1$.

(b) Let $u = \sin x$. Then $du = \cos x\,dx \;\Rightarrow\; \int \sin x\,\cos x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C_2$.

(c) $\int \sin x\,\cos x\,dx = \int \frac{1}{2}\sin 2x\,dx = -\frac{1}{4}\cos 2x + C_3$

(d) Let $u = \sin x$, $dv = \cos x\,dx$. Then $du = \cos x\,dx$, $v = \sin x$, so $\int \sin x\,\cos x\,dx = \sin^2 x - \int \sin x\,\cos x\,dx$, by Equation 7.1.2, so $\int \sin x\,\cos x\,dx = \frac{1}{2}\sin^2 x + C_4$.

Using $\cos^2 x = 1 - \sin^2 x$ and $\cos 2x = 1 - 2\sin^2 x$, we see that the answers differ only by a constant.

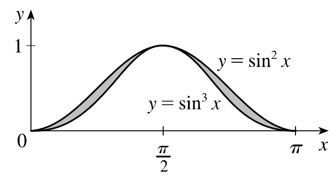
57. $A = \int_0^{\pi}(\sin^2 x - \sin^3 x)\,dx = \int_0^{\pi}\left[\frac{1}{2}(1-\cos 2x) - \sin x\,(1-\cos^2 x)\right]dx$

$= \int_0^{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx + \int_1^{-1}(1-u^2)\,du \qquad \begin{bmatrix} u = \cos x, \\ du = -\sin x\,dx \end{bmatrix}$

$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi} + 2\int_0^1 (u^2-1)\,du$

$= \left(\frac{1}{2}\pi - 0\right) - (0-0) + 2\left[\frac{1}{3}u^3 - u\right]_0^1$

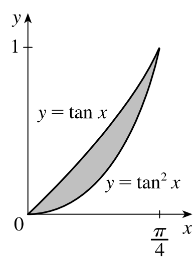
$= \frac{1}{2}\pi + 2\left(\frac{1}{3} - 1\right) = \frac{1}{2}\pi - \frac{4}{3}$



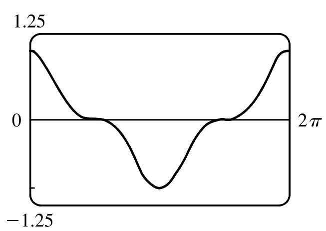
58. $A = \int_0^{\pi/4}(\tan x - \tan^2 x)\,dx = \int_0^{\pi/4}(\tan x - \sec^2 x + 1)\,dx$

$= \Big[\ln|\sec x| - \tan x + x\Big]_0^{\pi/4} = \left(\ln\sqrt{2} - 1 + \frac{\pi}{4}\right) - (\ln 1 - 0 + 0)$

$= \ln\sqrt{2} - 1 + \frac{\pi}{4}$

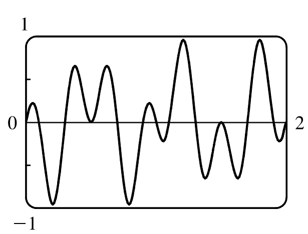


59.



It seems from the graph that $\int_0^{2\pi}\cos^3 x\,dx = 0$, since the area below the x-axis and above the graph looks about equal to the area above the axis and below the graph. By Example 1, the integral is $\left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{2\pi} = 0$. Note that due to symmetry, the integral of any odd power of $\sin x$ or $\cos x$ between limits which differ by $2n\pi$ (n any integer) is 0.

60.



It seems from the graph that $\int_0^2 \sin 2\pi x\,\cos 5\pi x\,dx = 0$, since each bulge above the x-axis seems to have a corresponding depression below the x-axis. To evaluate the integral, we use a trigonometric identity:

$\int_0^1 \sin 2\pi x\,\cos 5\pi x\,dx = \frac{1}{2}\int_0^2[\sin(2\pi x - 5\pi x) + \sin(2\pi x + 5\pi x)]\,dx$

$= \frac{1}{2}\int_0^2[\sin(-3\pi x) + \sin 7\pi x]\,dx$

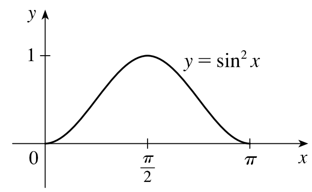
$= \frac{1}{2}\left[\frac{1}{3\pi}\cos(-3\pi x) - \frac{1}{7\pi}\cos 7\pi x\right]_0^2$

$= \frac{1}{2}\left[\frac{1}{3\pi}(1-1) - \frac{1}{7\pi}(1-1)\right] = 0$

61. Using disks, $V = \int_{\pi/2}^{\pi} \pi \sin^2 x\,dx = \pi \int_{\pi/2}^{\pi} \frac{1}{2}(1 - \cos 2x)\,dx = \pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_{\pi/2}^{\pi} = \pi\left(\frac{\pi}{2} - 0 - \frac{\pi}{4} + 0\right) = \frac{\pi^2}{4}$

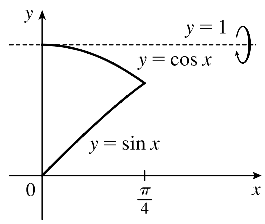
62. Using disks,

$$\begin{aligned} V &= \int_0^{\pi} \pi(\sin^2 x)^2\,dx = 2\pi \int_0^{\pi/2} \left[\tfrac{1}{2}(1 - \cos 2x)\right]^2 dx \\ &= \tfrac{\pi}{2} \int_0^{\pi/2} (1 - 2\cos 2x + \cos^2 2x)\,dx \\ &= \tfrac{\pi}{2} \int_0^{\pi/2} \left[1 - 2\cos 2x + \tfrac{1}{2}(1 - \cos 4x)\right] dx \\ &= \tfrac{\pi}{2} \int_0^{\pi/2} \left(\tfrac{3}{2} - 2\cos 2x - \tfrac{1}{2}\cos 4x\right) dx = \tfrac{\pi}{2}\left[\tfrac{3}{2}x - \sin 2x + \tfrac{1}{8}\sin 4x\right]_0^{\pi/2} \\ &= \tfrac{\pi}{2}\left[\left(\tfrac{3\pi}{4} - 0 + 0\right) - (0 - 0 + 0)\right] = \tfrac{3}{8}\pi^2 \end{aligned}$$



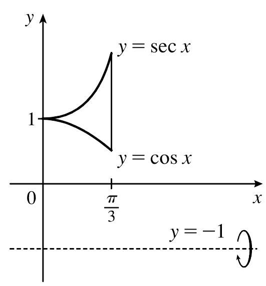
63. Using washers,

$$\begin{aligned} V &= \int_0^{\pi/4} \pi\left[(1 - \sin x)^2 - (1 - \cos x)^2\right] dx \\ &= \pi \int_0^{\pi/4} \left[(1 - 2\sin x + \sin^2 x) - (1 - 2\cos x + \cos^2 x)\right] dx \\ &= \pi \int_0^{\pi/4} (2\cos x - 2\sin x + \sin^2 x - \cos^2 x)\,dx \\ &= \pi \int_0^{\pi/4} (2\cos x - 2\sin x - \cos 2x)\,dx = \pi\left[2\sin x + 2\cos x - \tfrac{1}{2}\sin 2x\right]_0^{\pi/4} \\ &= \pi\left[\left(\sqrt{2} + \sqrt{2} - \tfrac{1}{2}\right) - (0 + 2 - 0)\right] = \pi\left(2\sqrt{2} - \tfrac{5}{2}\right) \end{aligned}$$



64. Using washers,

$$\begin{aligned} V &= \int_0^{\pi/3} \pi\left\{[\sec x - (-1)]^2 - [\cos x - (-1)]^2\right\} dx \\ &= \pi \int_0^{\pi/3} [(\sec^2 x + 2\sec x + 1) - (\cos^2 x + 2\cos x + 1)]\,dx \\ &= \pi \int_0^{\pi/3} \left[\sec^2 x + 2\sec x - \tfrac{1}{2}(1 + \cos 2x) - 2\cos x\right] dx \\ &= \pi\left[\tan x + 2\ln|\sec x + \tan x| - \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x - 2\sin x\right]_0^{\pi/3} \\ &= \pi\left[\left(\sqrt{3} + 2\ln\left(2 + \sqrt{3}\right) - \tfrac{\pi}{6} - \tfrac{1}{8}\sqrt{3} - \sqrt{3}\right) - 0\right] \\ &= 2\pi\ln\left(2 + \sqrt{3}\right) - \tfrac{1}{6}\pi^2 - \tfrac{1}{8}\pi\sqrt{3} \end{aligned}$$



65. $s = f(t) = \int_0^t \sin\omega u\,\cos^2\omega u\,du$. Let $y = \cos\omega u \;\Rightarrow\; dy = -\omega\sin\omega u\,du$. Then
$s = -\frac{1}{\omega}\int_1^{\cos\omega t} y^2 dy = -\frac{1}{\omega}\left[\frac{1}{3}y^3\right]_1^{\cos\omega t} = \frac{1}{3\omega}(1 - \cos^3\omega t)$.

66. (a) We want to calculate the square root of the average value of $[E(t)]^2 = [155\sin(120\pi t)]^2 = 155^2\sin^2(120\pi t)$. First, we calculate the average value itself, by integrating $[E(t)]^2$ over one cycle (between $t = 0$ and $t = \frac{1}{60}$, since there are 60 cycles per second) and dividing by $\left(\frac{1}{60} - 0\right)$:

$$\begin{aligned} [E(t)]^2_{\text{ave}} &= \frac{1}{1/60}\int_0^{1/60} [155^2\sin^2(120\pi t)]\,dt = 60\cdot 155^2\int_0^{1/60} \tfrac{1}{2}[1 - \cos(240\pi t)]\,dt \\ &= 60\cdot 155^2\left(\tfrac{1}{2}\right)\left[t - \tfrac{1}{240\pi}\sin(240\pi t)\right]_0^{1/60} = 60\cdot 155^2\left(\tfrac{1}{2}\right)\left[\left(\tfrac{1}{60} - 0\right) - (0 - 0)\right] = \tfrac{155^2}{2} \end{aligned}$$

The RMS value is just the square root of this quantity, which is $\frac{155}{\sqrt{2}} \approx 110$ V.

(b) $220 = \sqrt{[E(t)]^2_{\text{ave}}} \quad \Rightarrow$

$$220^2 = [E(t)]^2_{\text{ave}} = \tfrac{1}{1/60}\int_0^{1/60} A^2\sin^2(120\pi t)\,dt = 60A^2\int_0^{1/60}\tfrac{1}{2}[1-\cos(240\pi t)]\,dt$$

$$= 30A^2\left[t - \tfrac{1}{240\pi}\sin(240\pi t)\right]_0^{1/60} = 30A^2\left[\left(\tfrac{1}{60}-0\right)-(0-0)\right] = \tfrac{1}{2}A^2$$

Thus, $220^2 = \frac{1}{2}A^2 \quad \Rightarrow \quad A = 220\sqrt{2} \approx 311$ V.

67. Just note that the integrand is odd $[f(-x) = -f(x)]$.

Or: If $m \neq n$, calculate

$$\int_{-\pi}^{\pi} \sin mx\,\cos nx\,dx = \int_{-\pi}^{\pi}\tfrac{1}{2}[\sin(m-n)x + \sin(m+n)x]\,dx = \tfrac{1}{2}\left[-\frac{\cos(m-n)x}{m-n} - \frac{\cos(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$$

If $m = n$, then the first term in each set of brackets is zero.

68. $\int_{-\pi}^{\pi}\sin mx\,\sin nx\,dx = \int_{-\pi}^{\pi}\frac{1}{2}[\cos(m-n)x - \cos(m+n)x]\,dx$.

If $m \neq n$, this is equal to $\dfrac{1}{2}\left[\dfrac{\sin(m-n)x}{m-n} - \dfrac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$.

If $m = n$, we get $\int_{-\pi}^{\pi}\frac{1}{2}[1-\cos(m+n)x]\,dx = \left[\frac{1}{2}x\right]_{-\pi}^{\pi} - \left[\dfrac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi} = \pi - 0 = \pi$.

69. $\int_{-\pi}^{\pi}\cos mx\,\cos nx\,dx = \int_{-\pi}^{\pi}\frac{1}{2}[\cos(m-n)x + \cos(m+n)x]\,dx$.

If $m \neq n$, this is equal to $\dfrac{1}{2}\left[\dfrac{\sin(m-n)x}{m-n} + \dfrac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$.

If $m = n$, we get $\int_{-\pi}^{\pi}\frac{1}{2}[1+\cos(m+n)x]\,dx = \left[\frac{1}{2}x\right]_{-\pi}^{\pi} + \left[\dfrac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi} = \pi + 0 = \pi$.

70. $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,\sin mx\,dx = \dfrac{1}{\pi}\int_{-\pi}^{\pi}\left[\left(\sum_{n=1}^{m} a_n\sin nx\right)\sin mx\right]dx = \sum_{n=1}^{m}\dfrac{a_n}{\pi}\int_{-\pi}^{\pi}\sin mx\,\sin nx\,dx$. By Exercise 68, every term is zero except the mth one, and that term is $\dfrac{a_m}{\pi}\cdot\pi = a_m$.

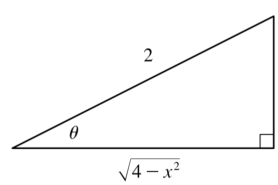
7.3 Trigonometric Substitution

1. Let $x = 2\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 2\cos\theta\,d\theta$ and

$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\,|\cos\theta| = 2\cos\theta$.

Thus, $\displaystyle\int\frac{dx}{x^2\sqrt{4-x^2}} = \int\frac{2\cos\theta}{4\sin^2\theta(2\cos\theta)}\,d\theta = \frac{1}{4}\int\csc^2\theta\,d\theta$

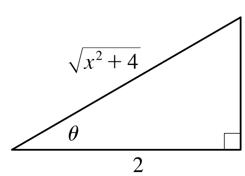
$$= -\frac{1}{4}\cot\theta + C = -\frac{\sqrt{4-x^2}}{4x} + C \quad \text{[see figure]}$$



2. Let $x = 2\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 2\sec^2\theta\,d\theta$ and

$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = \sqrt{4(\tan^2\theta+1)} = \sqrt{4\sec^2\theta} = 2\,|\sec\theta|$

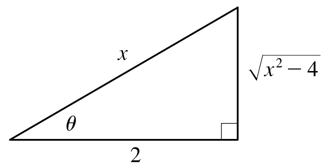
$= 2\sec\theta$ for the relevant values of θ.



[continued]

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+4}}\,dx &= \int \frac{8\tan^3\theta}{2\sec\theta}\,2\sec^2\theta\,d\theta = 8\int \tan^2\theta\,\sec\theta\,\tan\theta\,d\theta \\ &= 8\int(\sec^2\theta-1)\,\sec\theta\,\tan\theta\,d\theta = 8\int(u^2-1)\,du \qquad [u=\sec\theta] \\ &= 8\left(\frac{1}{3}u^3-u\right)+C = \frac{8}{3}\sec^3\theta - 8\sec\theta + C = \frac{8}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - 8\left(\frac{\sqrt{x^2+4}}{2}\right)+C \\ &= \tfrac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4}+C\end{aligned}$$

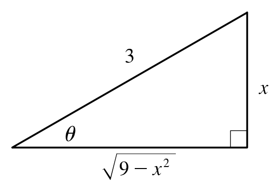
3. Let $x = 2\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = 2\sec\theta\,\tan\theta\,d\theta$ and



$$\begin{aligned}\sqrt{x^2-4} &= \sqrt{4\sec^2\theta-4} = \sqrt{4(\sec^2\theta-1)} \\ &= \sqrt{4\tan^2\theta} = 2\,|\tan\theta| = 2\tan\theta \quad \text{for the relevant values of } \theta\end{aligned}$$

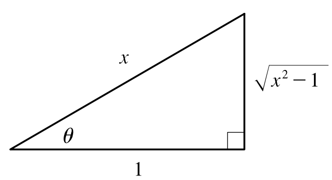
$$\begin{aligned}\int \frac{\sqrt{x^2-4}}{x}\,dx &= \int \frac{2\tan\theta}{2\sec\theta}\,2\sec\theta\,\tan\theta\,d\theta = 2\int \tan^2\theta\,d\theta \\ &= 2\int(\sec^2\theta-1)\,d\theta = 2\,(\tan\theta-\theta)+C = 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\left(\frac{x}{2}\right)\right]+C \\ &= \sqrt{x^2-4} - 2\sec^{-1}\left(\frac{x}{2}\right)+C\end{aligned}$$

4. Let $x = 3\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 3\cos\theta\,d\theta$ and $\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\,|\cos\theta| = 3\cos\theta$.



$$\begin{aligned}\int \frac{x^2}{\sqrt{9-x^2}}\,dx &= \int \frac{9\sin^2\theta}{3\cos\theta}\,3\cos\theta\,d\theta = 9\int \sin^2\theta\,d\theta \\ &= 9\int \tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{9}{2}\left(\theta - \tfrac{1}{2}\sin 2\theta\right)+C = \tfrac{9}{2}\theta - \tfrac{9}{4}(2\sin\theta\,\cos\theta)+C \\ &= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}+C = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2}+C\end{aligned}$$

5. Let $x = \sec\theta$, where $0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = \sec\theta\,\tan\theta\,d\theta$ and $\sqrt{x^2-1} = \sqrt{\sec^2\theta-1} = \sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$ for the relevant values of θ, so



$$\begin{aligned}\int \frac{\sqrt{x^2-1}}{x^4}\,dx &= \int \frac{\tan\theta}{\sec^4\theta}\,\sec\theta\,\tan\theta\,d\theta = \int \tan^2\theta\,\cos^3\theta\,d\theta \\ &= \int \sin^2\theta\,\cos\theta\,d\theta \overset{s}{=} \int u^2\,du = \tfrac{1}{3}u^3+C = \tfrac{1}{3}\sin^3\theta+C \\ &= \frac{1}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3+C = \frac{1}{3}\,\frac{(x^2-1)^{3/2}}{x^3}+C\end{aligned}$$

6. Let $u = 36-x^2$, so $du = -2x\,dx$. When $x=0$, $u=36$; when $x=3$, $u=27$. Thus,

$$\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx = \int_{36}^{27} \frac{1}{\sqrt{u}}\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\Big[2\sqrt{u}\Big]_{36}^{27} = -\left(\sqrt{27}-\sqrt{36}\right) = 6-3\sqrt{3}$$

[continued]

Another method: Let $x = 6\sin\theta$, so $dx = 6\cos\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 3 \;\Rightarrow\; \theta = \frac{\pi}{6}$. Then

$$\int_0^3 \frac{x}{\sqrt{36 - x^2}}\,dx = \int_0^{\pi/6} \frac{6\sin\theta}{\sqrt{36(1 - \sin^2\theta)}}\,6\cos\theta\,d\theta = \int_0^{\pi/6} \frac{6\sin\theta}{6\cos\theta}\,6\cos\theta\,d\theta = 6\int_0^{\pi/6} \sin\theta\,d\theta$$

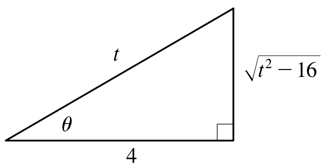
$$= 6\Big[-\cos\theta\Big]_0^{\pi/6} = 6\Big(-\tfrac{\sqrt{3}}{2} + 1\Big) = 6 - 3\sqrt{3}$$

7. Let $x = a\tan\theta$, where $a > 0$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = a\sec^2\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = a \;\Rightarrow\; \theta = \frac{\pi}{4}$. Thus,

$$\int_0^a \frac{dx}{(a^2 + x^2)^{3/2}} = \int_0^{\pi/4} \frac{a\sec^2\theta\,d\theta}{[a^2(1 + \tan^2\theta)]^{3/2}} = \int_0^{\pi/4} \frac{a\sec^2\theta\,d\theta}{a^3\sec^3\theta} = \frac{1}{a^2}\int_0^{\pi/4} \cos\theta\,d\theta = \frac{1}{a^2}\Big[\sin\theta\Big]_0^{\pi/4}$$

$$= \frac{1}{a^2}\left(\frac{\sqrt{2}}{2} - 0\right) = \frac{1}{\sqrt{2}\,a^2}.$$

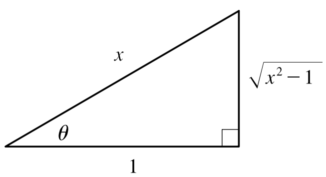
8. Let $t = 4\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dt = 4\sec\theta\,\tan\theta\,d\theta$ and $\sqrt{t^2 - 16} = \sqrt{16\sec^2\theta - 16} = \sqrt{16\tan^2\theta} = 4\tan\theta$ for the relevant values of θ, so



$$\int \frac{dt}{t^2\sqrt{t^2 - 16}} = \int \frac{4\sec\theta\,\tan\theta\,d\theta}{16\sec^2\theta \cdot 4\tan\theta} = \frac{1}{16}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{16}\int \cos\theta\,d\theta$$

$$= \frac{1}{16}\sin\theta + C = \frac{1}{16}\,\frac{\sqrt{t^2 - 16}}{t} + C = \frac{\sqrt{t^2 - 16}}{16t} + C$$

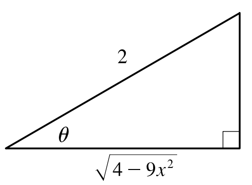
9. Let $x = \sec\theta$, so $dx = \sec\theta\,\tan\theta\,d\theta$, $x = 2 \;\Rightarrow\; \theta = \frac{\pi}{3}$, and $x = 3 \;\Rightarrow\; \theta = \sec^{-1} 3$. Then



$$\int_2^3 \frac{dx}{(x^2 - 1)^{3/2}} = \int_{\pi/3}^{\sec^{-1} 3} \frac{\sec\theta\,\tan\theta\,d\theta}{\tan^3\theta} = \int_{\pi/3}^{\sec^{-1} 3} \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$\overset{s}{=} \int_{\sqrt{3}/2}^{\sqrt{8}/3} \frac{1}{u^2}\,du = \left[-\frac{1}{u}\right]_{\sqrt{3}/2}^{\sqrt{8}/3} = \frac{-3}{\sqrt{8}} + \frac{2}{\sqrt{3}} = -\frac{3}{4}\sqrt{2} + \frac{2}{3}\sqrt{3}$$

10. Let $x = \frac{2}{3}\sin\theta$, so $dx = \frac{2}{3}\cos\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = \frac{2}{3} \;\Rightarrow\; \theta = \frac{\pi}{2}$. Thus,



$$\int_0^{2/3} \sqrt{4 - 9x^2}\,dx = \int_0^{\pi/2} \sqrt{4 - 9\cdot\frac{4}{9}\sin^2\theta}\;\frac{2}{3}\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 2\cos\theta \cdot \frac{2}{3}\cos\theta\,d\theta = \frac{4}{3}\int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\theta)\,d\theta = \frac{2}{3}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{2}{3}\left[\left(\frac{\pi}{2} + 0\right) - (0 + 0)\right] = \frac{\pi}{3}$$

11. $\int_0^{1/2} x\sqrt{1 - 4x^2}\,dx = \int_1^0 u^{1/2}\left(-\frac{1}{8}\,du\right)$ $\qquad \left[\begin{array}{c} u = 1 - 4x^2, \\ du = -8x\,dx \end{array}\right]$

$$= \tfrac{1}{8}\left[\tfrac{2}{3}u^{3/2}\right]_0^1 = \tfrac{1}{12}(1 - 0) = \tfrac{1}{12}$$

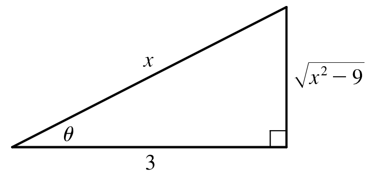
12. Let $t = 2\tan\theta$, so $dt = 2\sec^2\theta\,d\theta$, $t = 0 \;\Rightarrow\; \theta = 0$, and $t = 2 \;\Rightarrow\; \theta = \frac{\pi}{4}$. Thus,

$$\int_0^2 \frac{dt}{\sqrt{4+t^2}} = \int_0^{\pi/4} \frac{2\sec^2\theta\,d\theta}{\sqrt{4+4\tan^2\theta}} = \int_0^{\pi/4} \frac{2\sec^2\theta\,d\theta}{2\sec\theta} = \int_0^{\pi/4} \sec\theta\,d\theta = \Big[\ln|\sec\theta+\tan\theta|\Big]_0^{\pi/4}$$

$$= \ln\left|\sqrt{2}+1\right| - \ln|1+0| = \ln(\sqrt{2}+1)$$

13. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then

$dx = 3\sec\theta\,\tan\theta\,d\theta$ and $\sqrt{x^2-9} = 3\tan\theta$, so



$$\int \frac{\sqrt{x^2-9}}{x^3}\,dx = \int \frac{3\tan\theta}{27\sec^3\theta}\,3\sec\theta\,\tan\theta\,d\theta = \frac{1}{3}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$= \tfrac{1}{3}\int \sin^2\theta\,d\theta = \tfrac{1}{3}\int \tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{6}\theta - \tfrac{1}{12}\sin 2\theta + C = \tfrac{1}{6}\theta - \tfrac{1}{6}\sin\theta\,\cos\theta + C$$

$$= \frac{1}{6}\sec^{-1}\!\left(\frac{x}{3}\right) - \frac{1}{6}\frac{\sqrt{x^2-9}}{x}\,\frac{3}{x} + C = \frac{1}{6}\sec^{-1}\!\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C$$

14. Let $x = \tan\theta$, so $dx = \sec^2\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 1 \;\Rightarrow\; \theta = \frac{\pi}{4}$. Then

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{(\tan^2\theta+1)^2} = \int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^2}$$

$$= \int_0^{\pi/4} \cos^2\theta\,d\theta = \int_0^{\pi/4} \frac{1}{2}\left(1+\cos 2\theta\right)\,d\theta$$

$$= \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \tfrac{1}{2}\left[\left(\tfrac{\pi}{4}+\tfrac{1}{2}\right) - 0\right] = \tfrac{\pi}{8} + \tfrac{1}{4}$$

15. Let $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$ and $x = a \;\Rightarrow\; \theta = \frac{\pi}{2}$. Then

$$\textstyle\int_0^a x^2\sqrt{a^2-x^2}\,dx = \int_0^{\pi/2} a^2\sin^2\theta\,(a\cos\theta)\,a\cos\theta\,d\theta = a^4\int_0^{\pi/2}\sin^2\theta\,\cos^2\theta\,d\theta$$

$$= a^4\int_0^{\pi/2}\left[\tfrac{1}{2}(2\sin\theta\,\cos\theta)\right]^2\,d\theta = \frac{a^4}{4}\int_0^{\pi/2}\sin^2 2\theta\,d\theta = \frac{a^4}{4}\int_0^{\pi/2}\tfrac{1}{2}(1-\cos 4\theta)\,d\theta$$

$$= \frac{a^4}{8}\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{a^4}{8}\left[\left(\frac{\pi}{2}-0\right)-0\right] = \frac{\pi}{16}a^4$$

16. Let $x = \frac{1}{3}\sec\theta$, so $dx = \frac{1}{3}\sec\theta\,\tan\theta\,d\theta$, $x = \sqrt{2}/3 \;\Rightarrow\; \theta = \frac{\pi}{4}$, $x = \frac{2}{3} \;\Rightarrow\; \theta = \frac{\pi}{3}$. Then

$$\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}} = \int_{\pi/4}^{\pi/3} \frac{\frac{1}{3}\sec\theta\,\tan\theta\,d\theta}{\left(\frac{1}{3}\right)^5\sec^5\theta\,\tan\theta} = 3^4\int_{\pi/4}^{\pi/3}\cos^4\theta\,d\theta = 81\int_{\pi/4}^{\pi/3}\left[\tfrac{1}{2}(1+\cos 2\theta)\right]^2\,d\theta$$

$$= \tfrac{81}{4}\textstyle\int_{\pi/4}^{\pi/3}(1+2\cos 2\theta+\cos^2 2\theta)\,d\theta = \tfrac{81}{4}\int_{\pi/4}^{\pi/3}\left[1+2\cos 2\theta+\tfrac{1}{2}(1+\cos 4\theta)\right]d\theta$$

$$= \tfrac{81}{4}\textstyle\int_{\pi/4}^{\pi/3}\left(\tfrac{3}{2}+2\cos 2\theta+\tfrac{1}{2}\cos 4\theta\right)d\theta = \tfrac{81}{4}\left[\tfrac{3}{2}\theta+\sin 2\theta+\tfrac{1}{8}\sin 4\theta\right]_{\pi/4}^{\pi/3}$$

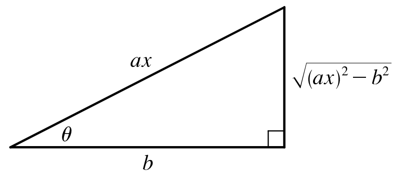
$$= \tfrac{81}{4}\left[\left(\tfrac{\pi}{2}+\tfrac{\sqrt{3}}{2}-\tfrac{\sqrt{3}}{16}\right)-\left(\tfrac{3\pi}{8}+1+0\right)\right] = \tfrac{81}{4}\left(\tfrac{\pi}{8}+\tfrac{7}{16}\sqrt{3}-1\right)$$

17. Let $u = x^2-7$, so $du = 2x\,dx$. Then $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx = \frac{1}{2}\int\frac{1}{\sqrt{u}}\,du = \tfrac{1}{2}\cdot 2\sqrt{u}+C = \sqrt{x^2-7}+C.$

18. Let $ax = b\sec\theta$, so $(ax)^2 = b^2\sec^2\theta \quad\Rightarrow$

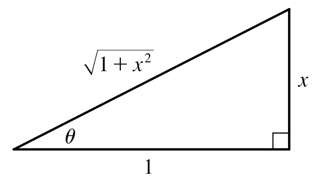
$(ax)^2 - b^2 = b^2\sec^2\theta - b^2 = b^2(\sec^2\theta - 1) = b^2\tan^2\theta$.

So $\sqrt{(ax)^2 - b^2} = b\tan\theta$, $dx = \dfrac{b}{a}\sec\theta\,\tan\theta\,d\theta$, and



$$\begin{aligned}\int\frac{dx}{[(ax)^2-b^2]^{3/2}} &= \int\frac{\frac{b}{a}\sec\theta\,\tan\theta}{b^3\tan^3\theta}\,d\theta = \frac{1}{ab^2}\int\frac{\sec\theta}{\tan^2\theta}\,d\theta \\ &= \frac{1}{ab^2}\int\frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{ab^2}\int\csc\theta\,\cot\theta\,d\theta \\ &= -\frac{1}{ab^2}\csc\theta + C = -\frac{1}{ab^2}\frac{ax}{\sqrt{(ax)^2-b^2}} + C \\ &= -\frac{x}{b^2\sqrt{(ax)^2-b^2}} + C\end{aligned}$$

19. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$ and $\sqrt{1+x^2} = \sec\theta$, so

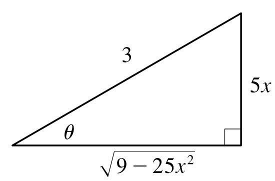


$$\begin{aligned}\int\frac{\sqrt{1+x^2}}{x}\,dx &= \int\frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = \int\frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\,d\theta \\ &= \int(\csc\theta + \sec\theta\,\tan\theta)\,d\theta \\ &= \ln|\csc\theta - \cot\theta| + \sec\theta + C \qquad \text{[by Exercise 7.2.39]} \\ &= \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \frac{\sqrt{1+x^2}}{1} + C = \ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + \sqrt{1+x^2} + C\end{aligned}$$

20. Let $u = 1 + x^2$, so $du = 2x\,dx$. Then

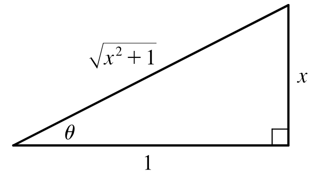
$$\int\frac{x}{\sqrt{1+x^2}}\,dx = \int\frac{1}{\sqrt{u}}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\cdot 2u^{1/2} + C = \sqrt{1+x^2} + C$$

21. Let $x = \frac{3}{5}\sin\theta$, so $dx = \frac{3}{5}\cos\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 0.6 \;\Rightarrow\; \theta = \frac{\pi}{2}$. Then



$$\begin{aligned}\int_0^{0.6}\frac{x^2}{\sqrt{9-25x^2}}\,dx &= \int_0^{\pi/2}\frac{\left(\frac{3}{5}\right)^2\sin^2\theta}{3\cos\theta}\left(\tfrac{3}{5}\cos\theta\,d\theta\right) = \tfrac{9}{125}\int_0^{\pi/2}\sin^2\theta\,d\theta \\ &= \tfrac{9}{125}\int_0^{\pi/2}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{9}{250}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} \\ &= \tfrac{9}{250}\left[\left(\tfrac{\pi}{2} - 0\right) - 0\right] = \tfrac{9}{500}\pi\end{aligned}$$

22. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$, $\sqrt{x^2+1} = \sec\theta$ and $x = 0 \;\Rightarrow\; \theta = 0$, $x = 1 \;\Rightarrow\; \theta = \frac{\pi}{4}$, so



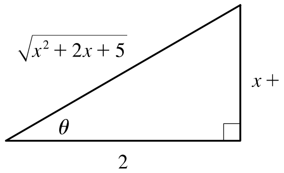
$$\begin{aligned}\int_0^1\sqrt{x^2+1}\,dx &= \int_0^{\pi/4}\sec\theta\,\sec^2\theta\,d\theta = \int_0^{\pi/4}\sec^3\theta\,d\theta \\ &= \tfrac{1}{2}\Big[\sec\theta\,\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4} \qquad \text{[by Example 7.2.8]} \\ &= \tfrac{1}{2}\left[\sqrt{2}\cdot 1 + \ln\left(1+\sqrt{2}\right) - 0 - \ln(1+0)\right] = \tfrac{1}{2}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]\end{aligned}$$

23. $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{dx}{\sqrt{(x+1)^2+4}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\tan^2\theta+4}}$ $\quad \left[\begin{array}{r} x+1 = 2\tan\theta, \\ dx = 2\sec^2\theta\, d\theta \end{array}\right]$

$$= \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta+\tan\theta| + C_1$$

$$= \ln\left|\frac{\sqrt{x^2+2x+5}}{2} + \frac{x+1}{2}\right| + C_1,$$

or $\ln\left|\sqrt{x^2+2x+5}+x+1\right| + C$, where $C = C_1 - \ln 2$.



24. $\displaystyle\int_0^1 \sqrt{x-x^2}\, dx = \int_0^1 \sqrt{\tfrac{1}{4} - \left(x^2 - x + \tfrac{1}{4}\right)}\, dx = \int_0^1 \sqrt{\tfrac{1}{4} - \left(x - \tfrac{1}{2}\right)^2}\, dx$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{\tfrac{1}{4} - \tfrac{1}{4}\sin^2\theta}\; \tfrac{1}{2}\cos\theta\, d\theta \qquad \left[\begin{array}{r} x - \frac{1}{2} = \frac{1}{2}\sin\theta, \\ dx = \frac{1}{2}\cos\theta\, d\theta \end{array}\right]$$

$$= 2\int_0^{\pi/2} \tfrac{1}{2}\cos\theta\; \tfrac{1}{2}\cos\theta\, d\theta = \tfrac{1}{2}\int_0^{\pi/2} \cos^2\theta\, d\theta = \tfrac{1}{2}\int_0^{\pi/2} \tfrac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \tfrac{1}{4}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{1}{4}\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{8}$$

25. $\int x^2\sqrt{3+2x-x^2}\, dx = \int x^2\sqrt{4-(x^2+2x+1)}\, dx = \int x^2\sqrt{2^2-(x-1)^2}\, dx$

$$= \int (1+2\sin\theta)^2\sqrt{4\cos^2\theta}\; 2\cos\theta\, d\theta \qquad \left[\begin{array}{r} x-1 = 2\sin\theta, \\ dx = 2\cos\theta\, d\theta \end{array}\right]$$

$$= \int (1+4\sin\theta+4\sin^2\theta)\, 4\cos^2\theta\, d\theta$$

$$= 4\int (\cos^2\theta + 4\sin\theta\,\cos^2\theta + 4\sin^2\theta\,\cos^2\theta)\, d\theta$$

$$= 4\int \tfrac{1}{2}(1+\cos 2\theta)\, d\theta + 4\int 4\sin\theta\,\cos^2\theta\, d\theta + 4\int (2\sin\theta\,\cos\theta)^2\, d\theta$$

$$= 2\int (1+\cos 2\theta)\, d\theta + 16\int \sin\theta\,\cos^2\theta\, d\theta + 4\int \sin^2 2\theta\, d\theta$$

$$= 2\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + 16\left(-\tfrac{1}{3}\cos^3\theta\right) + 4\int \tfrac{1}{2}(1-\cos 4\theta)\, d\theta$$

$$= 2\theta + \sin 2\theta - \tfrac{16}{3}\cos^3\theta + 2\left(\theta - \tfrac{1}{4}\sin 4\theta\right) + C$$

$$= 4\theta - \tfrac{1}{2}\sin 4\theta + \sin 2\theta - \tfrac{16}{3}\cos^3\theta + C$$

$$= 4\theta - \tfrac{1}{2}(2\sin 2\theta\,\cos 2\theta) + \sin 2\theta - \tfrac{16}{3}\cos^3\theta + C$$

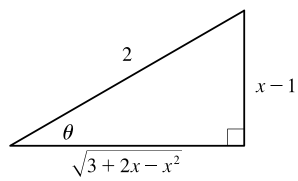
$$= 4\theta + \sin 2\theta(1-\cos 2\theta) - \tfrac{16}{3}\cos^3\theta + C$$

$$= 4\theta + (2\sin\theta\,\cos\theta)(2\sin^2\theta) - \tfrac{16}{3}\cos^3\theta + C$$

$$= 4\theta + 4\sin^3\theta\,\cos\theta - \tfrac{16}{3}\cos^3\theta + C$$

$$= 4\sin^{-1}\left(\frac{x-1}{2}\right) + 4\left(\frac{x-1}{2}\right)^3\frac{\sqrt{3+2x-x^2}}{2} - \frac{16}{3}\frac{(3+2x-x^2)^{3/2}}{2^3} + C$$

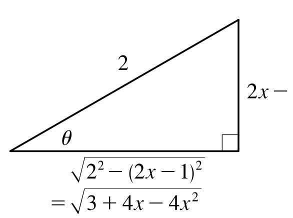
$$= 4\sin^{-1}\left(\frac{x-1}{2}\right) + \frac{1}{4}(x-1)^3\sqrt{3+2x-x^2} - \frac{2}{3}(3+2x-x^2)^{3/2} + C$$



26. $3+4x-4x^2 = -(4x^2-4x+1)+4 = 2^2-(2x-1)^2$.

Let $2x-1 = 2\sin\theta$, so $2\,dx = 2\cos\theta\, d\theta$ and $\sqrt{3+4x-4x^2} = 2\cos\theta$.

Then



$$\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx = \int \frac{\left[\frac{1}{2}(1+2\sin\theta)\right]^2}{(2\cos\theta)^3}\cos\theta\,d\theta$$

$$= \frac{1}{32}\int \frac{1+4\sin\theta+4\sin^2\theta}{\cos^2\theta}\,d\theta = \frac{1}{32}\int (\sec^2\theta + 4\tan\theta\,\sec\theta + 4\tan^2\theta)\,d\theta$$

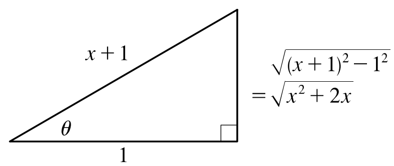
$$= \tfrac{1}{32}\int [\sec^2\theta + 4\tan\theta\,\sec\theta + 4(\sec^2\theta - 1)]\,d\theta$$

$$= \tfrac{1}{32}\int (5\sec^2\theta + 4\tan\theta\,\sec\theta - 4)\,d\theta = \tfrac{1}{32}(5\tan\theta + 4\sec\theta - 4\theta) + C$$

$$= \frac{1}{32}\left[5\cdot\frac{2x-1}{\sqrt{3+4x-4x^2}} + 4\cdot\frac{2}{\sqrt{3+4x-4x^2}} - 4\cdot\sin^{-1}\left(\frac{2x-1}{2}\right)\right] + C$$

$$= \frac{10x+3}{32\sqrt{3+4x-4x^2}} - \frac{1}{8}\sin^{-1}\left(\frac{2x-1}{2}\right) + C$$

27. $x^2+2x = (x^2+2x+1)-1 = (x+1)^2-1$. Let $x+1 = 1\sec\theta$, so $dx = \sec\theta\,\tan\theta\,d\theta$ and $\sqrt{x^2+2x} = \tan\theta$. Then



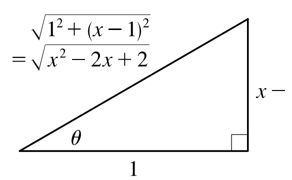
$$\int \sqrt{x^2+2x}\,dx = \int \tan\theta\,(\sec\theta\,\tan\theta\,d\theta) = \int \tan^2\theta\,\sec\theta\,d\theta$$

$$= \int (\sec^2\theta - 1)\,\sec\theta\,d\theta = \int \sec^3\theta\,d\theta - \int \sec\theta\,d\theta$$

$$= \tfrac{1}{2}\sec\theta\,\tan\theta + \tfrac{1}{2}\ln|\sec\theta+\tan\theta| - \ln|\sec\theta+\tan\theta| + C$$

$$= \tfrac{1}{2}\sec\theta\,\tan\theta - \tfrac{1}{2}\ln|\sec\theta+\tan\theta| + C = \tfrac{1}{2}(x+1)\sqrt{x^2+2x} - \tfrac{1}{2}\ln\left|x+1+\sqrt{x^2+2x}\right| + C$$

28. $x^2-2x+2 = (x^2-2x+1)+1 = (x-1)^2+1$. Let $x-1 = 1\tan\theta$, so $dx = \sec^2\theta\,d\theta$ and $\sqrt{x^2-2x+2} = \sec\theta$. Then



$$\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx = \int \frac{(\tan\theta+1)^2+1}{\sec^4\theta}\,\sec^2\theta\,d\theta$$

$$= \int \frac{\tan^2\theta + 2\tan\theta + 2}{\sec^2\theta}\,d\theta$$

$$= \int (\sin^2\theta + 2\sin\theta\,\cos\theta + 2\cos^2\theta)\,d\theta = \int (1 + 2\sin\theta\,\cos\theta + \cos^2\theta)\,d\theta$$

$$= \int \left[1 + 2\sin\theta\,\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right]\,d\theta = \int \left(\tfrac{3}{2} + 2\sin\theta\,\cos\theta + \tfrac{1}{2}\cos 2\theta\right)\,d\theta$$

$$= \tfrac{3}{2}\theta + \sin^2\theta + \tfrac{1}{4}\sin 2\theta + C = \tfrac{3}{2}\theta + \sin^2\theta + \tfrac{1}{2}\sin\theta\,\cos\theta + C$$

$$= \frac{3}{2}\tan^{-1}\left(\frac{x-1}{1}\right) + \frac{(x-1)^2}{x^2-2x+2} + \frac{1}{2}\frac{x-1}{\sqrt{x^2-2x+2}}\frac{1}{\sqrt{x^2-2x+2}} + C$$

$$= \frac{3}{2}\tan^{-1}(x-1) + \frac{2(x^2-2x+1)+x-1}{2(x^2-2x+2)} + C = \frac{3}{2}\tan^{-1}(x-1) + \frac{2x^2-3x+1}{2(x^2-2x+2)} + C$$

We can write the answer as

$$\frac{3}{2}\tan^{-1}(x-1) + \frac{(2x^2-4x+4)+x-3}{2(x^2-2x+2)} + C = \frac{3}{2}\tan^{-1}(x-1) + 1 + \frac{x-3}{2(x^2-2x+2)} + C$$

$$= \frac{3}{2}\tan^{-1}(x-1) + \frac{x-3}{2(x^2-2x+2)} + C_1, \text{ where } C_1 = 1 + C$$

29. Let $u = x^2$, $du = 2x\,dx$. Then

$$\int x\sqrt{1-x^4}\,dx = \int \sqrt{1-u^2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int \cos\theta\cdot\cos\theta\,d\theta \qquad \left[\begin{array}{c}\text{where } u=\sin\theta,\ du=\cos\theta\,d\theta,\\ \text{and } \sqrt{1-u^2}=\cos\theta\end{array}\right]$$

$$= \tfrac{1}{2}\int \tfrac{1}{2}(1+\cos 2\theta)\,d\theta = \tfrac{1}{4}\theta + \tfrac{1}{8}\sin 2\theta + C = \tfrac{1}{4}\theta + \tfrac{1}{4}\sin\theta\,\cos\theta + C$$

$$= \tfrac{1}{4}\sin^{-1}u + \tfrac{1}{4}u\sqrt{1-u^2} + C = \tfrac{1}{4}\sin^{-1}(x^2) + \tfrac{1}{4}x^2\sqrt{1-x^4} + C$$

30. Let $u = \sin t$, $du = \cos t\,dt$. Then

$$\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt = \int_0^1 \frac{1}{\sqrt{1+u^2}}\,du = \int_0^{\pi/4} \frac{1}{\sec\theta}\sec^2\theta\,d\theta \qquad \left[\begin{array}{c}\text{where } u=\tan\theta,\ du=\sec^2\theta\,d\theta,\\ \text{and } \sqrt{1+u^2}=\sec\theta\end{array}\right]$$

$$= \int_0^{\pi/4}\sec\theta\,d\theta = \Big[\ln|\sec\theta+\tan\theta|\Big]_0^{\pi/4} \qquad \text{[by (1) in Section 7.2]}$$

$$= \ln\left(\sqrt{2}+1\right) - \ln(1+0) = \ln\left(\sqrt{2}+1\right)$$

31. (a) Let $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $\sqrt{x^2+a^2} = a\sec\theta$ and

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta\,d\theta}{a\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C_1 = \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C_1$$

$$= \ln\left(x + \sqrt{x^2+a^2}\right) + C \quad \text{where } C = C_1 - \ln|a|$$

(b) Let $x = a\sinh t$, so that $dx = a\cosh t\,dt$ and $\sqrt{x^2+a^2} = a\cosh t$. Then

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\cosh t\,dt}{a\cosh t} = t + C = \sinh^{-1}\frac{x}{a} + C.$$

32. (a) Let $x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

$$I = \int \frac{x^2}{(x^2+a^2)^{3/2}}\,dx = \int \frac{a^2\tan^2\theta}{a^3\sec^3\theta}\,a\sec^2\theta\,d\theta = \int \frac{\tan^2\theta}{\sec\theta}\,d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta$$

$$= \int (\sec\theta - \cos\theta)\,d\theta = \ln|\sec\theta+\tan\theta| - \sin\theta + C$$

$$= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| - \frac{x}{\sqrt{x^2+a^2}} + C = \ln\left(x+\sqrt{x^2+a^2}\right) - \frac{x}{\sqrt{x^2+a^2}} + C_1$$

(b) Let $x = a\sinh t$. Then

$$I = \int \frac{a^2\sinh^2 t}{a^3\cosh^3 t}\,a\cosh t\,dt = \int \tanh^2 t\,dt = \int (1-\operatorname{sech}^2 t)\,dt = t - \tanh t + C$$

$$= \sinh^{-1}\frac{x}{a} - \frac{x}{\sqrt{a^2+x^2}} + C$$

33. The average value of $f(x) = \sqrt{x^2-1}/x$ on the interval $[1, 7]$ is

$$\frac{1}{7-1}\int_1^7 \frac{\sqrt{x^2-1}}{x}\,dx = \frac{1}{6}\int_0^\alpha \frac{\tan\theta}{\sec\theta}\cdot\sec\theta\tan\theta\,d\theta \qquad \left[\begin{array}{c}\text{where } x=\sec\theta,\ dx=\sec\theta\tan\theta\,d\theta,\\ \sqrt{x^2-1}=\tan\theta, \text{ and } \alpha=\sec^{-1}7\end{array}\right]$$

$$= \tfrac{1}{6}\int_0^\alpha \tan^2\theta\,d\theta = \tfrac{1}{6}\int_0^\alpha (\sec^2\theta - 1)\,d\theta = \tfrac{1}{6}\Big[\tan\theta - \theta\Big]_0^\alpha$$

$$= \tfrac{1}{6}(\tan\alpha - \alpha) = \tfrac{1}{6}\left(\sqrt{48} - \sec^{-1}7\right)$$

34. $9x^2 - 4y^2 = 36 \quad \Rightarrow \quad y = \pm\frac{3}{2}\sqrt{x^2-4} \quad \Rightarrow$

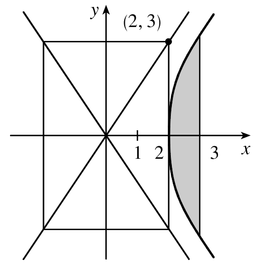
area $= 2\int_2^3 \frac{3}{2}\sqrt{x^2-4}\,dx = 3\int_2^3 \sqrt{x^2-4}\,dx$

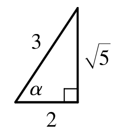
$$= 3\int_0^\alpha 2\tan\theta\, 2\sec\theta\,\tan\theta\,d\theta \qquad \left[\begin{array}{c}\text{where } x = 2\sec\theta,\\ dx = 2\sec\theta\,\tan\theta\,d\theta,\\ \alpha = \sec^{-1}\left(\frac{3}{2}\right)\end{array}\right]$$

$$= 12\int_0^\alpha \left(\sec^2\theta - 1\right)\sec\theta\,d\theta = 12\int_0^\alpha \left(\sec^3\theta - \sec\theta\right)d\theta$$

$$= 12\left[\tfrac{1}{2}(\sec\theta\,\tan\theta + \ln|\sec\theta+\tan\theta|) - \ln|\sec\theta+\tan\theta|\right]_0^\alpha$$

$$= 6\Big[\sec\theta\,\tan\theta - \ln|\sec\theta+\tan\theta|\Big]_0^\alpha = 6\left[\tfrac{3\sqrt{5}}{4} - \ln\left(\tfrac{3}{2}+\tfrac{\sqrt{5}}{2}\right)\right] = \tfrac{9\sqrt{5}}{2} - 6\ln\left(\tfrac{3+\sqrt{5}}{2}\right)$$





35. Area of $\triangle POQ = \frac{1}{2}(r\cos\theta)(r\sin\theta) = \frac{1}{2}r^2\sin\theta\,\cos\theta$. Area of region $PQR = \int_{r\cos\theta}^{r}\sqrt{r^2-x^2}\,dx$.

Let $x = r\cos u \quad \Rightarrow \quad dx = -r\sin u\,du$ for $\theta \le u \le \frac{\pi}{2}$. Then we obtain

$$\int\sqrt{r^2-x^2}\,dx = \int r\sin u\,(-r\sin u)\,du = -r^2\int\sin^2 u\,du = -\tfrac{1}{2}r^2(u - \sin u\,\cos u) + C$$
$$= -\tfrac{1}{2}r^2\cos^{-1}(x/r) + \tfrac{1}{2}x\sqrt{r^2-x^2} + C$$

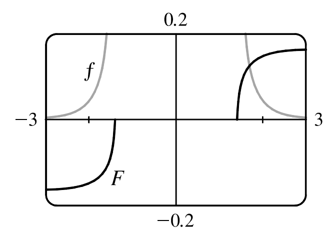
so

$$\text{area of region } PQR = \tfrac{1}{2}\left[-r^2\cos^{-1}(x/r) + x\sqrt{r^2-x^2}\,\right]_{r\cos\theta}^{r}$$
$$= \tfrac{1}{2}\left[0 - (-r^2\theta + r\cos\theta\, r\sin\theta)\right] = \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta\,\cos\theta$$

and thus, (area of sector POR) = (area of $\triangle POQ$) + (area of region PQR) $= \frac{1}{2}r^2\theta$.

36. Let $x = \sqrt{2}\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$, so $dx = \sqrt{2}\,\sec\theta\tan\theta\,d\theta$. Then

$$\int\frac{dx}{x^4\sqrt{x^2-2}} = \int\frac{\sqrt{2}\sec\theta\,\tan\theta\,d\theta}{4\sec^4\theta\,\sqrt{2}\,\tan\theta}$$
$$= \tfrac{1}{4}\int\cos^3\theta\,d\theta = \tfrac{1}{4}\int\left(1-\sin^2\theta\right)\cos\theta\,d\theta$$
$$= \tfrac{1}{4}\left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right] + C \qquad [\text{substitute } u = \sin\theta]$$
$$= \frac{1}{4}\left[\frac{\sqrt{x^2-2}}{x} - \frac{\left(x^2-2\right)^{3/2}}{3x^3}\right] + C$$

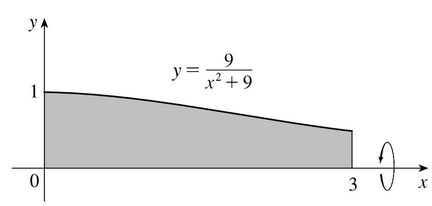


From the graph, it appears that our answer is reasonable. [Notice that $f(x)$ is large when F increases rapidly and small when F levels out.]

37. Use disks about the x-axis:

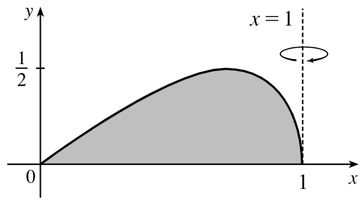
$$V = \int_0^3 \pi\left(\frac{9}{x^2+9}\right)^2 dx = 81\pi\int_0^3 \frac{1}{\left(x^2+9\right)^2}\,dx$$

Let $x = 3\tan\theta$, so $dx = 3\sec^2\theta\,d\theta$, $x = 0 \quad \Rightarrow \quad \theta = 0$ and $x = 3 \quad \Rightarrow \quad \theta = \frac{\pi}{4}$. Thus,



$$V = 81\pi\int_0^{\pi/4}\frac{1}{(9\sec^2\theta)^2}\,3\sec^2\theta\,d\theta = 3\pi\int_0^{\pi/4}\cos^2\theta\,d\theta = 3\pi\int_0^{\pi/4}\frac{1}{2}(1+\cos 2\theta)\,d\theta$$
$$= \tfrac{3\pi}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \tfrac{3\pi}{2}\left[\left(\tfrac{\pi}{4}+\tfrac{1}{2}\right) - 0\right] = \tfrac{3}{8}\pi^2 + \tfrac{3}{4}\pi$$

38. Use shells about $x = 1$:



$V = \int_0^1 2\pi(1-x)\,x\sqrt{1-x^2}\,dx$

$= 2\pi \int_0^1 x\sqrt{1-x^2}\,dx - 2\pi \int_0^1 x^2\sqrt{1-x^2}\,dx = 2\pi V_1 - 2\pi V_2$

For V_1, let $u = 1 - x^2$, so $du = -2x\,dx$, and

$V_1 = \int_1^0 \sqrt{u}\,\left(-\frac{1}{2}\,du\right) = \frac{1}{2}\int_0^1 u^{1/2}\,du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_0^1 = \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}.$

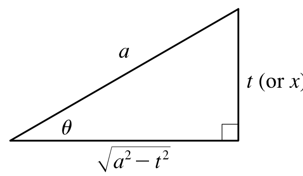
For V_2, let $x = \sin\theta$, so $dx = \cos\theta\,d\theta$, and

$$V_2 = \int_0^{\pi/2} \sin^2\theta\,\sqrt{\cos^2\theta}\,\cos\theta\,d\theta = \int_0^{\pi/2} \sin^2\theta\,\cos^2\theta\,d\theta = \int_0^{\pi/2} \tfrac{1}{4}(2\sin\theta\,\cos\theta)^2\,d\theta$$

$$= \tfrac{1}{4}\int_0^{\pi/2} \sin^2 2\theta\,d\theta = \tfrac{1}{4}\int_0^{\pi/2} \tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{8}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{1}{8}\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{16}$$

Thus, $V = 2\pi\left(\frac{1}{3}\right) - 2\pi\left(\frac{\pi}{16}\right) = \frac{2}{3}\pi - \frac{1}{8}\pi^2.$

39. (a) Let $t = a\sin\theta$, $dt = a\cos\theta\,d\theta$, $t = 0 \;\Rightarrow\; \theta = 0$ and $t = x \;\Rightarrow$ $\theta = \sin^{-1}(x/a)$. Then



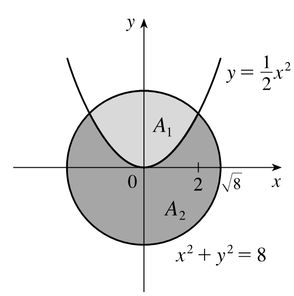
$$\int_0^x \sqrt{a^2 - t^2}\,dt = \int_0^{\sin^{-1}(x/a)} a\cos\theta\,(a\cos\theta\,d\theta) = a^2 \int_0^{\sin^{-1}(x/a)} \cos^2\theta\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\sin^{-1}(x/a)} (1+\cos 2\theta)\,d\theta = \frac{a^2}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\sin^{-1}(x/a)} = \frac{a^2}{2}\left[\theta + \sin\theta\,\cos\theta\right]_0^{\sin^{-1}(x/a)}$$

$$= \frac{a^2}{2}\left[\left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}\right) - 0\right] = \tfrac{1}{2}a^2\sin^{-1}(x/a) + \tfrac{1}{2}x\sqrt{a^2-x^2}$$

(b) The integral $\int_0^x \sqrt{a^2-t^2}\,dt$ represents the area under the curve $y = \sqrt{a^2-t^2}$ between the vertical lines $t = 0$ and $t = x$. The figure shows that this area consists of a triangular region and a sector of the circle $t^2 + y^2 = a^2$. The triangular region has base x and height $\sqrt{a^2-x^2}$, so its area is $\frac{1}{2}x\sqrt{a^2-x^2}$. The sector has area $\frac{1}{2}a^2\theta = \frac{1}{2}a^2\sin^{-1}(x/a)$.

40. The curves intersect when $x^2 + \left(\frac{1}{2}x^2\right)^2 = 8 \;\Leftrightarrow\; x^2 + \frac{1}{4}x^4 = 8 \;\Leftrightarrow\; x^4 + 4x^2 - 32 = 0 \;\Leftrightarrow$ $(x^2+8)(x^2-4) = 0 \;\Leftrightarrow\; x = \pm 2$. The area inside the circle and above the parabola is given by



$A_1 = \int_{-2}^2 \left(\sqrt{8-x^2} - \frac{1}{2}x^2\right)dx = 2\int_0^2 \sqrt{8-x^2}\,dx - 2\int_0^2 \frac{1}{2}x^2\,dx$

$= 2\left[\frac{1}{2}(8)\sin^{-1}\left(\frac{2}{\sqrt{8}}\right) + \frac{1}{2}(2)\sqrt{8-2^2} - \frac{1}{2}\left[\frac{1}{3}x^3\right]_0^2\right]$ [by Exercise 39]

$= 8\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + 2\sqrt{4} - \frac{8}{3} = 8\left(\frac{\pi}{4}\right) + 4 - \frac{8}{3} = 2\pi + \frac{4}{3}$

Since the area of the disk is $\pi\left(\sqrt{8}\right)^2 = 8\pi$, the area inside the circle and below the parabola ia $A_2 = 8\pi - \left(2\pi + \frac{4}{3}\right) = 6\pi - \frac{4}{3}$.

41. We use cylindrical shells and assume that $R > r$. $x^2 = r^2 - (y - R)^2 \Rightarrow x = \pm\sqrt{r^2 - (y-R)^2}$,

so $g(y) = 2\sqrt{r^2 - (y-R)^2}$ and

$$V = \int_{R-r}^{R+r} 2\pi y \cdot 2\sqrt{r^2 - (y-R)^2}\,dy = \int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\,du \qquad [\text{where } u = y - R]$$

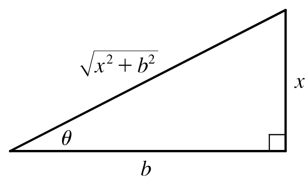
$$= 4\pi \int_{-r}^{r} u\sqrt{r^2-u^2}\,du + 4\pi R\int_{-r}^{r}\sqrt{r^2-u^2}\,du \qquad \left[\begin{array}{c}\text{where } u = r\sin\theta,\ du = r\cos\theta\,d\theta \\ \text{in the second integral}\end{array}\right]$$

$$= 4\pi\left[-\tfrac{1}{3}(r^2-u^2)^{3/2}\right]_{-r}^{r} + 4\pi R\int_{-\pi/2}^{\pi/2} r^2\cos^2\theta\,d\theta = -\tfrac{4\pi}{3}(0-0) + 4\pi Rr^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$

$$= 2\pi Rr^2\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta = 2\pi Rr^2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2\pi^2 Rr^2$$

Another method: Use washers instead of shells, so $V = 8\pi R\int_0^r \sqrt{r^2-y^2}\,dy$ as in Exercise 6.2.63(a), but evaluate the integral using $y = r\sin\theta$.

42. Let $x = b\tan\theta$, so that $dx = b\sec^2\theta\,d\theta$ and $\sqrt{x^2+b^2} = b\sec\theta$.

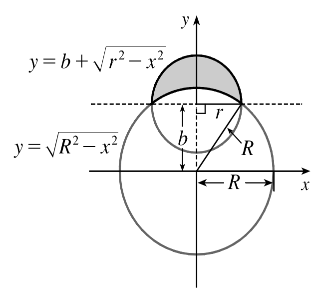


$$E(P) = \int_{-a}^{L-a} \frac{\lambda b}{4\pi\varepsilon_0(x^2+b^2)^{3/2}}\,dx = \frac{\lambda b}{4\pi\varepsilon_0}\int_{\theta_1}^{\theta_2}\frac{1}{(b\sec\theta)^3}\,b\sec^2\theta\,d\theta$$

$$= \frac{\lambda}{4\pi\varepsilon_0 b}\int_{\theta_1}^{\theta_2}\frac{1}{\sec\theta}\,d\theta = \frac{\lambda}{4\pi\varepsilon_0 b}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta = \frac{\lambda}{4\pi\varepsilon_0 b}\Big[\sin\theta\Big]_{\theta_1}^{\theta_2}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 b}\left[\frac{x}{\sqrt{x^2+b^2}}\right]_{-a}^{L-a} = \frac{\lambda}{4\pi\varepsilon_0 b}\left(\frac{L-a}{\sqrt{(L-a)^2+b^2}} + \frac{a}{\sqrt{a^2+b^2}}\right)$$

43. Let the equation of the large circle be $x^2 + y^2 = R^2$. Then the equation of the small circle is $x^2 + (y-b)^2 = r^2$, where $b = \sqrt{R^2 - r^2}$ is the distance between the centers of the circles. The desired area is



$$A = \int_{-r}^{r}\left[\left(b + \sqrt{r^2-x^2}\right) - \sqrt{R^2-x^2}\right]dx$$

$$= 2\int_0^r\left(b + \sqrt{r^2-x^2} - \sqrt{R^2-x^2}\right)dx$$

$$= 2\int_0^r b\,dx + 2\int_0^r\sqrt{r^2-x^2}\,dx - 2\int_0^r\sqrt{R^2-x^2}\,dx$$

The first integral is just $2br = 2r\sqrt{R^2-r^2}$. The second integral represents the area of a quarter-circle of radius r, so its value is $\frac{1}{4}\pi r^2$. To evaluate the other integral, note that

$$\int\sqrt{a^2-x^2}\,dx = \int a^2\cos^2\theta\,d\theta \quad [x = a\sin\theta,\ dx = a\cos\theta\,d\theta] \quad = \left(\tfrac{1}{2}a^2\right)\int(1+\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{2}a^2\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{2}a^2(\theta + \sin\theta\,\cos\theta) + C$$

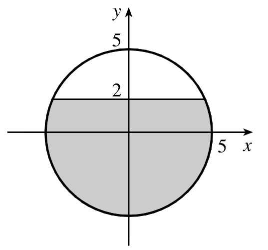
$$= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{a^2}{2}\left(\frac{x}{a}\right)\frac{\sqrt{a^2-x^2}}{a} + C = \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + C$$

Thus, the desired area is

$$A = 2r\sqrt{R^2-r^2} + 2\left(\tfrac{1}{4}\pi r^2\right) - \left[R^2\arcsin(x/R) + x\sqrt{R^2-x^2}\right]_0^r$$

$$= 2r\sqrt{R^2-r^2} + \tfrac{1}{2}\pi r^2 - \left[R^2\arcsin(r/R) + r\sqrt{R^2-r^2}\right] = r\sqrt{R^2-r^2} + \tfrac{\pi}{2}r^2 - R^2\arcsin(r/R)$$

44. Note that the circular cross-sections of the tank are the same everywhere, so the percentage of the total capacity that is being used is equal to the percentage of any cross-section that is under water. The underwater area is



$$\begin{aligned} A &= 2\int_{-5}^{2}\sqrt{25-y^2}\,dy \\ &= \left[25\arcsin(y/5) + y\sqrt{25-y^2}\right]_{-5}^{2} \qquad \text{[substitute } y = 5\sin\theta\text{]} \\ &= 25\arcsin\tfrac{2}{5} + 2\sqrt{21} + \tfrac{25}{2}\pi \approx 58.72\ \text{ft}^2 \end{aligned}$$

so the fraction of the total capacity in use is $\dfrac{A}{\pi(5)^2} \approx \dfrac{58.72}{25\pi} \approx 0.748$ or 74.8%.

7.4 Integration of Rational Functions by Partial Fractions

1. (a) $\dfrac{4+x}{(1+2x)(3-x)} = \dfrac{A}{1+2x} + \dfrac{B}{3-x}$

(b) $\dfrac{1-x}{x^3+x^4} = \dfrac{1-x}{x^3(1+x)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{1+x}$

2. (a) $\dfrac{x-6}{x^2+x-6} = \dfrac{x-6}{(x+3)(x-2)} = \dfrac{A}{x+3} + \dfrac{B}{x-2}$

(b) $\dfrac{x^2}{x^2+x+6} = \dfrac{(x^2+x+6)-(x+6)}{x^2+x+6} = 1 - \dfrac{x+6}{x^2+x+6}$

Notice that x^2+x+6 can't be factored because its discriminant is $b^2-4ac = -23 < 0$.

3. (a) $\dfrac{1}{x^2+x^4} = \dfrac{1}{x^2(1+x^2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{1+x^2}$

(b) $$\begin{aligned} \frac{x^3+1}{x^3-3x^2+2x} &= \frac{(x^3-3x^2+2x)+3x^2-2x+1}{x^3-3x^2+2x} = 1 + \frac{3x^2-2x+1}{x(x^2-3x+2)} \qquad \text{[or use long division]} \\ &= 1 + \frac{3x^2-2x+1}{x(x-1)(x-2)} = 1 + \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} \end{aligned}$$

4. (a) $$\begin{aligned} \frac{x^4-2x^3+x^2+2x-1}{x^2-2x+1} &= \frac{x^2(x^2-2x+1)+2x-1}{x^2-2x+1} = x^2 + \frac{2x-1}{(x-1)^2} \qquad \text{[or use long division]} \\ &= x^2 + \frac{A}{x-1} + \frac{B}{(x-1)^2} \end{aligned}$$

(b) $\dfrac{x^2-1}{x^3+x^2+x} = \dfrac{x^2-1}{x(x^2+x+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+x+1}$

5. (a) $$\begin{aligned} \frac{x^6}{x^2-4} &= x^4+4x^2+16+\frac{64}{(x+2)(x-2)} \qquad \text{[by long division]} \\ &= x^4+4x^2+16+\frac{A}{x+2}+\frac{B}{x-2} \end{aligned}$$

(b) $\dfrac{x^4}{(x^2-x+1)(x^2+2)^2} = \dfrac{Ax+B}{x^2-x+1} + \dfrac{Cx+D}{x^2+2} + \dfrac{Ex+F}{(x^2+2)^2}$

6. (a) $\dfrac{t^6+1}{t^6+t^3} = \dfrac{(t^6+t^3)-t^3+1}{t^6+t^3} = 1+\dfrac{-t^3+1}{t^3(t^3+1)} = 1+\dfrac{-t^3+1}{t^3(t+1)(t^2-t+1)} = 1+\dfrac{A}{t}+\dfrac{B}{t^2}+\dfrac{C}{t^3}+\dfrac{D}{t+1}+\dfrac{Ex+F}{t^2-t+1}$

(b) $\dfrac{x^5+1}{(x^2-x)(x^4+2x^2+1)} = \dfrac{x^5+1}{x(x-1)(x^2+1)^2} = \dfrac{A}{x}+\dfrac{B}{x-1}+\dfrac{Cx+D}{x^2+1}+\dfrac{Ex+F}{(x^2+1)^2}$

7. $\displaystyle\int \frac{x^4}{x-1}\,dx = \int \left(x^3+x^2+x+1+\frac{1}{x-1}\right) dx$ [by division] $= \dfrac{1}{4}x^4+\dfrac{1}{3}x^3+\dfrac{1}{2}x^2+x+\ln|x-1|+C$

8. $\displaystyle\int \frac{3t-2}{t+1}\,dt = \int \left(3-\frac{5}{t+1}\right) dt = 3t-5\ln|t+1|+C$

9. $\dfrac{5x+1}{(2x+1)(x-1)} = \dfrac{A}{2x+1}+\dfrac{B}{x-1}$. Multiply both sides by $(2x+1)(x-1)$ to get $5x+1 = A(x-1)+B(2x+1) \Rightarrow$

$5x+1 = Ax-A+2Bx+B \Rightarrow 5x+1 = (A+2B)x+(-A+B)$.

The coefficients of x must be equal and the constant terms are also equal, so $A+2B=5$ and $-A+B=1$. Adding these equations gives us $3B=6 \Leftrightarrow B=2$, and hence, $A=1$. Thus,

$$\int \frac{5x+1}{(2x+1)(x-1)}\,dx = \int \left(\frac{1}{2x+1}+\frac{2}{x-1}\right) dx = \tfrac{1}{2}\ln|2x+1|+2\ln|x-1|+C.$$

Another method: Substituting 1 for x in the equation $5x+1 = A(x-1)+B(2x+1)$ gives $6=3B \Leftrightarrow B=2$.

Substituting $-\frac{1}{2}$ for x gives $-\frac{3}{2} = -\frac{3}{2}A \Leftrightarrow A=1$.

10. $\dfrac{y}{(y+4)(2y-1)} = \dfrac{A}{y+4}+\dfrac{B}{2y-1}$. Multiply both sides by $(y+4)(2y-1)$ to get $y = A(2y-1)+B(y+4) \Rightarrow$

$y = 2Ay-A+By+4B \Rightarrow y=(2A+B)y+(-A+4B)$. The coefficients of y must be equal and the constant terms are also equal, so $2A+B=1$ and $-A+4B=0$. Adding 2 times the second equation and the first equation gives us $9B=1 \Leftrightarrow B=\frac{1}{9}$ and hence, $A=\frac{4}{9}$. Thus,

$$\begin{aligned}\int \frac{y\,dy}{(y+4)(2y-1)} &= \int \left(\frac{\frac{4}{9}}{y+4}+\frac{\frac{1}{9}}{2y-1}\right) dy = \frac{4}{9}\ln|y+4|+\frac{1}{9}\cdot\frac{1}{2}\ln|2y-1|+C \\ &= \tfrac{4}{9}\ln|y+4|+\tfrac{1}{18}\ln|2y-1|+C\end{aligned}$$

Another method: Substituting $\frac{1}{2}$ for y in the equation $y = A(2y-1)+B(y+4)$ gives $\frac{1}{2} = \frac{9}{2}B \Leftrightarrow B=\frac{1}{9}$.

Substituting -4 for y gives $-4=-9A \Leftrightarrow A=\frac{4}{9}$.

11. $\dfrac{2}{2x^2+3x+1} = \dfrac{2}{(2x+1)(x+1)} = \dfrac{A}{2x+1}+\dfrac{B}{x+1}$. Multiply both sides by $(2x+1)(x+1)$ to get

$2 = A(x+1)+B(2x+1)$. The coefficients of x must be equal and the constant terms are also equal, so $A+2B=0$ and $A+B=2$. Subtracting the second equation from the first gives $B=-2$, and hence, $A=4$. Thus,

$$\int_0^1 \frac{2}{2x^2+3x+1}\,dx = \int_0^1 \left(\frac{4}{2x+1}-\frac{2}{x+1}\right) dx = \left[\frac{4}{2}\ln|2x+1|-2\ln|x+1|\right]_0^1 = (2\ln 3-2\ln 2)-0 = 2\ln\frac{3}{2}.$$

Another method: Substituting -1 for x in the equation $2 = A(x+1)+B(2x+1)$ gives $2=-B \Leftrightarrow B=-2$.

Substituting $-\frac{1}{2}$ for x gives $2=\frac{1}{2}A \Leftrightarrow A=4$.

12. $\dfrac{x-4}{x^2-5x+6} = \dfrac{A}{x-2} + \dfrac{B}{x-3}$. Multiply both sides by $(x-2)(x-3)$ to get $x-4 = A(x-3)+B(x-2) \quad \Rightarrow$

$x-4 = Ax - 3A + Bx - 2B \quad \Rightarrow \quad x-4 = (A+B)x + (-3A-2B)$.

The coefficients of x must be equal and the constant terms are also equal, so $A+B=1$ and $-3A-2B=-4$.

Adding twice the first equation to the second gives us $-A=-2 \quad \Leftrightarrow \quad A=2$, and hence, $B=-1$.Thus,

$$\int_0^1 \frac{x-4}{x^2-5x+6}\,dx = \int_0^1 \left(\frac{2}{x-2} - \frac{1}{x-3}\right)dx = \left[2\ln|x-2| - \ln|x-3|\right]_0^1$$

$$= (0-\ln 2) - (2\ln 2 - \ln 3) = -3\ln 2 + \ln 3 \;\; [\text{or } \ln \tfrac{3}{8}]$$

Another method: Substituting 3 for x in the equation $x-4=A(x-3)+B(x-2)$ gives $-1=B$. Substituting 2 for x gives $-2=-A \quad \Leftrightarrow \quad A=2$.

13. $\displaystyle\int \frac{ax}{x^2-bx}\,dx = \int \frac{ax}{x(x-b)}\,dx = \int \frac{a}{x-b}\,dx = a\ln|x-b| + C$

14. If $a \neq b$, $\dfrac{1}{(x+a)(x+b)} = \dfrac{1}{b-a}\left(\dfrac{1}{x+a} - \dfrac{1}{x+b}\right)$, so if $a \neq b$, then

$$\int \frac{dx}{(x+a)(x+b)} = \frac{1}{b-a}\left(\ln|x+a| - \ln|x+b|\right) + C = \frac{1}{b-a}\ln\left|\frac{x+a}{x+b}\right| + C$$

If $a=b$, then $\displaystyle\int \frac{dx}{(x+a)^2} = -\frac{1}{x+a} + C$.

15. $\dfrac{x^3-4x+1}{x^2-3x+2} = x+3+\dfrac{3x-5}{(x-1)(x-2)}$. Write $\dfrac{3x-5}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$. Multiplying both sides by $(x-1)(x-2)$ gives $3x-5 = A(x-2)+B(x-1)$. Substituting 2 for x gives $1=B$. Substituting 1 for x gives $-2=-A \quad \Leftrightarrow \quad A=2$. Thus,

$$\int_{-1}^0 \frac{x^3-4x+1}{x^2-3x+2}\,dx = \int_{-1}^0 \left(x+3+\frac{2}{x-1}+\frac{1}{x-2}\right)dx = \left[\tfrac{1}{2}x^2 + 3x + 2\ln|x-1| + \ln|x-2|\right]_{-1}^0$$

$$= (0+0+0+\ln 2) - \left(\tfrac{1}{2} - 3 + 2\ln 2 + \ln 3\right) = \tfrac{5}{2} - \ln 2 - \ln 3, \text{ or } \tfrac{5}{2} - \ln 6$$

16. $\dfrac{x^3+4x^2+x-1}{x^3+x^2} = 1 + \dfrac{3x^2+x-1}{x^2(x+1)}$. Write $\dfrac{3x^2+x-1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$. Multiplying both sides by $x^2(x+1)$ gives $3x^2+x-1 = Ax(x+1)+B(x+1)+Cx^2$. Substituting 0 for x gives $-1=B$. Substituting -1 for x gives $1=C$. Equating coefficients of x^2 gives $3=A+C=A+1$, so $A=2$. Thus,

$$\int_1^2 \frac{x^3+4x^2+x-1}{x^3+x^2}\,dx = \int_1^2 \left(1+\frac{2}{x}-\frac{1}{x^2}+\frac{1}{x+1}\right)dx = \left[x + 2\ln|x| + \frac{1}{x} + \ln|x+1|\right]_1^2$$

$$= \left(2+2\ln 2+\tfrac{1}{2}+\ln 3\right) - (1+0+1+\ln 2) = \tfrac{1}{2} + \ln 2 + \ln 3, \text{ or } \tfrac{1}{2} + \ln 6.$$

17. $\dfrac{4y^2-7y-12}{y(y+2)(y-3)} = \dfrac{A}{y} + \dfrac{B}{y+2} + \dfrac{C}{y-3} \quad \Rightarrow \quad 4y^2-7y-12 = A(y+2)(y-3)+By(y-3)+Cy(y+2)$. Setting $y=0$ gives $-12=-6A$, so $A=2$. Setting $y=-2$ gives $18=10B$, so $B=\frac{9}{5}$. Setting $y=3$ gives $3=15C$, so $C=\frac{1}{5}$.

Now

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy = \int_1^2 \left(\frac{2}{y}+\frac{9/5}{y+2}+\frac{1/5}{y-3}\right)dy = \left[2\ln|y|+\tfrac{9}{5}\ln|y+2|+\tfrac{1}{5}\ln|y-3|\right]_1^2$$

$$= 2\ln 2+\tfrac{9}{5}\ln 4+\tfrac{1}{5}\ln 1-2\ln 1-\tfrac{9}{5}\ln 3-\tfrac{1}{5}\ln 2$$

$$= 2\ln 2+\tfrac{18}{5}\ln 2-\tfrac{1}{5}\ln 2-\tfrac{9}{5}\ln 3=\tfrac{27}{5}\ln 2-\tfrac{9}{5}\ln 3=\tfrac{9}{5}(3\ln 2-\ln 3)=\tfrac{9}{5}\ln\tfrac{8}{3}$$

18. $\dfrac{3x^2+6x+2}{x^2+3x+2}=3+\dfrac{-3x-4}{(x+1)(x+2)}$. Write $\dfrac{-3x-4}{(x+1)(x+2)}=\dfrac{A}{x+1}+\dfrac{B}{x+2}$. Multiplying both sides by $(x+1)(x+2)$ gives $-3x-4=A(x+2)+B(x+1)$. Substituting -2 for x gives $2=-B \;\Leftrightarrow\; B=-2$. Substituting -1 for x gives $-1=A$. Thus,

$$\int_1^2 \frac{3x^2+6x+2}{x^2+3x+2}\,dx=\int_1^2\left(3-\frac{1}{x+1}-\frac{2}{x+2}\right)dx=\Big[3x-\ln|x+1|-2\ln|x+2|\Big]_1^2$$

$$=(6-\ln 3-2\ln 4)-(3-\ln 2-2\ln 3)=3+\ln 2+\ln 3-2\ln 4,\text{ or } 3+\ln\tfrac{3}{8}$$

19. $\dfrac{x^2+x+1}{(x+1)^2(x+2)}=\dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}+\dfrac{C}{x+2}$. Multiplying both sides by $(x+1)^2(x+2)$ gives $x^2+x+1=A(x+1)(x+2)+B(x+2)+C(x+1)^2$. Substituting -1 for x gives $1=B$. Substituting -2 for x gives $3=C$. Equating coefficients of x^2 gives $1=A+C=A+3$, so $A=-2$. Thus,

$$\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx=\int_0^1\left(\frac{-2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x+2}\right)dx=\left[-2\ln|x+1|-\frac{1}{x+1}+3\ln|x+2|\right]_0^1$$

$$=\left(-2\ln 2-\tfrac{1}{2}+3\ln 3\right)-(0-1+3\ln 2)=\tfrac{1}{2}-5\ln 2+3\ln 3,\text{ or } \tfrac{1}{2}+\ln\tfrac{27}{32}$$

20. $\dfrac{x(3-5x)}{(3x-1)(x-1)^2}=\dfrac{A}{3x-1}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$. Multiplying both sides by $(3x-1)(x-1)^2$ gives $x(3-5x)=A(x-1)^2+B(x-1)(3x-1)+C(3x-1)$. Substituting 1 for x gives $-2=2C \;\Leftrightarrow\; C=-1$. Substituting $\frac{1}{3}$ for x gives $\frac{4}{9}=\frac{4}{9}A \;\Leftrightarrow\; A=1$. Substituting 0 for x gives $0=A+B-C=1+B+1$, so $B=-2$. Thus,

$$\int_2^3 \frac{x(3-5x)}{(3x-1)(x-1)^2}\,dx=\int_2^3\left[\frac{1}{3x-1}-\frac{2}{x-1}-\frac{1}{(x-1)^2}\right]dx=\left[\frac{1}{3}\ln|3x-1|-2\ln|x-1|+\frac{1}{x-1}\right]_2^3$$

$$=\left(\tfrac{1}{3}\ln 8-2\ln 2+\tfrac{1}{2}\right)-\left(\tfrac{1}{3}\ln 5-0+1\right)=-\ln 2-\tfrac{1}{3}\ln 5-\tfrac{1}{2}$$

21. $\dfrac{1}{(t^2-1)^2}=\dfrac{1}{(t+1)^2(t-1)^2}=\dfrac{A}{t+1}+\dfrac{B}{(t+1)^2}+\dfrac{C}{t-1}+\dfrac{D}{(t-1)^2}$. Multiplying both sides by $(t+1)^2(t-1)^2$ gives $1=A(t+1)(t-1)^2+B(t-1)^2+C(t-1)(t+1)^2+D(t+1)^2$. Substituting 1 for t gives $1=4D \;\Leftrightarrow\; D=\frac{1}{4}$. Substituting -1 for t gives $1=4B \;\Leftrightarrow\; B=\frac{1}{4}$. Substituting 0 for t gives $1=A+B-C+D=A+\frac{1}{4}-C+\frac{1}{4}$, so $\frac{1}{2}=A-C$. Equating coefficients of t^3 gives $0=A+C$. Adding the last two equations gives $2A=\frac{1}{2} \;\Leftrightarrow\; A=\frac{1}{4}$, and so $C=-\frac{1}{4}$. Thus,

$$\int\frac{dt}{(t^2-1)^2}=\int\left[\frac{1/4}{t+1}+\frac{1/4}{(t+1)^2}-\frac{1/4}{t-1}+\frac{1/4}{(t-1)^2}\right]dt$$

$$=\frac{1}{4}\left[\ln|t+1|-\frac{1}{t+1}-\ln|t-1|-\frac{1}{t-1}\right]+C,\text{ or } \frac{1}{4}\left(\ln\left|\frac{t+1}{t-1}\right|+\frac{2t}{1-t^2}\right)+C$$

22. $\displaystyle\int \frac{x^4+9x^2+x+2}{x^2+9}\,dx = \int\left(x^2+\frac{x+2}{x^2+9}\right)dx = \int\left(x^2+\frac{x}{x^2+9}+\frac{2}{x^2+9}\right)dx$

$$= \frac{1}{3}x^3 + \frac{1}{2}\ln(x^2+9) + \frac{2}{3}\tan^{-1}\frac{x}{3} + C$$

23. $\dfrac{10}{(x-1)(x^2+9)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9}$. Multiply both sides by $(x-1)(x^2+9)$ to get

$10 = A(x^2+9) + (Bx+C)(x-1)$ $(\star)$. Substituting 1 for x gives $10 = 10A \Leftrightarrow A = 1$. Substituting 0 for x gives $10 = 9A - C \Rightarrow C = 9(1) - 10 = -1$. The coefficients of the x^2-terms in $(\star)$ must be equal, so $0 = A + B \Rightarrow B = -1$. Thus,

$$\int \frac{10}{(x-1)(x^2+9)}\,dx = \int\left(\frac{1}{x-1}+\frac{-x-1}{x^2+9}\right)dx = \int\left(\frac{1}{x-1}-\frac{x}{x^2+9}-\frac{1}{x^2+9}\right)dx$$

$$= \ln|x-1| - \tfrac{1}{2}\ln(x^2+9) - \tfrac{1}{3}\tan^{-1}\left(\tfrac{x}{3}\right) + C$$

In the second term we used the substitution $u = x^2+9$ and in the last term we used Formula 10.

24. $\dfrac{x^2-x+6}{x^3+3x} = \dfrac{x^2-x+6}{x(x^2+3)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+3}$. Multiply by $x(x^2+3)$ to get $x^2-x+6 = A(x^2+3) + (Bx+C)x$.

Substituting 0 for x gives $6 = 3A \Leftrightarrow A = 2$. The coefficients of the x^2-terms must be equal, so $1 = A + B \Rightarrow B = 1 - 2 = -1$. The coefficients of the x-terms must be equal, so $-1 = C$. Thus,

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = \int\left(\frac{2}{x}+\frac{-x-1}{x^2+3}\right)dx = \int\left(\frac{2}{x}-\frac{x}{x^2+3}-\frac{1}{x^2+3}\right)dx$$

$$= 2\ln|x| - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

25. $\dfrac{4x}{x^3+x^2+x+1} = \dfrac{4x}{x^2(x+1)+1(x+1)} = \dfrac{4x}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$. Multiply both sides by

$(x+1)(x^2+1)$ to get $4x = A(x^2+1) + (Bx+C)(x+1) \Leftrightarrow 4x = Ax^2 + A + Bx^2 + Bx + Cx + C \Leftrightarrow 4x = (A+B)x^2 + (B+C)x + (A+C)$. Comparing coefficients gives us the following system of equations:

$$A + B = 0 \quad \textbf{(1)} \qquad B + C = 4 \quad \textbf{(2)} \qquad A + C = 0 \quad \textbf{(3)}$$

Subtracting equation **(1)** from equation **(2)** gives us $-A + C = 4$, and adding that equation to equation **(3)** gives us $2C = 4 \Leftrightarrow C = 2$, and hence $A = -2$ and $B = 2$. Thus,

$$\int \frac{4x}{x^3+x^2+x+1}\,dx = \int\left(\frac{-2}{x+1}+\frac{2x+2}{x^2+1}\right)dx = \int\left(\frac{-2}{x+1}+\frac{2x}{x^2+1}+\frac{2}{x^2+1}\right)dx$$

$$= -2\ln|x+1| + \ln(x^2+1) + 2\tan^{-1}x + C$$

26. $\displaystyle\int \frac{x^2+x+1}{(x^2+1)^2}\,dx = \int \frac{x^2+1}{(x^2+1)^2}\,dx + \int \frac{x}{(x^2+1)^2}\,dx = \int \frac{1}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{u^2}\,du \quad [u = x^2+1,\ du = 2x\,dx]$

$$= \tan^{-1}x + \frac{1}{2}\left(-\frac{1}{u}\right) + C = \tan^{-1}x - \frac{1}{2(x^2+1)} + C$$

27. $\dfrac{x^3+4x+3}{x^4+5x^2+4}=\dfrac{x^3+4x+3}{(x^2+1)(x^2+4)}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+4}$. Multiply both sides by $(x^2+1)(x^2+4)$

to get $x^3+4x+3=(Ax+B)(x^2+4)+(Cx+D)(x^2+1) \quad\Leftrightarrow$

$x^3+4x+3=Ax^3+Bx^2+4Ax+4B+Cx^3+Dx^2+Cx+D \quad\Leftrightarrow$

$x^3+4x+3=(A+C)x^3+(B+D)x^2+(4A+C)x+(4B+D)$. Comparing coefficients gives us the following system of equations:

$A+C=1$ **(1)** $\qquad B+D=0$ **(2)** $\qquad 4A+C=4$ **(3)** $\qquad 4B+D=3$ **(4)**

Subtracting equation **(1)** from equation **(3)** gives us $A=1$ and hence, $C=0$. Subtracting equation **(2)** from equation **(4)** gives us $B=1$ and hence, $D=-1$. Thus,

$$\int\frac{x^3+4x+3}{x^4+5x^2+4}\,dx=\int\left(\frac{x+1}{x^2+1}+\frac{-1}{x^2+4}\right)dx=\int\left(\frac{x}{x^2+1}+\frac{1}{x^2+1}-\frac{1}{x^2+4}\right)dx$$
$$=\frac{1}{2}\ln(x^2+1)+\tan^{-1}x-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$$

28. $\dfrac{x^3+6x-2}{x^4+6x^2}=\dfrac{x^3+6x-2}{x^2(x^2+6)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{Cx+D}{x^2+6}$. Multiply both sides by $x^2(x^2+6)$ to get

$x^3+6x-2=Ax(x^2+6)+B(x^2+6)+(Cx+D)x^2 \quad\Leftrightarrow$

$x^3+6x-2=Ax^3+6Ax+Bx^2+6B+Cx^3+Dx^2 \quad\Leftrightarrow\quad x^3+6x-2=(A+C)x^3+(B+D)x^2+6Ax+6B$.

Substituting 0 for x gives $-2=6B \quad\Leftrightarrow\quad B=-\frac{1}{3}$. Equating coefficients of x^2 gives $0=B+D$, so $D=\frac{1}{3}$. Equating coefficients of x gives $6=6A \quad\Leftrightarrow\quad A=1$. Equating coefficients of x^3 gives $1=A+C$, so $C=0$. Thus,

$$\int\frac{x^3+6x-2}{x^4+6x^2}\,dx=\int\left(\frac{1}{x}+\frac{-1/3}{x^2}+\frac{1/3}{x^2+6}\right)dx=\ln|x|+\frac{1}{3x}+\frac{1}{3\sqrt{6}}\tan^{-1}\left(\frac{x}{\sqrt{6}}\right)+C.$$

29. $$\int\frac{x+4}{x^2+2x+5}\,dx=\int\frac{x+1}{x^2+2x+5}\,dx+\int\frac{3}{x^2+2x+5}\,dx=\frac{1}{2}\int\frac{(2x+2)\,dx}{x^2+2x+5}+\int\frac{3\,dx}{(x+1)^2+4}$$
$$=\frac{1}{2}\ln\left|x^2+2x+5\right|+3\int\frac{2\,du}{4(u^2+1)}\qquad\left[\begin{array}{c}\text{where } x+1=2u,\\ \text{and } dx=2\,du\end{array}\right]$$
$$=\frac{1}{2}\ln(x^2+2x+5)+\frac{3}{2}\tan^{-1}u+C=\frac{1}{2}\ln(x^2+2x+5)+\frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$$

30. $\dfrac{x^3-2x^2+2x-5}{x^4+4x^2+3}=\dfrac{x^3-2x^2+2x-5}{(x^2+1)(x^2+3)}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+3}$. Multiply both sides by $(x^2+1)(x^2+3)$ to get

$x^3-2x^2+2x-5=(Ax+B)(x^2+3)+(Cx+D)(x^2+1) \quad\Leftrightarrow$

$x^3-2x^2+2x-5=Ax^3+Bx^2+3Ax+3B+Cx^3+Dx^2+Cx+D \quad\Leftrightarrow$

$x^3-2x^2+2x-5=(A+C)x^3+(B+D)x^2+(3A+C)x+(3B+D)$. Comparing coefficients gives us the following system of equations:

$A+C=1$ **(1)** $\qquad B+D=-2$ **(2)** $\qquad 3A+C=2$ **(3)** $\qquad 3B+D=-5$ **(4)**

Subtracting equation **(1)** from equation **(3)** gives us $2A=1 \quad\Leftrightarrow\quad A=\frac{1}{2}$, and hence, $C=\frac{1}{2}$. Subtracting equation **(2)** from equation **(4)** gives us $2B=-3 \quad\Leftrightarrow\quad B=-\frac{3}{2}$, and hence, $D=-\frac{1}{2}$.

Thus,

$$\int \frac{x^3 - 2x^2 + 2x - 5}{x^4 + 4x^2 + 3}\,dx = \int \left(\frac{\frac{1}{2}x - \frac{3}{2}}{x^2 + 1} + \frac{\frac{1}{2}x - \frac{1}{2}}{x^2 + 3} \right) dx = \int \left(\frac{\frac{1}{2}x}{x^2 + 1} - \frac{\frac{3}{2}}{x^2 + 1} + \frac{\frac{1}{2}x}{x^2 + 3} - \frac{\frac{1}{2}}{x^2 + 3} \right) dx$$

$$= \tfrac{1}{4}\ln(x^2 + 1) - \tfrac{3}{2}\tan^{-1} x + \tfrac{1}{4}\ln(x^2 + 3) - \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

31. $\dfrac{1}{x^3 - 1} = \dfrac{1}{(x - 1)(x^2 + x + 1)} = \dfrac{A}{x - 1} + \dfrac{Bx + C}{x^2 + x + 1} \quad\Rightarrow\quad 1 = A(x^2 + x + 1) + (Bx + C)(x - 1).$

Take $x = 1$ to get $A = \frac{1}{3}$. Equating coefficients of x^2 and then comparing the constant terms, we get $0 = \frac{1}{3} + B$, $1 = \frac{1}{3} - C$, so $B = -\frac{1}{3}$, $C = -\frac{2}{3} \quad\Rightarrow$

$$\int \frac{1}{x^3 - 1}dx = \int \frac{\frac{1}{3}}{x - 1}\,dx + \int \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2 + x + 1}\,dx = \tfrac{1}{3}\ln|x - 1| - \frac{1}{3}\int \frac{x + 2}{x^2 + x + 1}dx$$

$$= \tfrac{1}{3}\ln|x - 1| - \frac{1}{3}\int \frac{x + 1/2}{x^2 + x + 1}dx - \frac{1}{3}\int \frac{(3/2)\,dx}{(x + 1/2)^2 + 3/4}$$

$$= \tfrac{1}{3}\ln|x - 1| - \tfrac{1}{6}\ln(x^2 + x + 1) - \tfrac{1}{2}\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{x + \frac{1}{2}}{\sqrt{3}/2}\right) + K$$

$$= \tfrac{1}{3}\ln|x - 1| - \tfrac{1}{6}\ln(x^2 + x + 1) - \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{1}{\sqrt{3}}(2x + 1)\right) + K$$

32. $\displaystyle\int_0^1 \frac{x}{x^2 + 4x + 13}\,dx = \int_0^1 \frac{\frac{1}{2}(2x + 4)}{x^2 + 4x + 13}\,dx - 2\int_0^1 \frac{dx}{(x + 2)^2 + 9}$

$$= \frac{1}{2}\int_{13}^{18}\frac{dy}{y} - 2\int_{2/3}^{1}\frac{3\,du}{9u^2 + 9} \qquad \left[\begin{matrix}\text{where } y = x^2 + 4x + 13,\ dy = (2x + 4)\,dx, \\ x + 2 = 3u, \text{ and } dx = 3\,du\end{matrix}\right]$$

$$= \tfrac{1}{2}\left[\ln y\right]_{13}^{18} - \tfrac{2}{3}\left[\tan^{-1} u\right]_{2/3}^{1} = \tfrac{1}{2}\ln\tfrac{18}{13} - \tfrac{2}{3}\left(\tfrac{\pi}{4} - \tan^{-1}\left(\tfrac{2}{3}\right)\right)$$

$$= \tfrac{1}{2}\ln\tfrac{18}{13} - \tfrac{\pi}{6} + \tfrac{2}{3}\tan^{-1}\left(\tfrac{2}{3}\right)$$

33. Let $u = x^4 + 4x^2 + 3$, so that $du = (4x^3 + 8x)\,dx = 4(x^3 + 2x)\,dx$, $x = 0 \quad\Rightarrow\quad u = 3$, and $x = 1 \quad\Rightarrow\quad u = 8$.

Then $\displaystyle\int_0^1 \frac{x^3 + 2x}{x^4 + 4x^2 + 3}\,dx = \int_3^8 \frac{1}{u}\left(\frac{1}{4}\,du\right) = \frac{1}{4}\left[\ln|u|\right]_3^8 = \frac{1}{4}(\ln 8 - \ln 3) = \frac{1}{4}\ln\frac{8}{3}.$

34. $\dfrac{x^5 + x - 1}{x^3 + 1} = x^2 + \dfrac{-x^2 + x - 1}{x^3 + 1} = x^2 + \dfrac{-x^2 + x - 1}{(x + 1)(x^2 - x + 1)} = x^2 + \dfrac{-1}{x + 1}$, so

$$\int \frac{x^5 + x - 1}{x^3 + 1}\,dx = \int \left(x^2 - \frac{1}{x + 1}\right) dx = \frac{1}{3}x^3 - \ln|x + 1| + C$$

35. $\dfrac{5x^4 + 7x^2 + x + 2}{x(x^2 + 1)^2} = \dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 1} + \dfrac{Dx + E}{(x^2 + 1)^2}$. Multiply by $x(x^2 + 1)^2$ to get

$5x^4 + 7x^2 + x + 2 = A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x \quad\Leftrightarrow$

$5x^4 + 7x^2 + x + 2 = A(x^4 + 2x^2 + 1) + (Bx^2 + Cx)(x^2 + 1) + Dx^2 + Ex \quad\Leftrightarrow$

$5x^4 + 7x^2 + x + 2 = Ax^4 + 2Ax^2 + A + Bx^4 + Cx^3 + Bx^2 + Cx + Dx^2 + Ex \quad\Leftrightarrow$

$5x^4 + 7x^2 + x + 2 = (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A$. Equating coefficients gives us $C = 0$,

$A = 2$, $A + B = 5 \Rightarrow B = 3$, $C + E = 1 \Rightarrow E = 1$, and $2A + B + D = 7 \Rightarrow D = 0$. Thus,

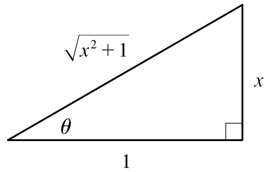
$\int \frac{5x^4 + 7x^2 + x + 2}{x(x^2+1)^2}\,dx = \int \left[\frac{2}{x} + \frac{3x}{x^2+1} + \frac{1}{(x^2+1)^2}\right] dx = I$. Now

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{\sec^2\theta\,d\theta}{(\tan^2\theta+1)^2} \qquad \begin{bmatrix} x = \tan\theta, \\ dx = \sec^2\theta\,d\theta \end{bmatrix}$$

$$= \int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \int \cos^2\theta\,d\theta = \int \tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + C = \tfrac{1}{2}\theta + \tfrac{1}{2}\sin\theta\,\cos\theta + C$$

$$= \frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{\sqrt{x^2+1}}\frac{1}{\sqrt{x^2+1}} + C$$



Therefore, $I = 2\ln|x| + \tfrac{3}{2}\ln(x^2+1) + \tfrac{1}{2}\tan^{-1}x + \dfrac{x}{2(x^2+1)} + C$.

36. Let $u = x^5 + 5x^3 + 5x$, so that $du = (5x^4 + 15x^2 + 5)dx = 5(x^4 + 3x^2 + 1)dx$. Then

$$\int \frac{x^4 + 3x^2 + 1}{x^5 + 5x^3 + 5x}\,dx = \int \frac{1}{u}\left(\frac{1}{5}\,du\right) = \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln\left|x^5 + 5x^3 + 5x\right| + C$$

37. $\dfrac{x^2 - 3x + 7}{(x^2 - 4x + 6)^2} = \dfrac{Ax + B}{x^2 - 4x + 6} + \dfrac{Cx + D}{(x^2 - 4x + 6)^2} \Rightarrow x^2 - 3x + 7 = (Ax + B)(x^2 - 4x + 6) + Cx + D \Rightarrow$

$x^2 - 3x + 7 = Ax^3 + (-4A + B)x^2 + (6A - 4B + C)x + (6B + D)$. So $A = 0$, $-4A + B = 1 \Rightarrow B = 1$,

$6A - 4B + C = -3 \Rightarrow C = 1$, $6B + D = 7 \Rightarrow D = 1$. Thus,

$$I = \int \frac{x^2 - 3x + 7}{(x^2 - 4x + 6)^2}\,dx = \int \left(\frac{1}{x^2 - 4x + 6} + \frac{x + 1}{(x^2 - 4x + 6)^2}\right) dx$$

$$= \int \frac{1}{(x-2)^2 + 2}\,dx + \int \frac{x - 2}{(x^2 - 4x + 6)^2}\,dx + \int \frac{3}{(x^2 - 4x + 6)^2}\,dx$$

$$= I_1 + I_2 + I_3.$$

$$I_1 = \int \frac{1}{(x-2)^2 + \left(\sqrt{2}\right)^2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + C_1$$

$$I_2 = \frac{1}{2}\int \frac{2x - 4}{(x^2 - 4x + 6)^2}\,dx = \tfrac{1}{2}\int \frac{1}{u^2}\,du = \tfrac{1}{2}\left(-\frac{1}{u}\right) + C_2 = -\frac{1}{2(x^2 - 4x + 6)} + C_2$$

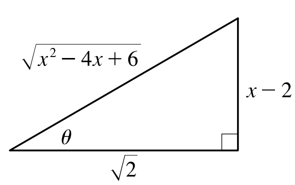
$$I_3 = 3\int \frac{1}{\left[(x-2)^2 + \left(\sqrt{2}\right)^2\right]^2}\,dx = 3\int \frac{1}{[2(\tan^2\theta + 1)]^2}\sqrt{2}\sec^2\theta\,d\theta \qquad \begin{bmatrix} x - 2 = \sqrt{2}\tan\theta, \\ dx = \sqrt{2}\,\sec^2\theta\,d\theta \end{bmatrix}$$

$$= \frac{3\sqrt{2}}{4}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \frac{3\sqrt{2}}{4}\int \cos^2\theta\,d\theta = \frac{3\sqrt{2}}{4}\int \tfrac{1}{2}(1 + \cos 2\theta)\,d\theta$$

$$= \frac{3\sqrt{2}}{8}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C_3 = \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3\sqrt{2}}{8}\left(\tfrac{1}{2}\cdot 2\sin\theta\,\cos\theta\right) + C_3$$

$$= \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3\sqrt{2}}{8}\cdot\frac{x-2}{\sqrt{x^2 - 4x + 6}}\cdot\frac{\sqrt{2}}{\sqrt{x^2 - 4x + 6}} + C_3$$

$$= \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3(x-2)}{4(x^2 - 4x + 6)} + C_3$$



So $I = I_1 + I_2 + I_3 \qquad [C = C_1 + C_2 + C_3]$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{-1}{2(x^2-4x+6)} + \frac{3\sqrt{2}}{8} \tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3(x-2)}{4(x^2-4x+6)} + C$$

$$= \left(\frac{4\sqrt{2}}{8} + \frac{3\sqrt{2}}{8}\right) \tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3(x-2)-2}{4(x^2-4x+6)} + C = \frac{7\sqrt{2}}{8} \tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3x-8}{4(x^2-4x+6)} + C$$

38. $\dfrac{x^3+2x^2+3x-2}{(x^2+2x+2)^2} = \dfrac{Ax+B}{x^2+2x+2} + \dfrac{Cx+D}{(x^2+2x+2)^2} \quad \Rightarrow$

$x^3+2x^2+3x-2 = (Ax+B)(x^2+2x+2) + Cx + D \quad \Rightarrow$

$x^3+2x^2+3x-2 = Ax^3 + (2A+B)x^2 + (2A+2B+C)x + 2B + D.$

So $A = 1, 2A + B = 2 \quad \Rightarrow \quad B = 0, 2A + 2B + C = 3 \quad \Rightarrow \quad C = 1$, and $2B + D = -2 \quad \Rightarrow \quad D = -2$. Thus,

$$I = \int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx = \int \left(\frac{x}{x^2+2x+2} + \frac{x-2}{(x^2+2x+2)^2}\right) dx$$

$$= \int \frac{x+1}{x^2+2x+2}\,dx + \int \frac{-1}{x^2+2x+2}\,dx + \int \frac{x+1}{(x^2+2x+2)^2}\,dx + \int \frac{-3}{(x^2+2x+2)^2}\,dx$$

$$= I_1 + I_2 + I_3 + I_4.$$

$$I_1 = \int \frac{x+1}{x^2+2x+2}\,dx = \int \frac{1}{u}\left(\frac{1}{2}\,du\right) \quad \begin{bmatrix} u = x^2+2x+2, \\ du = 2(x+1)\,dx \end{bmatrix} \quad = \frac{1}{2}\ln\left|x^2+2x+2\right| + C_1$$

$$I_2 = -\int \frac{1}{(x+1)^2+1}\,dx = -\frac{1}{1}\tan^{-1}\left(\frac{x+1}{1}\right) + C_2 = -\tan^{-1}(x+1) + C_2$$

$$I_3 = \int \frac{x+1}{(x^2+2x+2)^2}\,dx = \int \frac{1}{u^2}\left(\frac{1}{2}\,du\right) = -\frac{1}{2u} + C_3 = -\frac{1}{2(x^2+2x+2)} + C_3$$

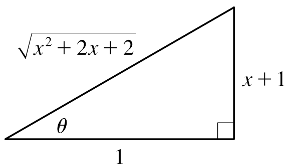
$$I_4 = -3\int \frac{1}{[(x+1)^2+1]^2}\,dx = -3\int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\,d\theta \quad \begin{bmatrix} x+1 = 1\tan\theta, \\ dx = \sec^2\theta\,d\theta \end{bmatrix}$$

$$= -3\int \frac{1}{\sec^2\theta}\,d\theta = -3\int \cos^2\theta\,d\theta = -\frac{3}{2}\int (1+\cos 2\theta)\,d\theta$$

$$= -\tfrac{3}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C_4 = -\tfrac{3}{2}\theta - \tfrac{3}{2}\left(\tfrac{1}{2}\cdot 2\sin\theta\,\cos\theta\right) + C_4$$

$$= -\frac{3}{2}\tan^{-1}\left(\frac{x+1}{1}\right) - \frac{3}{2}\cdot\frac{x+1}{\sqrt{x^2+2x+2}}\cdot\frac{1}{\sqrt{x^2+2x+2}} + C_4$$

$$= -\frac{3}{2}\tan^{-1}(x+1) - \frac{3(x+1)}{2(x^2+2x+2)} + C_4$$



So $I = I_1 + I_2 + I_3 + I_4 \qquad [C = C_1 + C_2 + C_3 + C_4]$

$$= \frac{1}{2}\ln(x^2+2x+2) - \tan^{-1}(x+1) - \frac{1}{2(x^2+2x+2)} - \frac{3}{2}\tan^{-1}(x+1) - \frac{3(x+1)}{2(x^2+2x+2)} + C$$

$$= \frac{1}{2}\ln(x^2+2x+2) - \frac{5}{2}\tan^{-1}(x+1) - \frac{3x+4}{2(x^2+2x+2)} + C$$

39. $\displaystyle\int \frac{dx}{x\sqrt{x-1}} = \int \frac{2u}{u(u^2+1)}\,du \qquad \begin{bmatrix} u=\sqrt{x-1},\ x=u^2+1 \\ u^2=x-1,\ \ dx=2u\,du \end{bmatrix}$

$\displaystyle = 2\int \frac{1}{u^2+1}\,du = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x-1} + C$

40. Let $u=\sqrt{x+3}$, so $u^2=x+3$ and $2u\,du=dx$. Then

$\displaystyle\int \frac{dx}{2\sqrt{x+3}+x} = \int \frac{2u\,du}{2u+(u^2-3)} = \int \frac{2u}{u^2+2u-3}\,du = \int \frac{2u}{(u+3)(u-1)}\,du$. Now

$\displaystyle\frac{2u}{(u+3)(u-1)} = \frac{A}{u+3} + \frac{B}{u-1} \quad\Rightarrow\quad 2u = A(u-1)+B(u+3)$. Setting $u=1$ gives $2=4B$, so $B=\frac{1}{2}$.

Setting $u=-3$ gives $-6=-4A$, so $A=\frac{3}{2}$. Thus,

$$\int \frac{2u}{(u+3)(u-1)}\,du = \int \left(\frac{\frac{3}{2}}{u+3} + \frac{\frac{1}{2}}{u-1}\,du\right)$$
$$= \tfrac{3}{2}\ln|u+3| + \tfrac{1}{2}\ln|u-1| + C = \tfrac{3}{2}\ln\!\left(\sqrt{x+3}+3\right) + \tfrac{1}{2}\ln\left|\sqrt{x+3}-1\right| + C$$

41. Let $u=\sqrt{x}$, so $u^2=x$ and $2u\,du=dx$. Then $\displaystyle\int \frac{dx}{x^2+x\sqrt{x}} = \int \frac{2u\,du}{u^4+u^3} = \int \frac{2\,du}{u^3+u^2} = \int \frac{2\,du}{u^2(u+1)}$.

$\displaystyle\frac{2}{u^2(u+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1} \quad\Rightarrow\quad 2 = Au(u+1)+B(u+1)+Cu^2$. Setting $u=0$ gives $B=2$. Setting $u=-1$

gives $C=2$. Equating coefficients of u^2, we get $0=A+C$, so $A=-2$. Thus,

$$\int \frac{2\,du}{u^2(u+1)} = \int \left(\frac{-2}{u} + \frac{2}{u^2} + \frac{2}{u+1}\right)du = -2\ln|u| - \frac{2}{u} + 2\ln|u+1| + C = -2\ln\sqrt{x} - \frac{2}{\sqrt{x}} + 2\ln\left(\sqrt{x}+1\right) + C.$$

42. Let $u=\sqrt[3]{x}$. Then $x=u^3$, $dx=3u^2\,du \quad\Rightarrow$

$$\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx = \int_0^1 \frac{3u^2\,du}{1+u} = \int_0^1 \left(3u-3+\frac{3}{1+u}\right)du = \left[\tfrac{3}{2}u^2 - 3u + 3\ln(1+u)\right]_0^1 = 3\left(\ln 2 - \tfrac{1}{2}\right).$$

43. Let $u=\sqrt[3]{x^2+1}$. Then $x^2=u^3-1$, $2x\,dx=3u^2\,du \quad\Rightarrow$

$$\int \frac{x^3\,dx}{\sqrt[3]{x^2+1}} = \int \frac{(u^3-1)\frac{3}{2}u^2\,du}{u} = \frac{3}{2}\int (u^4-u)\,du$$
$$= \tfrac{3}{10}u^5 - \tfrac{3}{4}u^2 + C = \tfrac{3}{10}(x^2+1)^{5/3} - \tfrac{3}{4}(x^2+1)^{2/3} + C$$

44. $\displaystyle\int \frac{dx}{(1+\sqrt{x}\,)^2} = \int \frac{2(u-1)}{u^2}\,du \qquad \begin{bmatrix} u=1+\sqrt{x}, \\ x=(u-1)^2,\ dx=2(u-1)\,du \end{bmatrix}$

$\displaystyle = 2\int \left(\frac{1}{u} - \frac{1}{u^2}\right)du = 2\ln|u| + \frac{2}{u} + C = 2\ln(1+\sqrt{x}\,) + \frac{2}{1+\sqrt{x}} + C$

45. If we were to substitute $u=\sqrt{x}$, then the square root would disappear but a cube root would remain. On the other hand, the substitution $u=\sqrt[3]{x}$ would eliminate the cube root but leave a square root. We can eliminate both roots by means of the substitution $u=\sqrt[6]{x}$. (Note that 6 is the least common multiple of 2 and 3.)

Let $u=\sqrt[6]{x}$. Then $x=u^6$, so $dx=6u^5\,du$ and $\sqrt{x}=u^3$, $\sqrt[3]{x}=u^2$. Thus,

$$\int \frac{dx}{\sqrt{x}-\sqrt[3]{x}} = \int \frac{6u^5\,du}{u^3-u^2} = 6\int \frac{u^5}{u^2(u-1)}\,du = 6\int \frac{u^3}{u-1}\,du$$

$$= 6\int \left(u^2+u+1+\frac{1}{u-1}\right)du \qquad \text{[by long division]}$$

$$= 6\left(\tfrac{1}{3}u^3+\tfrac{1}{2}u^2+u+\ln|u-1|\right)+C = 2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln\left|\sqrt[6]{x}-1\right|+C$$

46. Let $u=\sqrt{1+\sqrt{x}}$, so that $u^2=1+\sqrt{x}$, $x=(u^2-1)^2$, and $dx=2(u^2-1)\cdot 2u\,du=4u(u^2-1)\,du$. Then

$\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx = \int \frac{u}{(u^2-1)^2}\cdot 4u(u^2-1)\,du = \int \frac{4u^2}{u^2-1}\,du = \int \left(4+\frac{4}{u^2-1}\right)du$. Now

$\frac{4}{u^2-1}=\frac{A}{u+1}+\frac{B}{u-1} \;\Rightarrow\; 4=A(u-1)+B(u+1)$. Setting $u=1$ gives $4=2B$, so $B=2$. Setting $u=-1$ gives $4=-2A$, so $A=-2$. Thus,

$$\int\left(4+\frac{4}{u^2-1}\right)du = \int\left(4-\frac{2}{u+1}+\frac{2}{u-1}\right)du = 4u-2\ln|u+1|+2\ln|u-1|+C$$

$$= 4\sqrt{1+\sqrt{x}}-2\ln\left(\sqrt{1+\sqrt{x}}+1\right)+2\ln\left(\sqrt{1+\sqrt{x}}-1\right)+C$$

47. Let $u=e^x$. Then $x=\ln u$, $dx=\frac{du}{u} \;\Rightarrow$

$$\int \frac{e^{2x}\,dx}{e^{2x}+3e^x+2} = \int \frac{u^2\,(du/u)}{u^2+3u+2} = \int \frac{u\,du}{(u+1)(u+2)} = \int\left[\frac{-1}{u+1}+\frac{2}{u+2}\right]du$$

$$= 2\ln|u+2|-\ln|u+1|+C = \ln\frac{(e^x+2)^2}{e^x+1}+C$$

48. Let $u=\cos x$, so that $du=-\sin x\,dx$. Then $\int \frac{\sin x}{\cos^2 x-3\cos x}\,dx = \int \frac{1}{u^2-3u}(-du) = \int \frac{-1}{u(u-3)}\,du$.

$\frac{-1}{u(u-3)}=\frac{A}{u}+\frac{B}{u-3} \;\Rightarrow\; -1=A(u-3)+Bu$. Setting $u=3$ gives $B=-\frac{1}{3}$. Setting $u=0$ gives $A=\frac{1}{3}$.

Thus, $\int \frac{-1}{u(u-3)}\,du = \int\left(\frac{\frac{1}{3}}{u}-\frac{\frac{1}{3}}{u-3}\right)du = \frac{1}{3}\ln|u|-\frac{1}{3}\ln|u-3|+C = \frac{1}{3}\ln|\cos x|-\frac{1}{3}\ln|\cos x-3|+C$.

49. Let $u=\tan t$, so that $du=\sec^2 t\,dt$. Then $\int \frac{\sec^2 t}{\tan^2 t+3\tan t+2}\,dt = \int \frac{1}{u^2+3u+2}\,du = \int \frac{1}{(u+1)(u+2)}\,du$.

Now $\frac{1}{(u+1)(u+2)}=\frac{A}{u+1}+\frac{B}{u+2} \;\Rightarrow\; 1=A(u+2)+B(u+1)$.

Setting $u=-2$ gives $1=-B$, so $B=-1$. Setting $u=-1$ gives $1=A$.

Thus, $\int \frac{1}{(u+1)(u+2)}\,du = \int\left(\frac{1}{u+1}-\frac{1}{u+2}\right)du = \ln|u+1|-\ln|u+2|+C = \ln|\tan t+1|-\ln|\tan t+2|+C$.

50. Let $u=e^x$, so that $du=e^x\,dx$. Then $\int \frac{e^x}{(e^x-2)(e^{2x}+1)}\,dx = \int \frac{1}{(u-2)(u^2+1)}\,du$. Now

$\frac{1}{(u-2)(u^2+1)}=\frac{A}{u-2}+\frac{Bu+C}{u^2+1} \;\Rightarrow\; 1=A(u^2+1)+(Bu+C)(u-2)$. Setting $u=2$ gives $1=5A$, so $A=\frac{1}{5}$.

Setting $u = 0$ gives $1 = \frac{1}{5} - 2C$, so $C = -\frac{2}{5}$. Comparing coefficients of u^2 gives $0 = \frac{1}{5} + B$, so $B = -\frac{1}{5}$. Thus,

$$\begin{aligned}\int \frac{1}{(u-2)(u^2+1)}\,du &= \int \left(\frac{\frac{1}{5}}{u-2} + \frac{-\frac{1}{5}u - \frac{2}{5}}{u^2+1}\right) du = \frac{1}{5}\int \frac{1}{u-2}\,du - \frac{1}{5}\int \frac{u}{u^2+1}\,du - \frac{2}{5}\int \frac{1}{u^2+1}\,du \\ &= \tfrac{1}{5}\ln|u-2| - \tfrac{1}{5}\cdot\tfrac{1}{2}\ln\left|u^2+1\right| - \tfrac{2}{5}\tan^{-1}u + C \\ &= \tfrac{1}{5}\ln|e^x-2| - \tfrac{1}{10}\ln(e^{2x}+1) - \tfrac{2}{5}\tan^{-1}e^x + C\end{aligned}$$

51. Let $u = e^x$, so that $du = e^x\,dx$ and $dx = \dfrac{du}{u}$. Then $\displaystyle\int \frac{dx}{1+e^x} = \int \frac{du}{(1+u)u}$. $\dfrac{1}{u(u+1)} = \dfrac{A}{u} + \dfrac{B}{u+1} \Rightarrow$

$1 = A(u+1) + Bu$. Setting $u = -1$ gives $B = -1$. Setting $u = 0$ gives $A = 1$. Thus,

$$\int \frac{du}{u(u+1)} = \int \left(\frac{1}{u} - \frac{1}{u+1}\right) du = \ln|u| - \ln|u+1| + C = \ln e^x - \ln(e^x+1) + C = x - \ln(e^x+1) + C.$$

52. Let $u = \sinh t$, so that $du = \cosh t\,dt$. Then $\displaystyle\int \frac{\cosh t}{\sinh^2 t + \sinh^4 t}\,dt = \int \frac{1}{u^2+u^4}\,du = \int \frac{1}{u^2(u^2+1)}\,du.$

$\dfrac{1}{u^2(u^2+1)} = \dfrac{A}{u} + \dfrac{B}{u^2} + \dfrac{Cu+D}{u^2+1} \quad\Rightarrow\quad 1 = Au(u^2+1) + B(u^2+1) + (Cu+D)u^2$. Setting $u = 0$ gives $B = 1$.

Comparing coefficients of u^2, we get $0 = B + D$, so $D = -1$. Comparing coefficients of u, we get $0 = A$. Comparing coefficients of u^3, we get $0 = A + C$, so $C = 0$. Thus,

$$\begin{aligned}\int \frac{1}{u^2(u^2+1)}\,du &= \int \left(\frac{1}{u^2} - \frac{1}{u^2+1}\right) du = -\frac{1}{u} - \tan^{-1}u + C = -\frac{1}{\sinh t} - \tan^{-1}(\sinh t) + C \\ &= -\operatorname{csch} t - \tan^{-1}(\sinh t) + C\end{aligned}$$

53. Let $u = \ln(x^2 - x + 2)$, $dv = dx$. Then $du = \dfrac{2x-1}{x^2-x+2}\,dx$, $v = x$, and (by integration by parts)

$$\begin{aligned}\int \ln(x^2-x+2)\,dx &= x\ln(x^2-x+2) - \int \frac{2x^2-x}{x^2-x+2}\,dx = x\ln(x^2-x+2) - \int \left(2 + \frac{x-4}{x^2-x+2}\right) dx \\ &= x\ln(x^2-x+2) - 2x - \int \frac{\frac{1}{2}(2x-1)}{x^2-x+2}\,dx + \frac{7}{2}\int \frac{dx}{(x-\frac{1}{2})^2 + \frac{7}{4}} \\ &= x\ln(x^2-x+2) - 2x - \frac{1}{2}\ln(x^2-x+2) + \frac{7}{2}\int \frac{\frac{\sqrt{7}}{2}\,du}{\frac{7}{4}(u^2+1)} \qquad \left[\begin{array}{c}\text{where } x - \frac{1}{2} = \frac{\sqrt{7}}{2}u, \\ dx = \frac{\sqrt{7}}{2}\,du, \\ (x-\frac{1}{2})^2 + \frac{7}{4} = \frac{7}{4}(u^2+1)\end{array}\right] \\ &= (x - \tfrac{1}{2})\ln(x^2-x+2) - 2x + \sqrt{7}\tan^{-1}u + C \\ &= (x - \tfrac{1}{2})\ln(x^2-x+2) - 2x + \sqrt{7}\tan^{-1}\frac{2x-1}{\sqrt{7}} + C\end{aligned}$$

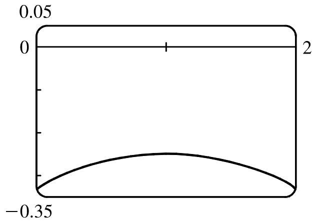
54. Let $u = \tan^{-1}x$, $dv = x\,dx \quad\Rightarrow\quad du = dx/(1+x^2)$, $v = \frac{1}{2}x^2$.

Then $\displaystyle\int x\tan^{-1}x\,dx = \tfrac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$. To evaluate the last integral, use long division or observe that

$\displaystyle\int \frac{x^2}{1+x^2}\,dx = \int \frac{(1+x^2)-1}{1+x^2}\,dx = \int 1\,dx - \int \frac{1}{1+x^2}\,dx = x - \tan^{-1}x + C_1$. So

$\int x\tan^{-1}x\,dx = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}(x - \tan^{-1}x + C_1) = \frac{1}{2}(x^2\tan^{-1}x + \tan^{-1}x - x) + C.$

55.



From the graph, we see that the integral will be negative, and we guess that the area is about the same as that of a rectangle with width 2 and height 0.3, so we estimate the integral to be $-(2 \cdot 0.3) = -0.6$. Now

$$\frac{1}{x^2-2x-3} = \frac{1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \quad \Leftrightarrow$$

$1 = (A+B)x + A - 3B$, so $A = -B$ and $A - 3B = 1 \quad \Leftrightarrow \quad A = \frac{1}{4}$ and $B = -\frac{1}{4}$, so the integral becomes

$$\int_0^2 \frac{dx}{x^2-2x-3} = \frac{1}{4}\int_0^2 \frac{dx}{x-3} - \frac{1}{4}\int_0^2 \frac{dx}{x+1} = \frac{1}{4}\Big[\ln|x-3| - \ln|x+1|\Big]_0^2 = \frac{1}{4}\left[\ln\left|\frac{x-3}{x+1}\right|\right]_0^2$$

$$= \tfrac{1}{4}\left(\ln\tfrac{1}{3} - \ln 3\right) = -\tfrac{1}{2}\ln 3 \approx -0.55$$

56. $\boldsymbol{k = 0}$: $\displaystyle\int \frac{dx}{x^2+k} = \int \frac{dx}{x^2} = -\frac{1}{x} + C$

$\boldsymbol{k > 0}$: $\displaystyle\int \frac{dx}{x^2+k} = \int \frac{dx}{x^2+(\sqrt{k}\,)^2} = \frac{1}{\sqrt{k}}\tan^{-1}\!\left(\frac{x}{\sqrt{k}}\right) + C$

$\boldsymbol{k < 0}$: $\displaystyle\int \frac{dx}{x^2+k} = \int \frac{dx}{x^2-(-k)} = \int \frac{dx}{x^2-\left(\sqrt{-k}\right)^2} = \frac{1}{2\sqrt{-k}}\ln\left|\frac{x-\sqrt{-k}}{x+\sqrt{-k}}\right| + C$ [by Example 3]

57. $\displaystyle\int \frac{dx}{x^2-2x} = \int \frac{dx}{(x-1)^2-1} = \int \frac{du}{u^2-1}$ [put $u = x-1$]

$\displaystyle= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C$ [by Equation 6] $\displaystyle= \frac{1}{2}\ln\left|\frac{x-2}{x}\right| + C$

58. $\displaystyle\int \frac{(2x+1)\,dx}{4x^2+12x-7} = \frac{1}{4}\int \frac{(8x+12)\,dx}{4x^2+12x-7} - \int \frac{2\,dx}{(2x+3)^2-16}$

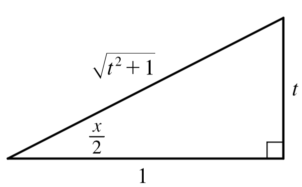
$\displaystyle= \tfrac{1}{4}\ln\left|4x^2+12x-7\right| - \int \frac{du}{u^2-16}$ [put $u = 2x+3$]

$= \tfrac{1}{4}\ln\left|4x^2+12x-7\right| - \tfrac{1}{8}\ln|(u-4)/(u+4)| + C$ [by Equation 6]

$= \tfrac{1}{4}\ln\left|4x^2+12x-7\right| - \tfrac{1}{8}\ln|(2x-1)/(2x+7)| + C$

59. (a) If $t = \tan\left(\frac{x}{2}\right)$, then $\frac{x}{2} = \tan^{-1} t$. The figure gives

$\displaystyle\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}}$ and $\displaystyle\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}$.



(b) $\displaystyle\cos x = \cos\left(2\cdot\frac{x}{2}\right) = 2\cos^2\left(\frac{x}{2}\right) - 1$

$\displaystyle= 2\left(\frac{1}{\sqrt{1+t^2}}\right)^2 - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$

(c) $\displaystyle\frac{x}{2} = \arctan t \quad\Rightarrow\quad x = 2\arctan t \quad\Rightarrow\quad dx = \frac{2}{1+t^2}\,dt$

60. Let $t = \tan(x/2)$. Then, by using the expressions in Exercise 59, we have

$$\int \frac{dx}{1-\cos x} = \int \frac{2\,dt/(1+t^2)}{1-(1-t^2)/(1+t^2)} = \int \frac{2\,dt}{(1+t^2)-(1-t^2)} = \int \frac{2\,dt}{2t^2} = \int \frac{1}{t^2}\,dt$$
$$= -\frac{1}{t} + C = -\frac{1}{\tan(x/2)} + C = -\cot(x/2) + C$$

Another method: $\displaystyle\int \frac{dx}{1-\cos x} = \int\left(\frac{1}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x}\right)dx = \int \frac{1+\cos x}{1-\cos^2 x}\,dx = \int \frac{1+\cos x}{\sin^2 x}\,dx$

$$= \int\left(\frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x}\right)dx = \int(\csc^2 x + \csc x\,\cot x)\,dx = -\cot x - \csc x + C$$

61. Let $t = \tan(x/2)$. Then, using the expressions in Exercise 59, we have

$$\int \frac{1}{3\sin x - 4\cos x}\,dx = \int \frac{1}{3\left(\dfrac{2t}{1+t^2}\right) - 4\left(\dfrac{1-t^2}{1+t^2}\right)}\,\frac{2\,dt}{1+t^2} = 2\int \frac{dt}{3(2t)-4(1-t^2)} = \int \frac{dt}{2t^2+3t-2}$$
$$= \int \frac{dt}{(2t-1)(t+2)} = \int\left[\frac{2}{5}\frac{1}{2t-1} - \frac{1}{5}\frac{1}{t+2}\right]dt \qquad \text{[using partial fractions]}$$
$$= \tfrac{1}{5}\Big[\ln|2t-1| - \ln|t+2|\Big] + C = \frac{1}{5}\ln\left|\frac{2t-1}{t+2}\right| + C = \frac{1}{5}\ln\left|\frac{2\tan(x/2)-1}{\tan(x/2)+2}\right| + C$$

62. Let $t = \tan(x/2)$. Then, by Exercise 59,

$$\int_{\pi/3}^{\pi/2} \frac{dx}{1+\sin x - \cos x} = \int_{1/\sqrt{3}}^{1} \frac{2\,dt/(1+t^2)}{1+2t/(1+t^2)-(1-t^2)/(1+t^2)} = \int_{1/\sqrt{3}}^{1} \frac{2\,dt}{1+t^2+2t-1+t^2}$$
$$= \int_{1/\sqrt{3}}^{1}\left[\frac{1}{t} - \frac{1}{t+1}\right]dt = \Big[\ln t - \ln(t+1)\Big]_{1/\sqrt{3}}^{1} = \ln\frac{1}{2} - \ln\frac{1}{\sqrt{3}+1} = \ln\frac{\sqrt{3}+1}{2}$$

63. Let $t = \tan(x/2)$. Then, by Exercise 59,

$$\int_0^{\pi/2} \frac{\sin 2x}{2+\cos x}\,dx = \int_0^{\pi/2} \frac{2\sin x\,\cos x}{2+\cos x}\,dx = \int_0^1 \frac{2\cdot\dfrac{2t}{1+t^2}\cdot\dfrac{1-t^2}{1+t^2}}{2+\dfrac{1-t^2}{1+t^2}}\,\frac{2}{1+t^2}\,dt = \int_0^1 \frac{\dfrac{8t(1-t^2)}{(1+t^2)^2}}{2(1+t^2)+(1-t^2)}\,dt$$
$$= \int_0^1 8t\cdot\frac{1-t^2}{(t^2+3)(t^2+1)^2}\,dt = I$$

If we now let $u = t^2$, then $\displaystyle\frac{1-t^2}{(t^2+3)(t^2+1)^2} = \frac{1-u}{(u+3)(u+1)^2} = \frac{A}{u+3} + \frac{B}{u+1} + \frac{C}{(u+1)^2} \quad\Rightarrow$

$1-u = A(u+1)^2 + B(u+3)(u+1) + C(u+3)$. Set $u=-1$ to get $2 = 2C$, so $C = 1$. Set $u = -3$ to get $4 = 4A$, so $A = 1$. Set $u = 0$ to get $1 = 1 + 3B + 3$, so $B = -1$. So

$$I = \int_0^1\left[\frac{8t}{t^2+3} - \frac{8t}{t^2+1} + \frac{8t}{(t^2+1)^2}\right]dt = \left[4\ln(t^2+3) - 4\ln(t^2+1) - \frac{4}{t^2+1}\right]_0^1$$
$$= (4\ln 4 - 4\ln 2 - 2) - (4\ln 3 - 0 - 4) = 8\ln 2 - 4\ln 2 - 4\ln 3 + 2 = 4\ln\tfrac{2}{3} + 2$$

64. $\displaystyle\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \quad\Rightarrow\quad 1 = A(x^2+1) + (Bx+C)x$. Set $x = 0$ to get $1 = A$. So

$1 = (1+B)x^2 + Cx + 1 \quad\Rightarrow\quad B + 1 = 0 \quad [B = -1]$ and $C = 0$. Thus, the area is

$$\int_1^2 \frac{1}{x^3+x}\,dx = \int_1^2 \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx = \left[\ln|x| - \tfrac{1}{2}\ln\left|x^2+1\right|\right]_1^2 = \left(\ln 2 - \tfrac{1}{2}\ln 5\right) - \left(0 - \tfrac{1}{2}\ln 2\right)$$
$$= \tfrac{3}{2}\ln 2 - \tfrac{1}{2}\ln 5 \quad \left[\text{or } \tfrac{1}{2}\ln\tfrac{8}{5}\right]$$

65. By long division, $\dfrac{x^2+1}{3x-x^2} = -1 + \dfrac{3x+1}{3x-x^2}$. Now

$\dfrac{3x+1}{3x-x^2} = \dfrac{3x+1}{x(3-x)} = \dfrac{A}{x} + \dfrac{B}{3-x} \quad\Rightarrow\quad 3x+1 = A(3-x)+Bx$. Set $x=3$ to get $10=3B$, so $B=\frac{10}{3}$. Set $x=0$ to get $1=3A$, so $A=\frac{1}{3}$. Thus, the area is

$$\int_1^2 \frac{x^2+1}{3x-x^2}\,dx = \int_1^2 \left(-1 + \frac{\frac{1}{3}}{x} + \frac{\frac{10}{3}}{3-x}\right) dx = \left[-x + \tfrac{1}{3}\ln|x| - \tfrac{10}{3}\ln|3-x|\right]_1^2$$
$$= \left(-2 + \tfrac{1}{3}\ln 2 - 0\right) - \left(-1 + 0 - \tfrac{10}{3}\ln 2\right) = -1 + \tfrac{11}{3}\ln 2$$

66. (a) We use disks, so the volume is $V = \pi\displaystyle\int_0^1 \left[\frac{1}{x^2+3x+2}\right]^2 dx = \pi\int_0^1 \frac{dx}{(x+1)^2(x+2)^2}$. To evaluate the integral, we use partial fractions: $\dfrac{1}{(x+1)^2(x+2)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+2} + \dfrac{D}{(x+2)^2} \quad\Rightarrow$

$1 = A(x+1)(x+2)^2 + B(x+2)^2 + C(x+1)^2(x+2) + D(x+1)^2$. We set $x=-1$, giving $B=1$, then set $x=-2$, giving $D=1$. Now equating coefficients of x^3 gives $A=-C$, and then equating constants gives $1 = 4A + 4 + 2(-A) + 1 \quad\Rightarrow\quad A=-2 \quad\Rightarrow\quad C=2$. So the expression becomes

$$V = \pi\int_0^1 \left[\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{2}{(x+2)} + \frac{1}{(x+2)^2}\right] dx = \pi\left[2\ln\left|\frac{x+2}{x+1}\right| - \frac{1}{x+1} - \frac{1}{x+2}\right]_0^1$$
$$= \pi\left[\left(2\ln\tfrac{3}{2} - \tfrac{1}{2} - \tfrac{1}{3}\right) - \left(2\ln 2 - 1 - \tfrac{1}{2}\right)\right] = \pi\left(2\ln\tfrac{3/2}{2} + \tfrac{2}{3}\right) = \pi\left(\tfrac{2}{3} + \ln\tfrac{9}{16}\right)$$

(b) In this case, we use cylindrical shells, so the volume is $V = 2\pi\displaystyle\int_0^1 \frac{x\,dx}{x^2+3x+2} = 2\pi\int_0^1 \frac{x\,dx}{(x+1)(x+2)}$. We use partial fractions to simplify the integrand: $\dfrac{x}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} \quad\Rightarrow\quad x = (A+B)x + 2A + B$. So $A+B=1$ and $2A+B=0 \quad\Rightarrow\quad A=-1$ and $B=2$. So the volume is

$$2\pi\int_0^1 \left[\frac{-1}{x+1} + \frac{2}{x+2}\right] dx = 2\pi\Big[-\ln|x+1| + 2\,\ln|x+2|\Big]_0^1$$
$$= 2\pi(-\ln 2 + 2\ln 3 + \ln 1 - 2\ln 2) = 2\pi(2\ln 3 - 3\ln 2) = 2\pi\ln\tfrac{9}{8}$$

67. $t = \displaystyle\int \frac{P+S}{P[(r-1)P-S]}\,dP = \int \frac{P+S}{P(0.1P-S)}\,dP \quad [r=1.1]$. Now $\dfrac{P+S}{P(0.1P-S)} = \dfrac{A}{P} + \dfrac{B}{0.1P-S} \quad\Rightarrow$

$P+S = A(0.1P-S) + BP$. Substituting 0 for P gives $S=-AS \quad\Rightarrow\quad A=-1$. Substituting $10S$ for P gives $11S = 10BS \quad\Rightarrow\quad B=\frac{11}{10}$. Thus, $t = \displaystyle\int\left(\frac{-1}{P} + \frac{11/10}{0.1P-S}\right) dP \quad\Rightarrow\quad t = -\ln P + 11\ln(0.1P-S) + C$.

When $t=0$, $P=10{,}000$ and $S=900$, so $0 = -\ln 10{,}000 + 11\ln(1000-900) + C \quad\Rightarrow$

$C = \ln 10{,}000 - 11\ln 100 \quad [= \ln 10^{-18} \approx -41.45]$.

Therefore, $t = -\ln P + 11\ln\left(\frac{1}{10}P - 900\right) + \ln 10{,}000 - 11\ln 100 \quad\Rightarrow\quad t = \ln\dfrac{10{,}000}{P} + 11\ln\dfrac{P-9000}{1000}$.

68. If we subtract and add $2x^2$, we get

$$x^4+1=x^4+2x^2+1-2x^2=\left(x^2+1\right)^2-2x^2=\left(x^2+1\right)^2-\left(\sqrt{2}\,x\right)^2$$
$$=\left[\left(x^2+1\right)-\sqrt{2}\,x\right]\left[\left(x^2+1\right)+\sqrt{2}\,x\right]=\left(x^2-\sqrt{2}\,x+1\right)\left(x^2+\sqrt{2}\,x+1\right)$$

So we can decompose $\dfrac{1}{x^4+1}=\dfrac{Ax+B}{x^2+\sqrt{2}\,x+1}+\dfrac{Cx+D}{x^2-\sqrt{2}\,x+1}\quad\Rightarrow$

$1=(Ax+B)\left(x^2-\sqrt{2}\,x+1\right)+(Cx+D)\left(x^2+\sqrt{2}\,x+1\right)$. Setting the constant terms equal gives $B+D=1$, then from the coefficients of x^3 we get $A+C=0$. Now from the coefficients of x we get $A+C+(B-D)\sqrt{2}=0\quad\Leftrightarrow$ $[(1-D)-D]\sqrt{2}=0\quad\Rightarrow\quad D=\frac{1}{2}\quad\Rightarrow\quad B=\frac{1}{2}$, and finally, from the coefficients of x^2 we get $\sqrt{2}\,(C-A)+B+D=0\quad\Rightarrow\quad C-A=-\frac{1}{\sqrt{2}}\quad\Rightarrow\quad C=-\frac{\sqrt{2}}{4}$ and $A=\frac{\sqrt{2}}{4}$. So we rewrite the integrand, splitting the terms into forms which we know how to integrate:

$$\frac{1}{x^4+1}=\frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+\sqrt{2}\,x+1}+\frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2-\sqrt{2}\,x+1}=\frac{1}{4\sqrt{2}}\left[\frac{2x+2\sqrt{2}}{x^2+\sqrt{2}\,x+1}-\frac{2x-2\sqrt{2}}{x^2-\sqrt{2}\,x+1}\right]$$
$$=\frac{\sqrt{2}}{8}\left[\frac{2x+\sqrt{2}}{x^2+\sqrt{2}\,x+1}-\frac{2x-\sqrt{2}}{x^2-\sqrt{2}\,x+1}\right]+\frac{1}{4}\left[\frac{1}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}+\frac{1}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\right]$$

Now we integrate: $\displaystyle\int\frac{dx}{x^4+1}=\frac{\sqrt{2}}{8}\ln\!\left(\frac{x^2+\sqrt{2}\,x+1}{x^2-\sqrt{2}\,x+1}\right)+\frac{\sqrt{2}}{4}\left[\tan^{-1}\!\left(\sqrt{2}\,x+1\right)+\tan^{-1}\!\left(\sqrt{2}\,x-1\right)\right]+C.$

69. (a) In Maple, we define $f(x)$, and then use `convert(f,parfrac,x);` to obtain

$$f(x)=\frac{24{,}110/4879}{5x+2}-\frac{668/323}{2x+1}-\frac{9438/80{,}155}{3x-7}+\frac{(22{,}098x+48{,}935)/260{,}015}{x^2+x+5}$$

In Mathematica, we use the command `Apart`, and in Derive, we use `Expand`.

(b) $\displaystyle\int f(x)\,dx=\tfrac{24{,}110}{4879}\cdot\tfrac{1}{5}\ln|5x+2|-\tfrac{668}{323}\cdot\tfrac{1}{2}\ln|2x+1|-\tfrac{9438}{80{,}155}\cdot\tfrac{1}{3}\ln|3x-7|$

$$+\frac{1}{260{,}015}\int\frac{22{,}098\left(x+\frac{1}{2}\right)+37{,}886}{\left(x+\frac{1}{2}\right)^2+\frac{19}{4}}\,dx+C$$

$$=\tfrac{24{,}110}{4879}\cdot\tfrac{1}{5}\ln|5x+2|-\tfrac{668}{323}\cdot\tfrac{1}{2}\ln|2x+1|-\tfrac{9438}{80{,}155}\cdot\tfrac{1}{3}\ln|3x-7|$$
$$+\tfrac{1}{260{,}015}\left[22{,}098\cdot\tfrac{1}{2}\ln\left(x^2+x+5\right)+37{,}886\cdot\sqrt{\tfrac{4}{19}}\tan^{-1}\left(\tfrac{1}{\sqrt{19/4}}\left(x+\tfrac{1}{2}\right)\right)\right]+C$$
$$=\tfrac{4822}{4879}\ln|5x+2|-\tfrac{334}{323}\ln|2x+1|-\tfrac{3146}{80{,}155}\ln|3x-7|+\tfrac{11{,}049}{260{,}015}\ln\left(x^2+x+5\right)$$
$$+\tfrac{75{,}772}{260{,}015\sqrt{19}}\tan^{-1}\left[\tfrac{1}{\sqrt{19}}\left(2x+1\right)\right]+C$$

Using a CAS, we get

$$\frac{4822\ln(5x+2)}{4879}-\frac{334\ln(2x+1)}{323}-\frac{3146\ln(3x-7)}{80{,}155}$$
$$+\frac{11{,}049\ln(x^2+x+5)}{260{,}015}+\frac{3988\sqrt{19}}{260{,}015}\tan^{-1}\left[\frac{\sqrt{19}}{19}(2x+1)\right]$$

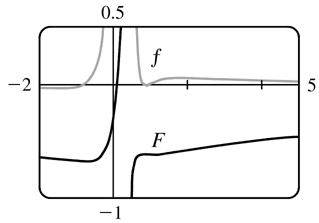
The main difference in this answer is that the absolute value signs and the constant of integration have been omitted. Also, the fractions have been reduced and the denominators rationalized.

70. (a) In Maple, we define $f(x)$, and then use `convert(f,parfrac,x);` to get

$$f(x) = \frac{5828/1815}{(5x-2)^2} - \frac{59{,}096/19{,}965}{5x-2} + \frac{2(2843x+816)/3993}{2x^2+1} + \frac{(313x-251)/363}{(2x^2+1)^2}.$$

In Mathematica, we use the command `Apart`, and in Derive, we use `Expand`.

(b) As we saw in Exercise 69, computer algebra systems omit the absolute value signs in $\int (1/y)\,dy = \ln|y|$. So we use the CAS to integrate the expression in part (a) and add the necessary absolute value signs and constant of integration to get



$$\int f(x)\,dx = -\frac{5828}{9075(5x-2)} - \frac{59{,}096\ln|5x-2|}{99{,}825} + \frac{2843\ln(2x^2+1)}{7986} + \frac{503}{15{,}972}\sqrt{2}\tan^{-1}(\sqrt{2}\,x) - \frac{1}{2904}\frac{1004x+626}{2x^2+1} + C$$

(c) From the graph, we see that f goes from negative to positive at $x \approx -0.78$, then back to negative at $x \approx 0.8$, and finally back to positive at $x = 1$. Also, $\lim_{x\to 0.4} f(x) = \infty$. So we see (by the First Derivative Test) that $\int f(x)\,dx$ has minima at $x \approx -0.78$ and $x = 1$, and a maximum at $x \approx 0.80$, and that $\int f(x)\,dx$ is unbounded as $x \to 0.4$. Note also that just to the right of $x = 0.4$, f has large values, so $\int f(x)\,dx$ increases rapidly, but slows down as f drops toward 0. $\int f(x)\,dx$ decreases from about 0.8 to 1, then increases slowly since f stays small and positive.

71. $\dfrac{x^4(1-x)^4}{1+x^2} = \dfrac{x^4(1-4x+6x^2-4x^3+x^4)}{1+x^2} = \dfrac{x^8-4x^7+6x^6-4x^5+x^4}{1+x^2} = x^6-4x^5+5x^4-4x^2+4-\dfrac{4}{1+x^2}$, so

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \left[\frac{1}{7}x^7 - \frac{2}{3}x^6 + x^5 - \frac{4}{3}x^3 + 4x - 4\tan^{-1}x\right]_0^1 = \left(\frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4\cdot\frac{\pi}{4}\right) - 0 = \frac{22}{7} - \pi.$$

72. (a) Let $u = (x^2+a^2)^{-n}$, $dv = dx \quad\Rightarrow\quad du = -n(x^2+a^2)^{-n-1}\,2x\,dx$, $v = x$.

$$\begin{aligned} I_n = \int \frac{dx}{(x^2+a^2)^n} &= \frac{x}{(x^2+a^2)^n} - \int \frac{-2nx^2}{(x^2+a^2)^{n+1}}\,dx \qquad \text{[by parts]} \\ &= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{(x^2+a^2)-a^2}{(x^2+a^2)^{n+1}}\,dx \\ &= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{dx}{(x^2+a^2)^n} - 2na^2\int \frac{dx}{(x^2+a^2)^{n+1}} \end{aligned}$$

Recognizing the last two integrals as I_n and I_{n+1}, we can solve for I_{n+1} in terms of I_n.

$$2na^2 I_{n+1} = \frac{x}{(x^2+a^2)^n} + 2nI_n - I_n \quad\Rightarrow\quad I_{n+1} = \frac{x}{2a^2n(x^2+a^2)^n} + \frac{2n-1}{2a^2n}I_n \quad\Rightarrow$$

$$I_n = \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)}I_{n-1}$$ [decrease n-values by 1], which is the desired result.

(b) Using part (a) with $a = 1$ and $n = 2$, we get

$$\int \frac{dx}{(x^2+1)^2} = \frac{x}{2(x^2+1)} + \frac{1}{2}\int \frac{dx}{x^2+1} = \frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1}x + C$$

Using part (a) with $a = 1$ and $n = 3$, we get

$$\int \frac{dx}{(x^2+1)^3} = \frac{x}{2(2)(x^2+1)^2} + \frac{3}{2(2)} \int \frac{dx}{(x^2+1)^2} = \frac{x}{4(x^2+1)^2} + \frac{3}{4}\left[\frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1} x\right] + C$$
$$= \frac{x}{4(x^2+1)^2} + \frac{3x}{8(x^2+1)} + \frac{3}{8}\tan^{-1} x + C$$

73. There are only finitely many values of x where $Q(x) = 0$ (assuming that Q is not the zero polynomial). At all other values of x, $F(x)/Q(x) = G(x)/Q(x)$, so $F(x) = G(x)$. In other words, the values of F and G agree at all except perhaps finitely many values of x. By continuity of F and G, the polynomials F and G must agree at those values of x too.

More explicitly: if a is a value of x such that $Q(a) = 0$, then $Q(x) \neq 0$ for all x sufficiently close to a. Thus,

$$\begin{aligned} F(a) &= \lim_{x\to a} F(x) && \text{[by continuity of } F] \\ &= \lim_{x\to a} G(x) && [\text{whenever } Q(x) \neq 0] \\ &= G(a) && \text{[by continuity of } G] \end{aligned}$$

74. Let $f(x) = ax^2 + bx + c$. We calculate the partial fraction decomposition of $\dfrac{f(x)}{x^2(x+1)^3}$. Since $f(0) = 1$, we must have $c = 1$, so $\dfrac{f(x)}{x^2(x+1)^3} = \dfrac{ax^2+bx+1}{x^2(x+1)^3} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2} + \dfrac{E}{(x+1)^3}$. Now in order for the integral not to contain any logarithms (that is, in order for it to be a rational function), we must have $A = C = 0$, so $ax^2 + bx + 1 = B(x+1)^3 + Dx^2(x+1) + Ex^2$. Equating constant terms gives $B = 1$, then equating coefficients of x gives $3B = b \;\Rightarrow\; b = 3$. This is the quantity we are looking for, since $f'(0) = b$.

75. If $a \neq 0$ and n is a positive integer, then $f(x) = \dfrac{1}{x^n(x-a)} = \dfrac{A_1}{x} + \dfrac{A_2}{x^2} + \cdots + \dfrac{A_n}{x^n} + \dfrac{B}{x-a}$. Multiply both sides by $x^n(x-a)$ to get $1 = A_1x^{n-1}(x-a) + A_2x^{n-2}(x-a) + \cdots + A_n(x-a) + Bx^n$. Let $x = a$ in the last equation to get $1 = Ba^n \;\Rightarrow\; B = 1/a^n$. So

$$\begin{aligned} f(x) - \frac{B}{x-a} &= \frac{1}{x^n(x-a)} - \frac{1}{a^n(x-a)} = \frac{a^n - x^n}{x^n a^n (x-a)} = -\frac{x^n - a^n}{a^n x^n (x-a)} \\ &= -\frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})}{a^n x^n (x-a)} \\ &= -\left(\frac{x^{n-1}}{a^n x^n} + \frac{x^{n-2}a}{a^n x^n} + \frac{x^{n-3}a^2}{a^n x^n} + \cdots + \frac{xa^{n-2}}{a^n x^n} + \frac{a^{n-1}}{a^n x^n}\right) \\ &= -\frac{1}{a^n x} - \frac{1}{a^{n-1}x^2} - \frac{1}{a^{n-2}x^3} - \cdots - \frac{1}{a^2 x^{n-1}} - \frac{1}{ax^n} \end{aligned}$$

Thus, $f(x) = \dfrac{1}{x^n(x-a)} = -\dfrac{1}{a^n x} - \dfrac{1}{a^{n-1}x^2} - \cdots - \dfrac{1}{ax^n} + \dfrac{1}{a^n(x-a)}$.

7.5 Strategy for Integration

1. Let $u = 1 - \sin x$. Then $du = -\cos x\, dx \;\Rightarrow$

$$\int \frac{\cos x}{1 - \sin x}\, dx = \int \frac{1}{u}(-du) = -\ln|u| + C = -\ln|1 - \sin x| + C = -\ln(1 - \sin x) + C$$

2. Let $u = 3x + 1$. Then $du = 3\,dx \quad \Rightarrow$

$$\int_0^1 (3x+1)^{\sqrt{2}}\,dx = \int_1^4 u^{\sqrt{2}}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\left[\frac{1}{\sqrt{2}+1}\,u^{\sqrt{2}+1}\right]_1^4 = \frac{1}{3\big(\sqrt{2}+1\big)}\left(4^{\sqrt{2}+1} - 1\right)$$

3. Let $u = \ln y$, $dv = \sqrt{y}\,dy \quad \Rightarrow \quad du = \dfrac{1}{y}\,dy$, $v = \dfrac{2}{3}y^{3/2}$. Then

$$\int_1^4 \sqrt{y}\ln y\,dy = \left[\frac{2}{3}y^{3/2}\ln y\right]_1^4 - \int_1^4 \frac{2}{3}y^{1/2}\,dy = \frac{2}{3}\cdot 8\ln 4 - 0 - \left[\frac{4}{9}y^{3/2}\right]_1^4 = \frac{16}{3}(2\ln 2) - \left(\frac{4}{9}\cdot 8 - \frac{4}{9}\right) = \frac{32}{3}\ln 2 - \frac{28}{9}$$

4. $$\int \frac{\sin^3 x}{\cos x}\,dx = \int \frac{\sin^2 x\,\sin x}{\cos x}\,dx = \int \frac{(1-\cos^2 x)\sin x}{\cos x}\,dx = \int \frac{1-u^2}{u}\,(-du) \qquad \begin{bmatrix} u = \cos x \\ du = -\sin x\,dx \end{bmatrix}$$

$$= \int \left(u - \tfrac{1}{u}\right)du = \tfrac{1}{2}u^2 - \ln|u| + C = \tfrac{1}{2}\cos^2 x - \ln|\cos x| + C$$

5. Let $u = t^2$. Then $du = 2t\,dt \quad \Rightarrow$

$$\int \frac{t}{t^4+2}\,dt = \int \frac{1}{u^2+2}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C \;\; [\text{by Formula 17}] \; = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t^2}{\sqrt{2}}\right) + C$$

6. Let $u = 2x + 1$. Then $du = 2\,dx \quad \Rightarrow$

$$\int_0^1 \frac{x}{(2x+1)^3}\,dx = \int_1^3 \frac{(u-1)/2}{u^3}\left(\frac{1}{2}\,du\right) = \frac{1}{4}\int_1^3 \left(\frac{1}{u^2} - \frac{1}{u^3}\right)du = \frac{1}{4}\left[-\frac{1}{u} + \frac{1}{2u^2}\right]_1^3$$

$$= \tfrac{1}{4}\left[\left(-\tfrac{1}{3} + \tfrac{1}{18}\right) - \left(-1 + \tfrac{1}{2}\right)\right] = \tfrac{1}{4}\left(\tfrac{2}{9}\right) = \tfrac{1}{18}$$

7. Let $u = \arctan y$. Then $du = \dfrac{dy}{1+y^2} \quad \Rightarrow \quad \displaystyle\int_{-1}^1 \frac{e^{\arctan y}}{1+y^2}\,dy = \int_{-\pi/4}^{\pi/4} e^u\,du = \left[e^u\right]_{-\pi/4}^{\pi/4} = e^{\pi/4} - e^{-\pi/4}$.

8. $$\int t\sin t\,\cos t\,dt = \int t\cdot\tfrac{1}{2}(2\sin t\,\cos t)\,dt = \tfrac{1}{2}\int t\sin 2t\,dt$$

$$= \tfrac{1}{2}\left(-\tfrac{1}{2}t\cos 2t - \int -\tfrac{1}{2}\cos 2t\,dt\right) \qquad \begin{bmatrix} u = t, & dv = \sin 2t\,dt \\ du = dt, & v = -\tfrac{1}{2}\cos 2t \end{bmatrix}$$

$$= -\tfrac{1}{4}t\cos 2t + \tfrac{1}{4}\int \cos 2t\,dt = -\tfrac{1}{4}t\cos 2t + \tfrac{1}{8}\sin 2t + C$$

9. $\dfrac{x+2}{x^2+3x-4} = \dfrac{x+2}{(x+4)(x-1)} = \dfrac{A}{x+4} + \dfrac{B}{x-1}$. Multiply by $(x+4)(x-1)$ to get $x + 2 = A(x-1) + B(x+4)$.

Substituting 1 for x gives $3 = 5B \quad \Leftrightarrow \quad B = \frac{3}{5}$. Substituting -4 for x gives $-2 = -5A \quad \Leftrightarrow \quad A = \frac{2}{5}$. Thus,

$$\int_2^4 \frac{x+2}{x^2+3x-4}\,dx = \int_2^4 \left(\frac{2/5}{x+4} + \frac{3/5}{x-1}\right)dx = \left[\frac{2}{5}\ln|x+4| + \frac{3}{5}\ln|x-1|\right]_2^4$$

$$= \left(\tfrac{2}{5}\ln 8 + \tfrac{3}{5}\ln 3\right) - \left(\tfrac{2}{5}\ln 6 + 0\right) = \tfrac{2}{5}(3\ln 2) + \tfrac{3}{5}\ln 3 - \tfrac{2}{5}(\ln 2 + \ln 3)$$

$$= \tfrac{4}{5}\ln 2 + \tfrac{1}{5}\ln 3, \text{ or } \tfrac{1}{5}\ln 48$$

10. Let $u = \dfrac{1}{x}$, $dv = \dfrac{\cos(1/x)}{x^2} \quad \Rightarrow \quad du = -\dfrac{1}{x^2}\,dx$, $v = -\sin\left(\dfrac{1}{x}\right)$. Then

$$\int \frac{\cos(1/x)}{x^3}\,dx = -\frac{1}{x}\sin\left(\frac{1}{x}\right) - \int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx = -\frac{1}{x}\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) + C.$$

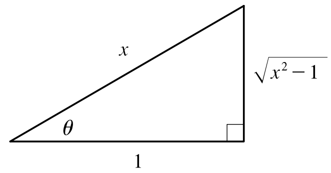
11. Let $x = \sec\theta$, where $0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = \sec\theta\,\tan\theta\,d\theta$ and

$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$ for the relevant values of θ, so

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx = \int \frac{\sec\theta\,\tan\theta}{\sec^3\theta\,\tan\theta}\,d\theta = \int \cos^2\theta\,d\theta = \int \tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + C = \tfrac{1}{2}\theta + \tfrac{1}{2}\sin\theta\,\cos\theta + C$$

$$= \frac{1}{2}\sec^{-1}x + \frac{1}{2}\frac{\sqrt{x^2-1}}{x}\frac{1}{x} + C = \frac{1}{2}\sec^{-1}x + \frac{\sqrt{x^2-1}}{2x^2} + C$$



12. $\dfrac{2x-3}{x^3+3x} = \dfrac{2x-3}{x(x^2+3)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+3}$. Multiply by $x(x^2+3)$ to get $2x-3 = A(x^2+3) + (Bx+C)x \quad\Leftrightarrow$

$2x-3 = (A+B)x^2 + Cx + 3A$. Equating coefficients gives us $C = 2$, $3A = -3 \quad\Leftrightarrow\quad A = -1$, and $A + B = 0$, so

$B = 1$. Thus,

$$\int \frac{2x-3}{x^3+3x}\,dx = \int\left(\frac{-1}{x} + \frac{x+2}{x^2+3}\right)dx = \int\left(-\frac{1}{x} + \frac{x}{x^2+3} + \frac{2}{x^2+3}\right)dx$$

$$= -\ln|x| + \frac{1}{2}\ln(x^2+3) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

13. $\int \sin^5 t\,\cos^4 t\,dt = \int \sin^4 t\,\cos^4 t\,\sin t\,dt = \int (\sin^2 t)^2\cos^4 t\,\sin t\,dt$

$$= \int (1-\cos^2 t)^2\cos^4 t\,\sin t\,dt = \int (1-u^2)^2 u^4\,(-du) \quad [u = \cos t,\ du = -\sin t\,dt]$$

$$= \int (-u^4 + 2u^6 - u^8)\,du = -\tfrac{1}{5}u^5 + \tfrac{2}{7}u^7 - \tfrac{1}{9}u^9 + C = -\tfrac{1}{5}\cos^5 t + \tfrac{2}{7}\cos^7 t - \tfrac{1}{9}\cos^9 t + C$$

14. Let $u = \ln(1+x^2)$, $dv = dx \quad\Rightarrow\quad du = \dfrac{2x}{1+x^2}\,dx$, $v = x$. Then

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx = x\ln(1+x^2) - 2\int \frac{(x^2+1)-1}{1+x^2}\,dx$$

$$= x\ln(1+x^2) - 2\int\left(1 - \frac{1}{1+x^2}\right)dx = x\ln(1+x^2) - 2x + 2\tan^{-1}x + C$$

15. Let $u = x$, $dv = \sec x\,\tan x\,dx \quad\Rightarrow\quad du = dx$, $v = \sec x$. Then

$\int x\sec x\,\tan x\,dx = x\sec x - \int \sec x\,dx = x\sec x - \ln|\sec x + \tan x| + C$.

16. $\displaystyle\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/4} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta \quad \begin{bmatrix} u = \sin\theta, \\ du = \cos\theta\,d\theta \end{bmatrix}$

$$= \int_0^{\pi/4} \tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \tfrac{1}{2}\left[\left(\tfrac{\pi}{4} - \tfrac{1}{2}\right) - (0-0)\right] = \tfrac{\pi}{8} - \tfrac{1}{4}$$

17. $\int_0^\pi t\,\cos^2 t\,dt = \int_0^\pi t\left[\tfrac{1}{2}(1+\cos 2t)\right]dt = \tfrac{1}{2}\int_0^\pi t\,dt + \tfrac{1}{2}\int_0^\pi t\cos 2t\,dt$

$$= \tfrac{1}{2}\left[\tfrac{1}{2}t^2\right]_0^\pi + \tfrac{1}{2}\left[\tfrac{1}{2}t\sin 2t\right]_0^\pi - \tfrac{1}{2}\int_0^\pi \tfrac{1}{2}\sin 2t\,dt \quad \begin{bmatrix} u = t, & dv = \cos 2t\,dt \\ du = dt, & v = \tfrac{1}{2}\sin 2t \end{bmatrix}$$

$$= \tfrac{1}{4}\pi^2 + 0 - \tfrac{1}{4}\left[-\tfrac{1}{2}\cos 2t\right]_0^\pi = \tfrac{1}{4}\pi^2 + \tfrac{1}{8}(1-1) = \tfrac{1}{4}\pi^2$$

18. Let $u = \sqrt{t}$. Then $du = \dfrac{1}{2\sqrt{t}}\,dt \;\Rightarrow\; \displaystyle\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt = \int_1^2 e^u\,(2\,du) = 2\Big[e^u\Big]_1^2 = 2(e^2 - e)$.

19. Let $u = e^x$. Then $\int e^{x+e^x}\,dx = \int e^{e^x} e^x\,dx = \int e^u\,du = e^u + C = e^{e^x} + C$.

20. Since e^2 is a constant, $\int e^2\,dx = e^2 x + C$.

21. Let $t = \sqrt{x}$, so that $t^2 = x$ and $2t\,dt = dx$. Then $\int \arctan\sqrt{x}\,dx = \int \arctan t\,(2t\,dt) = I$. Now use parts with $u = \arctan t$, $dv = 2t\,dt \;\Rightarrow\; du = \dfrac{1}{1+t^2}\,dt$, $v = t^2$. Thus,

$$I = t^2\arctan t - \int \frac{t^2}{1+t^2}\,dt = t^2\arctan t - \int\left(1 - \frac{1}{1+t^2}\right)dt = t^2\arctan t - t + \arctan t + C$$

$$= x\arctan\sqrt{x} - \sqrt{x} + \arctan\sqrt{x} + C \quad \left[\text{or } (x+1)\arctan\sqrt{x} - \sqrt{x} + C\right]$$

22. Let $u = 1 + (\ln x)^2$, so that $du = \dfrac{2\ln x}{x}\,dx$. Then

$$\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\left(2\sqrt{u}\right) + C = \sqrt{1+(\ln x)^2} + C.$$

23. Let $u = 1 + \sqrt{x}$. Then $x = (u-1)^2$, $dx = 2(u-1)\,du \;\Rightarrow$

$$\textstyle\int_0^1 \left(1+\sqrt{x}\right)^8 dx = \int_1^2 u^8\cdot 2(u-1)\,du = 2\int_1^2 (u^9 - u^8)\,du = \left[\frac{1}{5}u^{10} - 2\cdot\frac{1}{9}u^9\right]_1^2 = \frac{1024}{5} - \frac{1024}{9} - \frac{1}{5} + \frac{2}{9} = \frac{4097}{45}.$$

24. $\int (1+\tan x)^2 \sec x\,dx = \int (1 + 2\tan x + \tan^2 x)\sec x\,dx$

$$= \int [\sec x + 2\sec x\,\tan x + (\sec^2 x - 1)\sec x]\,dx = \int (2\sec x\,\tan x + \sec^3 x)\,dx$$

$$= 2\sec x + \tfrac{1}{2}(\sec x\,\tan x + \ln|\sec x + \tan x| + C) \quad \text{[by Example 7.2.8]}$$

25. $\displaystyle\int_0^1 \frac{1+12t}{1+3t}\,dt = \int_0^1 \frac{(12t+4)-3}{3t+1}\,dt = \int_0^1\left(4 - \frac{3}{3t+1}\right)dt = \Big[4t - \ln|3t+1|\Big]_0^1 = (4 - \ln 4) - (0 - 0) = 4 - \ln 4$

26. $\dfrac{3x^2+1}{x^3+x^2+x+1} = \dfrac{3x^2+1}{(x^2+1)(x+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$. Multiply by $(x+1)(x^2+1)$ to get $3x^2 + 1 = A(x^2+1) + (Bx+C)(x+1) \;\Leftrightarrow\; 3x^2+1 = (A+B)x^2 + (B+C)x + (A+C)$. Substituting -1 for x gives $4 = 2A \;\Leftrightarrow\; A = 2$. Equating coefficients of x^2 gives $3 = A + B = 2 + B \;\Leftrightarrow\; B = 1$. Equating coefficients of x gives $0 = B + C = 1 + C \;\Leftrightarrow\; C = -1$. Thus,

$$\int_0^1 \frac{3x^2+1}{x^3+x^2+x+1}\,dx = \int_0^1\left(\frac{2}{x+1} + \frac{x-1}{x^2+1}\right)dx = \int_0^1\left(\frac{2}{x+1} + \frac{x}{x^2+1} - \frac{1}{x^2+1}\right)dx$$

$$= \Big[2\ln|x+1| + \tfrac{1}{2}\ln(x^2+1) - \tan^{-1}x\Big]_0^1 = (2\ln 2 + \tfrac{1}{2}\ln 2 - \tfrac{\pi}{4}) - (0 + 0 - 0)$$

$$= \tfrac{5}{2}\ln 2 - \tfrac{\pi}{4}$$

27. Let $u = 1 + e^x$, so that $du = e^x\,dx = (u-1)\,dx$. Then $\displaystyle\int \frac{1}{1+e^x}\,dx = \int \frac{1}{u}\cdot\frac{du}{u-1} = \int \frac{1}{u(u-1)}\,du = I$. Now $\dfrac{1}{u(u-1)} = \dfrac{A}{u} + \dfrac{B}{u-1} \;\Rightarrow\; 1 = A(u-1) + Bu$. Set $u = 1$ to get $1 = B$. Set $u = 0$ to get $1 = -A$, so $A = -1$.

Thus, $I = \int \left(\frac{-1}{u} + \frac{1}{u-1} \right) du = -\ln|u| + \ln|u-1| + C = -\ln(1+e^x) + \ln e^x + C = x - \ln(1+e^x) + C.$

Another method: Multiply numerator and denominator by e^{-x} and let $u = e^{-x} + 1$. This gives the answer in the form $-\ln(e^{-x}+1) + C$.

28. $\int \sin\sqrt{at}\,dt = \int \sin u \cdot \frac{2}{a} u\,du \quad [u = \sqrt{at},\, u^2 = at,\, 2u\,du = a\,dt] \quad = \frac{2}{a}\int u \sin u\,du$

$= \frac{2}{a}[-u\cos u + \sin u] + C \quad$ [integration by parts] $\quad = -\frac{2}{a}\sqrt{at}\,\cos\sqrt{at} + \frac{2}{a}\,\sin\sqrt{at} + C$

$= -2\sqrt{\frac{t}{a}}\,\cos\sqrt{at} + \frac{2}{a}\,\sin\sqrt{at} + C$

29. Use integration by parts with $u = \ln\left(x + \sqrt{x^2-1}\right)$, $dv = dx \quad \Rightarrow$

$du = \frac{1}{x+\sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right) dx = \frac{1}{x+\sqrt{x^2-1}}\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right) dx = \frac{1}{\sqrt{x^2-1}}\,dx$, $v = x$. Then

$$\int \ln\left(x+\sqrt{x^2-1}\right) dx = x\ln\left(x+\sqrt{x^2-1}\right) - \int \frac{x}{\sqrt{x^2-1}}\,dx = x\ln\left(x+\sqrt{x^2-1}\right) - \sqrt{x^2-1} + C.$$

30. $|e^x - 1| = \begin{cases} e^x - 1 & \text{if } e^x - 1 \ge 0 \\ -(e^x-1) & \text{if } e^x - 1 < 0 \end{cases} = \begin{cases} e^x - 1 & \text{if } x \ge 0 \\ 1 - e^x & \text{if } x < 0 \end{cases}$

Thus, $\int_{-1}^{2} |e^x - 1|\,dx = \int_{-1}^{0}(1-e^x)\,dx + \int_0^2 (e^x - 1)\,dx = \Big[x - e^x\Big]_{-1}^{0} + \Big[e^x - x\Big]_0^2$

$= (0-1) - (-1-e^{-1}) + (e^2 - 2) - (1 - 0) = e^2 + e^{-1} - 3$

31. As in Example 5,

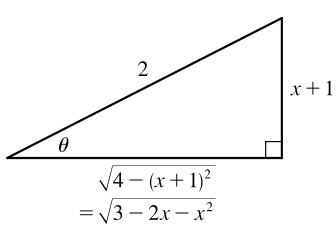
$$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1}x - \sqrt{1-x^2} + C.$$

Another method: Substitute $u = \sqrt{(1+x)/(1-x)}$.

32. $\int_1^3 \frac{e^{3/x}}{x^2}\,dx = \int_3^1 e^u\,\left(-\frac{1}{3}\,du\right) \qquad \begin{bmatrix} u = 3/x, \\ du = -3/x^2\,dx \end{bmatrix}$

$= -\frac{1}{3}\Big[e^u\Big]_3^1 = -\frac{1}{3}(e - e^3) = \frac{1}{3}(e^3 - e)$

33. $3 - 2x - x^2 = -(x^2 + 2x + 1) + 4 = 4 - (x+1)^2$. Let $x + 1 = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\cos\theta\,d\theta$ and



$\int \sqrt{3 - 2x - x^2}\,dx = \int \sqrt{4 - (x+1)^2}\,dx = \int \sqrt{4 - 4\sin^2\theta}\,2\cos\theta\,d\theta$

$= 4\int \cos^2\theta\,d\theta = 2\int(1 + \cos 2\theta)\,d\theta$

$= 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C$

$= 2\sin^{-1}\left(\frac{x+1}{2}\right) + 2 \cdot \frac{x+1}{2} \cdot \frac{\sqrt{3-2x-x^2}}{2} + C$

$= 2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{x+1}{2}\sqrt{3-2x-x^2} + C$

34. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x}\,dx = \int_{\pi/4}^{\pi/2} \left[\frac{(1+4\cos x/\sin x)}{(4-\cos x/\sin x)}\cdot\frac{\sin x}{\sin x}\right]dx = \int_{\pi/4}^{\pi/2} \frac{\sin x+4\cos x}{4\sin x-\cos x}\,dx$

$$= \int_{3/\sqrt{2}}^{4} \frac{1}{u}\,du \quad \left[\begin{array}{l} u = 4\sin x-\cos x, \\ du = (4\cos x+\sin x)\,dx \end{array}\right]$$

$$= \Big[\ln|u|\Big]_{3/\sqrt{2}}^{4} = \ln 4 - \ln\frac{3}{\sqrt{2}} = \ln\frac{4}{3/\sqrt{2}} = \ln\left(\frac{4}{3}\sqrt{2}\right)$$

35. The integrand is an odd function, so $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{x}{1+\cos^2 x}\,dx = 0$ [by 5.5.7(b)].

36. $\displaystyle\int \frac{1+\sin x}{1+\cos x}\,dx = \int \frac{(1+\sin x)(1-\cos x)}{(1+\cos x)(1-\cos x)}\,dx = \int \frac{1-\cos x+\sin x-\sin x\,\cos x}{\sin^2 x}\,dx$

$$= \int\left(\csc^2 x - \frac{\cos x}{\sin^2 x} + \csc x - \frac{\cos x}{\sin x}\right)dx$$

$$\overset{\text{s}}{=} -\cot x + \frac{1}{\sin x} + \ln|\csc x-\cot x| - \ln|\sin x| + C \quad \text{[by Exercise 7.2.39]}$$

The answer can be written as $\dfrac{1-\cos x}{\sin x} - \ln(1+\cos x) + C$.

37. Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta \;\Rightarrow\; \int_0^{\pi/4} \tan^3\theta\,\sec^2\theta\,d\theta = \int_0^1 u^3\,du = \left[\frac{1}{4}u^4\right]_0^1 = \frac{1}{4}$.

38. $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin\theta\,\cot\theta}{\sec\theta}\,d\theta = \int_{\pi/6}^{\pi/3} \cos^2\theta\,d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/3}(1+\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/3}$

$$= \frac{1}{2}\left[\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right) - \left(\frac{\pi}{6}+\frac{\sqrt{3}}{4}\right)\right] = \frac{1}{2}\left(\frac{\pi}{6}\right) = \frac{\pi}{12}$$

39. Let $u = \sec\theta$, so that $du = \sec\theta\,\tan\theta\,d\theta$. Then $\displaystyle\int \frac{\sec\theta\,\tan\theta}{\sec^2\theta-\sec\theta}\,d\theta = \int \frac{1}{u^2-u}\,du = \int \frac{1}{u(u-1)}\,du = I$. Now

$\dfrac{1}{u(u-1)} = \dfrac{A}{u} + \dfrac{B}{u-1} \;\Rightarrow\; 1 = A(u-1)+Bu$. Set $u=1$ to get $1 = B$. Set $u = 0$ to get $1 = -A$, so $A = -1$.

Thus, $I = \displaystyle\int\left(\frac{-1}{u}+\frac{1}{u-1}\right)du = -\ln|u| + \ln|u-1| + C = \ln|\sec\theta-1| - \ln|\sec\theta| + C$ [or $\ln|1-\cos\theta| + C$].

40. Using product formula 2(a) in Section 7.2, $\sin 6x\,\cos 3x = \frac{1}{2}[\sin(6x-3x)+\sin(6x+3x)] = \frac{1}{2}(\sin 3x+\sin 9x)$. Thus,

$$\int_0^\pi \sin 6x\,\cos 3x\,dx = \int_0^\pi \tfrac{1}{2}(\sin 3x+\sin 9x)\,dx = \tfrac{1}{2}\left[-\tfrac{1}{3}\cos 3x - \tfrac{1}{9}\cos 9x\right]_0^\pi$$

$$= \tfrac{1}{2}\left[\left(\tfrac{1}{3}+\tfrac{1}{9}\right) - \left(-\tfrac{1}{3}-\tfrac{1}{9}\right)\right] = \tfrac{1}{2}\left(\tfrac{4}{9}+\tfrac{4}{9}\right) = \tfrac{4}{9}$$

41. Let $u = \theta$, $dv = \tan^2\theta\,d\theta = \left(\sec^2\theta - 1\right)d\theta \;\Rightarrow\; du = d\theta$ and $v = \tan\theta - \theta$. So

$$\int \theta\tan^2\theta\,d\theta = \theta(\tan\theta-\theta) - \int(\tan\theta-\theta)\,d\theta = \theta\tan\theta - \theta^2 - \ln|\sec\theta| + \tfrac{1}{2}\theta^2 + C$$

$$= \theta\tan\theta - \tfrac{1}{2}\theta^2 - \ln|\sec\theta| + C$$

42. Let $u = \tan^{-1} x$, $dv = \dfrac{1}{x^2}\,dx \;\Rightarrow\; du = \dfrac{1}{1+x^2}\,dx$, $v = -\dfrac{1}{x}$. Then

$$I = \int \frac{\tan^{-1} x}{x^2}\,dx = -\frac{1}{x}\tan^{-1} x - \int\left(-\frac{1}{x(1+x^2)}\right)dx = -\frac{1}{x}\tan^{-1} x + \int\left(\frac{A}{x} + \frac{Bx+C}{1+x^2}\right)dx$$

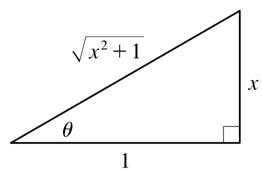
$\dfrac{1}{x(1+x^2)} = \dfrac{A}{x} + \dfrac{Bx+C}{1+x^2} \;\Rightarrow\; 1 = A(1+x^2) + (Bx+C)x \;\Rightarrow\; 1 = (A+B)x^2 + Cx + A$, so $C = 0$, $A = 1$,

and $A + B = 0 \;\Rightarrow\; B = -1$. Thus,

$$\begin{aligned} I &= -\frac{1}{x}\tan^{-1} x + \int\left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx = -\frac{1}{x}\tan^{-1} x + \ln|x| - \frac{1}{2}\ln\left|1+x^2\right| + C \\ &= -\frac{\tan^{-1} x}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C \end{aligned}$$

Or: Let $x = \tan\theta$, so that $dx = \sec^2\theta\,d\theta$. Then $\displaystyle\int \frac{\tan^{-1} x}{x^2}\,dx = \int \frac{\theta}{\tan^2\theta}\sec^2\theta\,d\theta = \int \theta\csc^2\theta\,d\theta = I$. Now use parts

with $u = \theta$, $dv = \csc^2\theta\,d\theta \;\Rightarrow\; du = d\theta$, $v = -\cot\theta$. Thus,



$$\begin{aligned} I &= -\theta\cot\theta - \textstyle\int(-\cot\theta)\,d\theta = -\theta\cot\theta + \ln|\sin\theta| + C \\ &= -\tan^{-1} x \cdot \frac{1}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C = -\frac{\tan^{-1} x}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C \end{aligned}$$

43. Let $u = \sqrt{x}$ so that $du = \dfrac{1}{2\sqrt{x}}\,dx$. Then

$$\begin{aligned} \int \frac{\sqrt{x}}{1+x^3}\,dx &= \int \frac{u}{1+u^6}\,(2u\,du) = 2\int \frac{u^2}{1+(u^3)^2}\,du = 2\int \frac{1}{1+t^2}\left(\frac{1}{3}\,dt\right) \qquad \left[\begin{matrix} t = u^3 \\ dt = 3u^2\,du \end{matrix}\right] \\ &= \tfrac{2}{3}\tan^{-1} t + C = \tfrac{2}{3}\tan^{-1} u^3 + C = \tfrac{2}{3}\tan^{-1}(x^{3/2}) + C \end{aligned}$$

Another method: Let $u = x^{3/2}$ so that $u^2 = x^3$ and $du = \frac{3}{2}x^{1/2}\,dx \;\Rightarrow\; \sqrt{x}\,dx = \frac{2}{3}\,du$. Then

$$\int \frac{\sqrt{x}}{1+x^3}\,dx = \int \frac{\frac{2}{3}}{1+u^2}\,du = \frac{2}{3}\tan^{-1} u + C = \frac{2}{3}\tan^{-1}(x^{3/2}) + C.$$

44. Let $u = \sqrt{1+e^x}$. Then $u^2 = 1 + e^x$, $2u\,du = e^x\,dx = (u^2-1)\,dx$, and $dx = \dfrac{2u}{u^2-1}\,du$, so

$$\begin{aligned} \int \sqrt{1+e^x}\,dx &= \int u \cdot \frac{2u}{u^2-1}\,du = \int \frac{2u^2}{u^2-1}\,du = \int\left(2 + \frac{2}{u^2-1}\right)du = \int\left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right)du \\ &= 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{1+e^x} + \ln\left(\sqrt{1+e^x}-1\right) - \ln\left(\sqrt{1+e^x}+1\right) + C \end{aligned}$$

45. Let $t = x^3$. Then $dt = 3x^2\,dx \;\Rightarrow\; I = \int x^5 e^{-x^3}\,dx = \frac{1}{3}\int te^{-t}\,dt$. Now integrate by parts with $u = t$, $dv = e^{-t}\,dt$:

$I = -\frac{1}{3}te^{-t} + \frac{1}{3}\int e^{-t}\,dt = -\frac{1}{3}te^{-t} - \frac{1}{3}e^{-t} + C = -\frac{1}{3}e^{-x^3}(x^3+1) + C.$

46. Use integration by parts with $u = (x-1)e^x$, $dv = \dfrac{1}{x^2}\,dx \;\Rightarrow\; du = [(x-1)e^x + e^x]\,dx = xe^x\,dx$, $v = -\dfrac{1}{x}$. Then

$$\int \frac{(x-1)e^x}{x^2}\,dx = (x-1)e^x\left(-\frac{1}{x}\right) - \int -e^x\,dx = -e^x + \frac{e^x}{x} + e^x + C = \frac{e^x}{x} + C.$$

47. Let $u = x - 1$, so that $du = dx$. Then

$$\begin{aligned}\textstyle\int x^3(x-1)^{-4}\,dx &= \textstyle\int (u+1)^3 u^{-4}\,du = \int (u^3+3u^2+3u+1)u^{-4}\,du = \int (u^{-1}+3u^{-2}+3u^{-3}+u^{-4})\,du \\ &= \ln|u| - 3u^{-1} - \tfrac{3}{2}u^{-2} - \tfrac{1}{3}u^{-3} + C = \ln|x-1| - 3(x-1)^{-1} - \tfrac{3}{2}(x-1)^{-2} - \tfrac{1}{3}(x-1)^{-3} + C\end{aligned}$$

48. Let $u = \sqrt{1-x^2}$, so $u^2 = 1 - x^2$, and $2u\,du = -2x\,dx$. Then $\int_0^1 x\sqrt{2-\sqrt{1-x^2}}\,dx = \int_1^0 \sqrt{2-u}\,(-u\,du)$.

Now let $v = \sqrt{2-u}$, so $v^2 = 2-u$, and $2v\,dv = -du$. Thus,

$$\begin{aligned}\int_1^0 \sqrt{2-u}\,(-u\,du) &= \int_1^{\sqrt{2}} v(2-v^2)\,(2v\,dv) = \int_1^{\sqrt{2}} (4v^2 - 2v^4)\,dv = \left[\tfrac{4}{3}v^3 - \tfrac{2}{5}v^5\right]_1^{\sqrt{2}} \\ &= \left(\tfrac{8}{3}\sqrt{2} - \tfrac{8}{5}\sqrt{2}\right) - \left(\tfrac{4}{3} - \tfrac{2}{5}\right) = \tfrac{16}{15}\sqrt{2} - \tfrac{14}{15}\end{aligned}$$

49. Let $u = \sqrt{4x+1} \;\Rightarrow\; u^2 = 4x+1 \;\Rightarrow\; 2u\,du = 4\,dx \;\Rightarrow\; dx = \frac{1}{2}u\,du$. So

$$\begin{aligned}\int \frac{1}{x\sqrt{4x+1}}\,dx &= \int \frac{\frac{1}{2}u\,du}{\frac{1}{4}(u^2-1)\,u} = 2\int \frac{du}{u^2-1} = 2\left(\tfrac{1}{2}\right)\ln\left|\frac{u-1}{u+1}\right| + C \qquad \text{[by Formula 19]} \\ &= \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C\end{aligned}$$

50. As in Exercise 49, let $u = \sqrt{4x+1}$. Then $\displaystyle\int \frac{dx}{x^2\sqrt{4x+1}} = \int \frac{\frac{1}{2}u\,du}{\left[\frac{1}{4}(u^2-1)\right]^2 u} = 8\int \frac{du}{(u^2-1)^2}$. Now

$$\frac{1}{(u^2-1)^2} = \frac{1}{(u+1)^2(u-1)^2} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-1} + \frac{D}{(u-1)^2} \quad\Rightarrow$$

$1 = A(u+1)(u-1)^2 + B(u-1)^2 + C(u-1)(u+1)^2 + D(u+1)^2$. $u = 1 \;\Rightarrow\; D = \frac{1}{4}$, $u = -1 \;\Rightarrow\; B = \frac{1}{4}$.

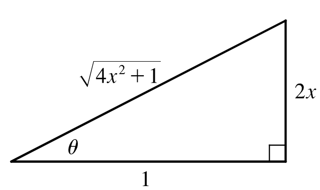
Equating coefficients of u^3 gives $A + C = 0$, and equating coefficients of 1 gives $1 = A + B - C + D \;\Rightarrow$

$1 = A + \frac{1}{4} - C + \frac{1}{4} \;\Rightarrow\; \frac{1}{2} = A - C$. So $A = \frac{1}{4}$ and $C = -\frac{1}{4}$. Therefore,

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{4x+1}} &= 8\int \left[\frac{1/4}{u+1} + \frac{1/4}{(u+1)^2} + \frac{-1/4}{u-1} + \frac{1/4}{(u-1)^2}\right] du \\ &= \int \left[\frac{2}{u+1} + 2(u+1)^{-2} - \frac{2}{u-1} + 2(u-1)^{-2}\right] du \\ &= 2\ln|u+1| - \frac{2}{u+1} - 2\ln|u-1| - \frac{2}{u-1} + C \\ &= 2\ln\left(\sqrt{4x+1}+1\right) - \frac{2}{\sqrt{4x+1}+1} - 2\ln\left|\sqrt{4x+1}-1\right| - \frac{2}{\sqrt{4x+1}-1} + C\end{aligned}$$

51. Let $2x = \tan\theta \;\Rightarrow\; x = \frac{1}{2}\tan\theta$, $dx = \frac{1}{2}\sec^2\theta\,d\theta$, $\sqrt{4x^2+1} = \sec\theta$, so

$$\begin{aligned}\int \frac{dx}{x\sqrt{4x^2+1}} &= \int \frac{\frac{1}{2}\sec^2\theta\,d\theta}{\frac{1}{2}\tan\theta\,\sec\theta} = \int \frac{\sec\theta}{\tan\theta}\,d\theta = \int \csc\theta\,d\theta \\ &= -\ln|\csc\theta + \cot\theta| + C \qquad [\text{or } \ln|\csc\theta - \cot\theta| + C] \\ &= -\ln\left|\frac{\sqrt{4x^2+1}}{2x} + \frac{1}{2x}\right| + C \qquad \left[\text{or } \ln\left|\frac{\sqrt{4x^2+1}}{2x} - \frac{1}{2x}\right| + C\right]\end{aligned}$$



52. Let $u = x^2$. Then $du = 2x\,dx \quad \Rightarrow$

$$\int \frac{dx}{x(x^4+1)} = \int \frac{x\,dx}{x^2(x^4+1)} = \frac{1}{2}\int \frac{du}{u(u^2+1)} = \frac{1}{2}\int \left[\frac{1}{u} - \frac{u}{u^2+1}\right] du = \tfrac{1}{2}\ln|u| - \tfrac{1}{4}\ln(u^2+1) + C$$

$$= \tfrac{1}{2}\ln(x^2) - \tfrac{1}{4}\ln(x^4+1) + C = \tfrac{1}{4}\left[\ln(x^4) - \ln(x^4+1)\right] + C = \frac{1}{4}\ln\left(\frac{x^4}{x^4+1}\right) + C$$

Or: Write $I = \displaystyle\int \frac{x^3\,dx}{x^4(x^4+1)}$ and let $u = x^4$.

53. $\displaystyle\int x^2\sinh(mx)dx = \frac{1}{m}x^2\cosh(mx) - \frac{2}{m}\int x\cosh(mx)\,dx \qquad \left[\begin{array}{ll} u = x^2, & dv = \sinh(mx)\,dx, \\ du = 2x\,dx & v = \frac{1}{m}\cosh(mx) \end{array}\right]$

$$= \frac{1}{m}x^2\cosh(mx) - \frac{2}{m}\left(\tfrac{1}{m}x\sinh(mx) - \frac{1}{m}\textstyle\int \sinh(mx)\,dx\right) \qquad \left[\begin{array}{ll} U = x, & dV = \cosh(mx)\,dx, \\ dU = dx & V = \frac{1}{m}\sinh(mx) \end{array}\right]$$

$$= \frac{1}{m}x^2\cosh(mx) - \frac{2}{m^2}x\sinh(mx) + \frac{2}{m^3}\cosh(mx) + C$$

54. $\int (x+\sin x)^2\ dx = \int \left(x^2 + 2x\sin x + \sin^2 x\right) dx = \tfrac{1}{3}x^3 + 2(\sin x - x\cos x) + \tfrac{1}{2}(x - \sin x\cos x) + C$

$$= \tfrac{1}{3}x^3 + \tfrac{1}{2}x + 2\sin x - \tfrac{1}{2}\sin x\cos x - 2x\cos x + C$$

55. Let $u = \sqrt{x}$, so that $x = u^2$ and $dx = 2u\,du$. Then $\displaystyle\int \frac{dx}{x + x\sqrt{x}} = \int \frac{2u\,du}{u^2 + u^2\cdot u} = \int \frac{2}{u(1+u)}\,du = I.$

Now $\dfrac{2}{u(1+u)} = \dfrac{A}{u} + \dfrac{B}{1+u} \quad \Rightarrow \quad 2 = A(1+u) + Bu$. Set $u = -1$ to get $2 = -B$, so $B = -2$. Set $u = 0$ to get $2 = A$.

Thus, $I = \displaystyle\int \left(\frac{2}{u} - \frac{2}{1+u}\right) du = 2\ln|u| - 2\ln|1+u| + C = 2\ln\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C.$

56. Let $u = \sqrt{x}$, so that $x = u^2$ and $dx = 2u\,du$. Then

$$\int \frac{dx}{\sqrt{x} + x\sqrt{x}} = \int \frac{2u\,du}{u + u^2\cdot u} = \int \frac{2}{1+u^2}\,du = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x} + C.$$

57. Let $u = \sqrt[3]{x+c}$. Then $x = u^3 - c \quad \Rightarrow$

$$\int x\sqrt[3]{x+c}\,dx = \int (u^3 - c)u\cdot 3u^2\,du = 3\int (u^6 - cu^3)\,du = \tfrac{3}{7}u^7 - \tfrac{3}{4}cu^4 + C = \tfrac{3}{7}(x+c)^{7/3} - \tfrac{3}{4}c(x+c)^{4/3} + C$$

58. Let $t = \sqrt{x^2-1}$. Then $dt = \left(x/\sqrt{x^2-1}\right)dx$, $x^2 - 1 = t^2$, $x = \sqrt{t^2+1}$, so

$I = \displaystyle\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx = \int \ln\sqrt{t^2+1}\,dt = \tfrac{1}{2}\int \ln(t^2+1)\,dt$. Now use parts with $u = \ln(t^2+1)$, $dv = dt$:

$$I = \tfrac{1}{2}t\ln\left(t^2+1\right) - \int \frac{t^2}{t^2+1}\,dt = \tfrac{1}{2}t\ln\left(t^2+1\right) - \int \left[1 - \frac{1}{t^2+1}\right] dt$$

$$= \tfrac{1}{2}t\ln\left(t^2+1\right) - t + \tan^{-1}t + C = \sqrt{x^2-1}\,\ln x - \sqrt{x^2-1} + \tan^{-1}\sqrt{x^2-1} + C$$

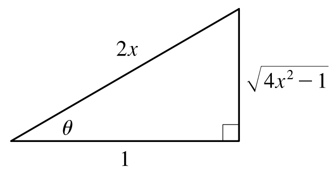
Another method: First integrate by parts with $u = \ln x$, $dv = \left(x/\sqrt{x^2-1}\right)dx$ and then use substitution $\left(x = \sec\theta \text{ or } u = \sqrt{x^2-1}\right)$.

59. $\dfrac{1}{x^4-16}=\dfrac{1}{(x^2-4)(x^2+4)}=\dfrac{1}{(x-2)(x+2)(x^2+4)}=\dfrac{A}{x-2}+\dfrac{B}{x+2}+\dfrac{Cx+D}{x^2+4}$. Multiply by $(x-2)(x+2)(x^2+4)$ to get $1=A(x+2)(x^2+4)+B(x-2)(x^2+4)+(Cx+D)(x-2)(x+2)$. Substituting 2 for x gives $1=32A \quad\Leftrightarrow\quad A=\frac{1}{32}$. Substituting -2 for x gives $1=-32B \quad\Leftrightarrow\quad B=-\frac{1}{32}$. Equating coefficients of x^3 gives $0=A+B+C=\frac{1}{32}-\frac{1}{32}+C$, so $C=0$. Equating constant terms gives $1=8A-8B-4D=\frac{1}{4}+\frac{1}{4}-4D$, so $\frac{1}{2}=-4D \quad\Leftrightarrow\quad D=-\frac{1}{8}$. Thus,

$$\int\frac{dx}{x^4-16}=\int\left(\frac{1/32}{x-2}-\frac{1/32}{x+2}-\frac{1/8}{x^2+4}\right)dx=\frac{1}{32}\ln|x-2|-\frac{1}{32}\ln|x+2|-\frac{1}{8}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$$

$$=\frac{1}{32}\ln\left|\frac{x-2}{x+2}\right|-\frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right)+C$$

60. Let $2x=\sec\theta$, so that $2\,dx=\sec\theta\,\tan\theta\,d\theta$. Then



$$\int\frac{dx}{x^2\sqrt{4x^2-1}}=\int\frac{\frac{1}{2}\sec\theta\,\tan\theta\,d\theta}{\frac{1}{4}\sec^2\theta\sqrt{\sec^2\theta-1}}=\int\frac{2\tan\theta\,d\theta}{\sec\theta\,\tan\theta}$$

$$=2\int\cos\theta\,d\theta=2\sin\theta+C$$

$$=2\cdot\frac{\sqrt{4x^2-1}}{2x}+C=\frac{\sqrt{4x^2-1}}{x}+C$$

61. $$\int\frac{d\theta}{1+\cos\theta}=\int\left(\frac{1}{1+\cos\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta}\right)d\theta=\int\frac{1-\cos\theta}{1-\cos^2\theta}\,d\theta=\int\frac{1-\cos\theta}{\sin^2\theta}\,d\theta=\int\left(\frac{1}{\sin^2\theta}-\frac{\cos\theta}{\sin^2\theta}\right)d\theta$$

$$=\int(\csc^2\theta-\cot\theta\,\csc\theta)\,d\theta=-\cot\theta+\csc\theta+C$$

Another method: Use the substitutions in Exercise 7.4.59.

$$\int\frac{d\theta}{1+\cos\theta}=\int\frac{2/(1+t^2)\,dt}{1+(1-t^2)/(1+t^2)}=\int\frac{2\,dt}{(1+t^2)+(1-t^2)}=\int dt=t+C=\tan\left(\frac{\theta}{2}\right)+C$$

62. $$\int\frac{d\theta}{1+\cos^2\theta}=\int\frac{(1/\cos^2\theta)\,d\theta}{(1+\cos^2\theta)/\cos^2\theta}=\int\frac{\sec^2\theta}{\sec^2\theta+1}\,d\theta=\int\frac{\sec^2\theta}{\tan^2\theta+2}\,d\theta=\int\frac{1}{u^2+2}\,du\quad\begin{bmatrix}u=\tan\theta,\\du=\sec^2\theta\,d\theta\end{bmatrix}$$

$$=\int\frac{1}{u^2+\left(\sqrt{2}\right)^2}\,du=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+C=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan\theta}{\sqrt{2}}\right)+C$$

63. Let $y=\sqrt{x}$ so that $dy=\dfrac{1}{2\sqrt{x}}\,dx \quad\Rightarrow\quad dx=2\sqrt{x}\,dy=2y\,dy$. Then

$$\int\sqrt{x}\,e^{\sqrt{x}}\,dx=\int ye^y(2y\,dy)=\int 2y^2e^y\,dy\quad\begin{bmatrix}u=2y^2, & dv=e^y\,dy,\\du=4y\,dy & v=e^y\end{bmatrix}$$

$$=2y^2e^y-\int 4ye^y\,dy\quad\begin{bmatrix}U=4y, & dV=e^y\,dy,\\dU=4\,dy & V=e^y\end{bmatrix}$$

$$=2y^2e^y-\left(4ye^y-\int 4e^y\,dy\right)=2y^2e^y-4ye^y+4e^y+C$$

$$=2(y^2-2y+2)e^y+C=2\left(x-2\sqrt{x}+2\right)e^{\sqrt{x}}+C$$

64. Let $u = \sqrt{x} + 1$, so that $x = (u-1)^2$ and $dx = 2(u-1)\,du$. Then

$$\int \frac{1}{\sqrt{\sqrt{x}+1}}\,dx = \int \frac{2(u-1)\,du}{\sqrt{u}} = \int (2u^{1/2} - 2u^{-1/2})\,du = \frac{4}{3}u^{3/2} - 4u^{1/2} + C = \frac{4}{3}(\sqrt{x}+1)^{3/2} - 4\sqrt{\sqrt{x}+1} + C.$$

65. Let $u = \cos^2 x$, so that $du = 2\cos x\,(-\sin x)\,dx$. Then

$$\int \frac{\sin 2x}{1+\cos^4 x}\,dx = \int \frac{2\sin x\,\cos x}{1+(\cos^2 x)^2}\,dx = \int \frac{1}{1+u^2}\,(-du) = -\tan^{-1} u + C = -\tan^{-1}(\cos^2 x) + C.$$

66. Let $u = \tan x$. Then

$$\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)\,dx}{\sin x\,\cos x} = \int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\tan x}\,\sec^2 x\,dx = \int_1^{\sqrt{3}} \frac{\ln u}{u}\,du = \left[\tfrac{1}{2}(\ln u)^2\right]_1^{\sqrt{3}} = \tfrac{1}{2}\left(\ln\sqrt{3}\right)^2 = \tfrac{1}{8}(\ln 3)^2.$$

67. $$\int \frac{dx}{\sqrt{x+1}+\sqrt{x}} = \int \left(\frac{1}{\sqrt{x+1}+\sqrt{x}} \cdot \frac{\sqrt{x+1}-\sqrt{x\sqrt{x}}}{\sqrt{x+1}-\sqrt{x}}\right) dx = \int \left(\sqrt{x+1}-\sqrt{x}\right) dx$$
$$= \tfrac{2}{3}\left[(x+1)^{3/2} - x^{3/2}\right] + C$$

68. $$\int \frac{x^2}{x^6+3x^3+2}\,dx = \int \frac{x^2\,dx}{(x^3+1)(x^3+2)} = \int \frac{\frac{1}{3}\,du}{(u+1)(u+2)} \quad \left[\begin{matrix} u = x^3, \\ du = 3x^2\,dx \end{matrix}\right].$$

Now $\dfrac{1}{(u+1)(u+2)} = \dfrac{A}{u+1} + \dfrac{B}{u+2} \quad\Rightarrow\quad 1 = A(u+2) + B(u+1)$. Setting $u = -2$ gives $B = -1$. Setting $u = -1$ gives $A = 1$. Thus,

$$\frac{1}{3}\int \frac{du}{(u+1)(u+2)} = \frac{1}{3}\int \left(\frac{1}{u+1} - \frac{1}{u+2}\right) du = \frac{1}{3}\ln|u+1| - \frac{1}{3}\ln|u+2| + C$$
$$= \tfrac{1}{3}\ln\left|x^3+1\right| - \tfrac{1}{3}\ln\left|x^3+2\right| + C$$

69. Let $x = \tan\theta$, so that $dx = \sec^2\theta\,d\theta$, $x = \sqrt{3} \;\Rightarrow\; \theta = \frac{\pi}{3}$, and $x = 1 \;\Rightarrow\; \theta = \frac{\pi}{4}$. Then

$$\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2}\,dx = \int_{\pi/4}^{\pi/3} \frac{\sec\theta}{\tan^2\theta}\,\sec^2\theta\,d\theta = \int_{\pi/4}^{\pi/3} \frac{\sec\theta\,(\tan^2\theta+1)}{\tan^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \left(\frac{\sec\theta\,\tan^2\theta}{\tan^2\theta} + \frac{\sec\theta}{\tan^2\theta}\right) d\theta$$
$$= \int_{\pi/4}^{\pi/3} (\sec\theta + \csc\theta\,\cot\theta)\,d\theta = \Big[\ln|\sec\theta + \tan\theta| - \csc\theta\Big]_{\pi/4}^{\pi/3}$$
$$= \left(\ln\left|2+\sqrt{3}\right| - \tfrac{2}{\sqrt{3}}\right) - \left(\ln\left|\sqrt{2}+1\right| - \sqrt{2}\right) = \sqrt{2} - \tfrac{2}{\sqrt{3}} + \ln\left(2+\sqrt{3}\right) - \ln\left(1+\sqrt{2}\right)$$

70. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \;\Rightarrow$

$$\int \frac{dx}{1+2e^x-e^{-x}} = \int \frac{du/u}{1+2u-1/u} = \int \frac{du}{2u^2+u-1} = \int \left[\frac{2/3}{2u-1} - \frac{1/3}{u+1}\right] du$$
$$= \tfrac{1}{3}\ln|2u-1| - \tfrac{1}{3}\ln|u+1| + C = \tfrac{1}{3}\ln|(2e^x-1)/(e^x+1)| + C$$

71. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \quad \Rightarrow$

$$\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{u^2}{1+u}\,\frac{du}{u} = \int \frac{u}{1+u}\,du = \int \left(1 - \frac{1}{1+u}\right) du = u - \ln|1+u| + C = e^x - \ln(1+e^x) + C.$$

72. Use parts with $u = \ln(x+1)$, $dv = dx/x^2$:

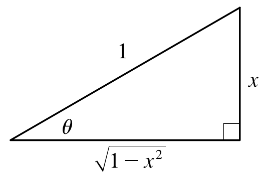
$$\int \frac{\ln(x+1)}{x^2}\,dx = -\frac{1}{x}\ln(x+1) + \int \frac{dx}{x(x+1)} = -\frac{1}{x}\ln(x+1) + \int \left[\frac{1}{x} - \frac{1}{x+1}\right] dx$$

$$= -\frac{1}{x}\ln(x+1) + \ln|x| - \ln(x+1) + C = -\left(1 + \frac{1}{x}\right)\ln(x+1) + \ln|x| + C$$

73. Let $\theta = \arcsin x$, so that $d\theta = \dfrac{1}{\sqrt{1-x^2}}\,dx$ and $x = \sin\theta$. Then

$$\int \frac{x + \arcsin x}{\sqrt{1-x^2}}\,dx = \int (\sin\theta + \theta)\,d\theta = -\cos\theta + \tfrac{1}{2}\theta^2 + C$$

$$= -\sqrt{1-x^2} + \tfrac{1}{2}(\arcsin x)^2 + C$$



74. $$\int \frac{4^x + 10^x}{2^x}\,dx = \int \left(\frac{4^x}{2^x} + \frac{10^x}{2^x}\right) dx = \int (2^x + 5^x)\,dx = \frac{2^x}{\ln 2} + \frac{5^x}{\ln 5} + C$$

75. $$\int \frac{dx}{x\ln x - x} = \int \frac{dx}{x(\ln x - 1)} = \int \frac{du}{u} \quad \left[\begin{matrix} u = \ln x - 1, \\ du = (1/x)\,dx \end{matrix}\right]$$

$$= \ln|u| + C = \ln|\ln x - 1| + C$$

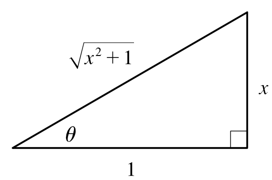
76. $$\int \frac{x^2}{\sqrt{x^2+1}}\,dx = \int \frac{\tan^2\theta}{\sec\theta}\sec^2\theta\,d\theta \quad \left[\begin{matrix} x = \tan\theta, \\ dx = \sec^2\theta\,d\theta \end{matrix}\right]$$

$$= \textstyle\int \tan^2\theta\,\sec\theta\,d\theta = \int (\sec^2\theta - 1)\sec\theta\,d\theta$$

$$= \textstyle\int (\sec^3\theta - \sec\theta)\,d\theta$$

$$= \tfrac{1}{2}(\sec\theta\,\tan\theta + \ln|\sec\theta + \tan\theta|) - \ln|\sec\theta + \tan\theta| + C \quad \text{[by (1) and Example 7.2.8]}$$

$$= \tfrac{1}{2}(\sec\theta\,\tan\theta - \ln|\sec\theta + \tan\theta|) + C = \tfrac{1}{2}\left[x\sqrt{x^2+1} - \ln(\sqrt{x^2+1} + x)\right] + C$$



77. Let $y = \sqrt{1+e^x}$, so that $y^2 = 1 + e^x$, $2y\,dy = e^x\,dx$, $e^x = y^2 - 1$, and $x = \ln(y^2-1)$. Then

$$\int \frac{xe^x}{\sqrt{1+e^x}}\,dx = \int \frac{\ln(y^2-1)}{y}(2y\,dy) = 2\int [\ln(y+1) + \ln(y-1)]\,dy$$

$$= 2[(y+1)\ln(y+1) - (y+1) + (y-1)\ln(y-1) - (y-1)] + C \quad \text{[by Example 7.1.2]}$$

$$= 2[y\ln(y+1) + \ln(y+1) - y - 1 + y\ln(y-1) - \ln(y-1) - y + 1] + C$$

$$= 2[y(\ln(y+1) + \ln(y-1)) + \ln(y+1) - \ln(y-1) - 2y] + C$$

$$= 2\left[y\ln(y^2-1) + \ln\frac{y+1}{y-1} - 2y\right] + C = 2\left[\sqrt{1+e^x}\,\ln(e^x) + \ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} - 2\sqrt{1+e^x}\right] + C$$

$$= 2x\sqrt{1+e^x} + 2\ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} - 4\sqrt{1+e^x} + C = 2(x-2)\sqrt{1+e^x} + 2\ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} + C$$

78. $\dfrac{1+\sin x}{1-\sin x}=\dfrac{1+\sin x}{1-\sin x}\cdot\dfrac{1+\sin x}{1+\sin x}=\dfrac{1+2\sin x+\sin^2 x}{1-\sin^2 x}=\dfrac{1+2\sin x+\sin^2 x}{\cos^2 x}=\dfrac{1}{\cos^2 x}+\dfrac{2\sin x}{\cos^2 x}+\dfrac{\sin^2 x}{\cos^2 x}$

$=\sec^2 x+2\sec x\,\tan x+\tan^2 x=\sec^2 x+2\sec x\,\tan x+\sec^2 x-1=2\sec^2 x+2\sec x\,\tan x-1$

Thus, $\displaystyle\int\frac{1+\sin x}{1-\sin x}\,dx=\int(2\sec^2 x+2\sec x\,\tan x-1)\,dx=2\tan x+2\sec x-x+C$

79. Let $u=x$, $dv=\sin^2 x\cos x\,dx \quad\Rightarrow\quad du=dx$, $v=\frac{1}{3}\sin^3 x$. Then

$$\begin{aligned}\int x\sin^2 x\,\cos x\,dx&=\tfrac{1}{3}x\sin^3 x-\int\tfrac{1}{3}\sin^3 x\,dx=\tfrac{1}{3}x\sin^3 x-\tfrac{1}{3}\int(1-\cos^2 x)\sin x\,dx\\&=\frac{1}{3}x\sin^3 x+\frac{1}{3}\int(1-y^2)\,dy\qquad\begin{bmatrix}u=\cos x,\\du=-\sin x\,dx\end{bmatrix}\\&=\tfrac{1}{3}x\sin^3 x+\tfrac{1}{3}y-\tfrac{1}{9}y^3+C=\tfrac{1}{3}x\sin^3 x+\tfrac{1}{3}\cos x-\tfrac{1}{9}\cos^3 x+C\end{aligned}$$

80. $$\begin{aligned}\int\frac{\sec x\,\cos 2x}{\sin x+\sec x}\,dx&=\int\frac{\sec x\,\cos 2x}{\sin x+\sec x}\cdot\frac{2\cos x}{2\cos x}\,dx=\int\frac{2\cos 2x}{2\sin x\cos x+2}\,dx\\&=\int\frac{2\cos 2x}{\sin 2x+2}\,dx=\int\frac{1}{u}\,du\qquad\begin{bmatrix}u=\sin 2x+2,\\du=2\cos 2x\,dx\end{bmatrix}\\&=\ln|u|+C=\ln|\sin 2x+2|+C=\ln(\sin 2x+2)+C\end{aligned}$$

81. $$\begin{aligned}\int\sqrt{1-\sin x}\,dx&=\int\sqrt{\frac{1-\sin x}{1}\cdot\frac{1+\sin x}{1+\sin x}}\,dx=\int\sqrt{\frac{1-\sin^2 x}{1+\sin x}}\,dx\\&=\int\sqrt{\frac{\cos^2 x}{1+\sin x}}\,dx=\int\frac{\cos x\,dx}{\sqrt{1+\sin x}}\qquad[\text{assume }\cos x>0]\\&=\int\frac{du}{\sqrt{u}}\qquad\begin{bmatrix}u=1+\sin x,\\du=\cos x\,dx\end{bmatrix}\\&=2\sqrt{u}+C=2\sqrt{1+\sin x}+C\end{aligned}$$

Another method: Let $u=\sin x$ so that $du=\cos x\,dx=\sqrt{1-\sin^2 x}\,dx=\sqrt{1-u^2}\,dx$. Then

$$\int\sqrt{1-\sin x}\,dx=\int\sqrt{1-u}\left(\frac{du}{\sqrt{1-u^2}}\right)=\int\frac{1}{\sqrt{1+u}}\,du=2\sqrt{1+u}+C=2\sqrt{1+\sin x}+C.$$

82. $$\begin{aligned}\int\frac{\sin x\,\cos x}{\sin^4 x+\cos^4 x}\,dx&=\int\frac{\sin x\,\cos x}{(\sin^2 x)^2+(\cos^2 x)^2}\,dx=\int\frac{\sin x\,\cos x}{(\sin^2 x)^2+(1-\sin^2 x)^2}\,dx\\&=\int\frac{1}{u^2+(1-u)^2}\left(\frac{1}{2}\,du\right)\qquad\begin{bmatrix}u=\sin^2 x,\\du=2\sin x\,\cos x\,dx\end{bmatrix}\\&=\int\frac{1}{4u^2-4u+2}\,du=\int\frac{1}{(4u^2-4u+1)+1}\,du\\&=\int\frac{1}{(2u-1)^2+1}\,du=\frac{1}{2}\int\frac{1}{y^2+1}\,dy\qquad\begin{bmatrix}y=2u-1,\\dy=2\,du\end{bmatrix}\\&=\tfrac{1}{2}\tan^{-1}y+C=\tfrac{1}{2}\tan^{-1}(2u-1)+C=\tfrac{1}{2}\tan^{-1}(2\sin^2 x-1)+C\end{aligned}$$

Another solution:
$$\int \frac{\sin x\,\cos x}{\sin^4 x+\cos^4 x}\,dx = \int \frac{(\sin x\,\cos x)/\cos^4 x}{(\sin^4 x+\cos^4 x)/\cos^4 x}\,dx = \int \frac{\tan x\,\sec^2 x}{\tan^4 x+1}\,dx$$
$$= \int \frac{1}{u^2+1}\left(\frac{1}{2}\,du\right) \quad \left[\begin{array}{l} u=\tan^2 x, \\ du = 2\tan x\,\sec^2 x\,dx \end{array}\right]$$
$$= \tfrac{1}{2}\tan^{-1} u + C = \tfrac{1}{2}\tan^{-1}(\tan^2 x) + C$$

83. The function $y = 2xe^{x^2}$ *does* have an elementary antiderivative, so we'll use this fact to help evaluate the integral.

$$\int (2x^2+1)e^{x^2}\,dx = \int 2x^2 e^{x^2}\,dx + \int e^{x^2}\,dx = \int x\left(2xe^{x^2}\right)dx + \int e^{x^2}\,dx$$
$$= xe^{x^2} - \int e^{x^2}\,dx + \int e^{x^2}\,dx \quad \left[\begin{array}{ll} u=x, & dv = 2xe^{x^2}\,dx, \\ du = dx & v = e^{x^2} \end{array}\right] \quad = xe^{x^2} + C$$

84. (a) $$\int_1^2 \frac{e^x}{x}\,dx = \int_0^{\ln 2} \frac{e^{e^t}}{e^t}\,e^t\,dt \quad \left[\begin{array}{l} x=e^t, \\ dx = e^t\,dt \end{array}\right] \quad = \int_0^{\ln 2} e^{e^t}\,dt = F(\ln 2)$$

(b) $$\int_2^3 \frac{1}{\ln x}\,dx = \int_{\ln 2}^{\ln 3} \frac{1}{u}\,(e^u\,du) \quad \left[\begin{array}{l} u=\ln x, \\ du = \frac{1}{x}\,dx \end{array}\right] \quad = \int_{\ln\ln 2}^{\ln\ln 3} \frac{e^{e^v}}{e^v}\,e^v\,dv \quad \left[\begin{array}{l} u=e^v, \\ du = e^v\,dv \end{array}\right]$$
$$= \int_{\ln\ln 2}^{0} e^{e^v}\,dv + \int_0^{\ln\ln 3} e^{e^v}\,dv \quad [\text{note that } \ln\ln 2 < 0]$$
$$= \int_0^{\ln\ln 3} e^{e^v}\,dv - \int_0^{\ln\ln 2} e^{e^v}\,dv = F(\ln\ln 3) - F(\ln\ln 2)$$

Another method: Substitute $x = e^{e^t}$ in the original integral.

7.6 Integration Using Tables and Computer Algebra Systems

Keep in mind that there are several ways to approach many of these exercises, and different methods can lead to different forms of the answer.

1. $$\int_0^{\pi/2} \cos 5x\,\cos 2x\,dx \overset{80}{=} \left[\frac{\sin(5-2)x}{2(5-2)} + \frac{\sin(5+2)x}{2(5+2)}\right]_0^{\pi/2} \quad \left[\begin{array}{l} a=5, \\ b=2 \end{array}\right]$$
$$= \left[\frac{\sin 3x}{6} + \frac{\sin 7x}{14}\right]_0^{\pi/2} = \left(-\frac{1}{6} - \frac{1}{14}\right) - 0 = \frac{-7-3}{42} = -\frac{5}{21}$$

2. $$\int_0^1 \sqrt{x-x^2}\,dx = \int_0^1 \sqrt{2\left(\tfrac{1}{2}\right)x - x^2}\,dx \overset{113}{=} \left[\frac{x-\frac{1}{2}}{2}\sqrt{2\left(\tfrac{1}{2}\right)x - x^2} + \frac{\left(\frac{1}{2}\right)^2}{2}\cos^{-1}\left(\frac{\frac{1}{2}-x}{\frac{1}{2}}\right)\right]_0^1$$
$$= \left[\frac{2x-1}{4}\sqrt{x-x^2} + \frac{1}{8}\cos^{-1}(1-2x)\right]_0^1 = \left(0 + \frac{1}{8}\cdot\pi\right) - \left(0 + \frac{1}{8}\cdot 0\right) = \frac{1}{8}\pi$$

3. $$\int_1^2 \sqrt{4x^2-3}\,dx = \tfrac{1}{2}\int_2^4 \sqrt{u^2 - \left(\sqrt{3}\right)^2}\,du \quad [u = 2x,\ du = 2\,dx]$$
$$\overset{39}{=} \frac{1}{2}\left[\frac{u}{2}\sqrt{u^2-\left(\sqrt{3}\right)^2} - \frac{\left(\sqrt{3}\right)^2}{2}\ln\left|u + \sqrt{u^2-\left(\sqrt{3}\right)^2}\right|\right]_2^4$$
$$= \tfrac{1}{2}\left[2\sqrt{13} - \tfrac{3}{2}\ln\left(4+\sqrt{13}\right)\right] - \tfrac{1}{2}\left(1 - \tfrac{3}{2}\ln 3\right) = \sqrt{13} - \tfrac{3}{4}\ln\left(4+\sqrt{13}\right) - \tfrac{1}{2} + \tfrac{3}{4}\ln 3$$

4. $\displaystyle\int_0^1 \tan^3\left(\frac{\pi}{6}x\right)dx = \frac{6}{\pi}\int_0^{\pi/6}\tan^3 u\,du \qquad [u = (\pi/6)x,\ du = (\pi/6)\,dx]$

$$\overset{69}{=} \frac{6}{\pi}\left[\frac{1}{2}\tan^2 u + \ln|\cos u|\right]_0^{\pi/6} = \frac{6}{\pi}\left[\left(\frac{1}{2}\left(\frac{1}{\sqrt{3}}\right)^2 + \ln\frac{\sqrt{3}}{2}\right) - (0 + \ln 1)\right] = \frac{1}{\pi} + \frac{6}{\pi}\ln\frac{\sqrt{3}}{2}$$

5. $\int_0^{\pi/8} \arctan 2x\,dx = \frac{1}{2}\int_0^{\pi/4}\arctan u\,du \qquad [u = 2x,\ du = 2\,dx]$

$$\overset{89}{=} \frac{1}{2}\left[u\arctan u - \frac{1}{2}\ln(1+u^2)\right]_0^{\pi/4} = \frac{1}{2}\left\{\left[\frac{\pi}{4}\arctan\frac{\pi}{4} - \frac{1}{2}\ln\left(1 + \frac{\pi^2}{16}\right)\right] - 0\right\}$$

$$= \frac{\pi}{8}\arctan\frac{\pi}{4} - \frac{1}{4}\ln\left(1 + \frac{\pi^2}{16}\right)$$

6. $\displaystyle\int_0^2 x^2\sqrt{4 - x^2}\,dx \overset{31}{=} \left[\frac{x}{8}(2x^2 - 4)\sqrt{4 - x^2} + \frac{16}{8}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 = \left(0 + 2\cdot\frac{\pi}{2}\right) - 0 = \pi$

7. $\displaystyle\int \frac{\cos x}{\sin^2 x - 9}\,dx = \int\frac{1}{u^2 - 9}\,du \quad \begin{bmatrix} u = \sin x, \\ du = \cos x\,dx\end{bmatrix} \quad \overset{20}{=} \frac{1}{2(3)}\ln\left|\frac{u-3}{u+3}\right| + C = \frac{1}{6}\ln\left|\frac{\sin x - 3}{\sin x + 3}\right| + C$

8. $\displaystyle\int \frac{e^x}{4 - e^{2x}}\,dx = \int\frac{1}{4 - u^2}\,du \quad \begin{bmatrix} u = e^x, \\ du = e^x\,dx\end{bmatrix} \quad \overset{19}{=} \frac{1}{2(2)}\ln\left|\frac{u+2}{u-2}\right| + C = \frac{1}{4}\ln\left|\frac{e^x + 2}{e^x - 2}\right| + C$

9. $\displaystyle\int \frac{\sqrt{9x^2 + 4}}{x^2}\,dx = \int \frac{\sqrt{u^2 + 4}}{u^2/9}\left(\frac{1}{3}du\right) \qquad \begin{bmatrix} u = 3x, \\ du = 3\,dx\end{bmatrix}$

$$= 3\int\frac{\sqrt{4 + u^2}}{u^2}\,du \overset{24}{=} 3\left[-\frac{\sqrt{4 + u^2}}{u} + \ln(u + \sqrt{4 + u^2})\right] + C$$

$$= -\frac{3\sqrt{4 + 9x^2}}{3x} + 3\ln(3x + \sqrt{4 + 9x^2}) + C = -\frac{\sqrt{9x^2 + 4}}{x} + 3\ln(3x + \sqrt{9x^2 + 4}) + C$$

10. Let $u = \sqrt{2}\,y$ and $a = \sqrt{3}$. Then $du = \sqrt{2}\,dy$ and

$$\int \frac{\sqrt{2y^2 - 3}}{y^2}\,dy = \int\frac{\sqrt{u^2 - a^2}}{\frac{1}{2}u^2}\,\frac{du}{\sqrt{2}} = \sqrt{2}\int\frac{\sqrt{u^2 - a^2}}{u^2}\,du \overset{42}{=} \sqrt{2}\left(-\frac{\sqrt{u^2 - a^2}}{u} + \ln\left|u + \sqrt{u^2 - a^2}\right|\right) + C$$

$$= \sqrt{2}\left(-\frac{\sqrt{2y^2 - 3}}{\sqrt{2}\,y} + \ln\left|\sqrt{2}\,y + \sqrt{2y^2 - 3}\right|\right) + C$$

$$= -\frac{\sqrt{2y^2 - 3}}{y} + \sqrt{2}\ln\left|\sqrt{2}\,y + \sqrt{2y^2 - 3}\right| + C$$

11. $\int_0^\pi \cos^6\theta\,d\theta \overset{74}{=} \left[\frac{1}{6}\cos^5\theta\,\sin\theta\right]_0^\pi + \frac{5}{6}\int_0^\pi \cos^4\theta\,d\theta \overset{74}{=} 0 + \frac{5}{6}\left\{\left[\frac{1}{4}\cos^3\theta\,\sin\theta\right]_0^\pi + \frac{3}{4}\int_0^\pi\cos^2\theta\,d\theta\right\}$

$$\overset{64}{=} \tfrac{5}{6}\left\{0 + \tfrac{3}{4}\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^\pi\right\} = \tfrac{5}{6}\cdot\tfrac{3}{4}\cdot\tfrac{\pi}{2} = \tfrac{5\pi}{16}$$

12. $\int x\sqrt{2 + x^4}\,dx = \int\sqrt{2 + u^2}\,(\frac{1}{2}\,du) \qquad \begin{bmatrix} u = x^2, \\ du = 2x\,dx\end{bmatrix}$

$$\overset{21}{=} \frac{1}{2}\left[\frac{u}{2}\sqrt{2 + u^2} + \frac{2}{2}\ln\left(u + \sqrt{2 + u^2}\right)\right] + C = \frac{x^2}{4}\sqrt{2 + x^4} + \frac{1}{2}\ln(x^2 + \sqrt{2 + x^4}) + C$$

13. $\displaystyle\int \frac{\arctan\sqrt{x}}{\sqrt{x}}\,dx = \int \arctan u\,(2\,du) \qquad \left[\begin{array}{l} u=\sqrt{x}, \\ du = 1/(2\sqrt{x})\,dx \end{array}\right]$

$\overset{89}{=} 2\left[u\arctan u - \frac{1}{2}\ln(1+u^2)\right] + C = 2\sqrt{x}\arctan\sqrt{x} - \ln(1+x) + C$

14. $\int_0^\pi x^3 \sin x\,dx \overset{84}{=} \Big[-x^3\cos x\Big]_0^\pi + 3\int_0^\pi x^2\cos x\,dx \overset{85}{=} -\pi^3(-1) + 3\Big\{\Big[x^2\sin x\Big]_0^\pi - 2\int_0^\pi x\sin x\,dx\Big\}$

$= \pi^3 - 6\int_0^\pi x\sin x\,dx \overset{84}{=} \pi^3 - 6\Big\{\Big[-x\cos x\Big]_0^\pi + \int_0^\pi \cos x\,dx\Big\}$

$= \pi^3 - 6[\pi] - 6\Big[\sin x\Big]_0^\pi = \pi^3 - 6\pi$

15. $\displaystyle\int \frac{\coth(1/y)}{y^2}\,dy = \int \coth u\,(-du) \qquad \left[\begin{array}{l} u = 1/y, \\ du = -1/y^2\,dy \end{array}\right]$

$\overset{106}{=} -\ln|\sinh u| + C = -\ln|\sinh(1/y)| + C$

16. $\displaystyle\int \frac{e^{3t}}{\sqrt{e^{2t}-1}}\,dt = \int \frac{e^{2t}}{\sqrt{e^{2t}-1}}\,(e^t\,dt) = \int \frac{u^2}{\sqrt{u^2-1}}\,du \qquad \left[\begin{array}{l} u = e^t, \\ du = e^t\,dt \end{array}\right]$

$\displaystyle\overset{44}{=} \frac{u}{2}\sqrt{u^2-1} + \tfrac{1}{2}\ln\left|u+\sqrt{u^2-1}\right| + C = \tfrac{1}{2}e^t\sqrt{e^{2t}-1} + \tfrac{1}{2}\ln\left(e^t + \sqrt{e^{2t}-1}\right) + C$

17. Let $z = 6 + 4y - 4y^2 = 6 - (4y^2 - 4y + 1) + 1 = 7 - (2y-1)^2$, $u = 2y - 1$, and $a = \sqrt{7}$.

Then $z = a^2 - u^2$, $du = 2\,dy$, and

$\int y\sqrt{6+4y-4y^2}\,dy = \int y\sqrt{z}\,dy = \int \frac{1}{2}(u+1)\sqrt{a^2-u^2}\,\frac{1}{2}\,du = \frac{1}{4}\int u\sqrt{a^2-u^2}\,du + \frac{1}{4}\int\sqrt{a^2-u^2}\,du$

$= \frac{1}{4}\int\sqrt{a^2-u^2}\,du - \frac{1}{8}\int(-2u)\sqrt{a^2-u^2}\,du$

$\displaystyle\overset{30}{=} \frac{u}{8}\sqrt{a^2-u^2} + \frac{a^2}{8}\sin^{-1}\left(\frac{u}{a}\right) - \frac{1}{8}\int\sqrt{w}\,dw \qquad \left[\begin{array}{l} w = a^2-u^2, \\ dw = -2u\,du \end{array}\right]$

$\displaystyle= \frac{2y-1}{8}\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}} - \frac{1}{8}\cdot\frac{2}{3}w^{3/2} + C$

$\displaystyle= \frac{2y-1}{8}\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}} - \frac{1}{12}(6+4y-4y^2)^{3/2} + C$

This can be rewritten as

$\displaystyle\sqrt{6+4y-4y^2}\left[\frac{1}{8}(2y-1) - \frac{1}{12}(6+4y-4y^2)\right] + \frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}} + C$

$\displaystyle= \left(\frac{1}{3}y^2 - \frac{1}{12}y - \frac{5}{8}\right)\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\left(\frac{2y-1}{\sqrt{7}}\right) + C$

$\displaystyle= \frac{1}{24}(8y^2 - 2y - 15)\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\left(\frac{2y-1}{\sqrt{7}}\right) + C$

18. $\displaystyle\int \frac{dx}{2x^3 - 3x^2} = \int \frac{dx}{x^2(-3+2x)} \overset{50}{=} -\frac{1}{-3x} + \frac{2}{(-3)^2}\ln\left|\frac{-3+2x}{x}\right| + C = \frac{1}{3x} + \tfrac{2}{9}\ln\left|\frac{2x-3}{x}\right| + C$

19. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$\displaystyle\int \sin^2 x\,\cos x\,\ln(\sin x)\,dx = \int u^2\ln u\,du \overset{101}{=} \frac{u^{2+1}}{(2+1)^2}\left[(2+1)\ln u - 1\right] + C = \tfrac{1}{9}u^3(3\ln u - 1) + C$

$= \frac{1}{9}\sin^3 x\,[3\ln(\sin x) - 1] + C$

20. Let $u = \sin\theta$, so that $du = \cos\theta\,d\theta$. Then

$$\int \frac{\sin 2\theta}{\sqrt{5-\sin\theta}}\,d\theta = \int \frac{2\sin\theta\,\cos\theta}{\sqrt{5-\sin\theta}}d\theta = 2\int \frac{u}{\sqrt{5-u}}\,du \overset{55}{=} 2\cdot\frac{2}{3(-1)^2}\left[-1u-2(5)\right]\sqrt{5-u}+C$$
$$= \tfrac{4}{3}(-u-10)\sqrt{5-u}+C = -\tfrac{4}{3}(\sin\theta+10)\sqrt{5-\sin\theta}+C$$

21. Let $u = e^x$ and $a = \sqrt{3}$. Then $du = e^x\,dx$ and

$$\int \frac{e^x}{3-e^{2x}}\,dx = \int \frac{du}{a^2-u^2} \overset{19}{=} \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C = \frac{1}{2\sqrt{3}}\ln\left|\frac{e^x+\sqrt{3}}{e^x-\sqrt{3}}\right| + C.$$

22. Let $u = x^2$ and $a = 2$. Then $du = 2x\,dx$ and

$$\int_0^2 x^3\sqrt{4x^2-x^4}\,dx = \tfrac{1}{2}\int_0^2 x^2\sqrt{2\cdot 2\cdot x^2-(x^2)^2}\cdot 2x\,dx = \tfrac{1}{2}\int_0^4 u\sqrt{2au-u^2}\,du$$
$$\overset{114}{=} \left[\frac{2u^2-au-3a^2}{12}\sqrt{2au-u^2}+\frac{a^3}{4}\cos^{-1}\left(\frac{a-u}{a}\right)\right]_0^4$$
$$= \left[\frac{2u^2-2u-12}{12}\sqrt{4u-u^2}+\frac{8}{4}\cos^{-1}\left(\frac{2-u}{2}\right)\right]_0^4$$
$$= \left[\frac{u^2-u-6}{6}\sqrt{4u-u^2}+2\cos^{-1}\left(\frac{2-u}{2}\right)\right]_0^4$$
$$= \left[0+2\cos^{-1}(-1)\right]-\left(0+2\cos^{-1}1\right) = 2\cdot\pi-2\cdot 0 = 2\pi$$

23. $\int \sec^5 x\,dx \overset{77}{=} \tfrac{1}{4}\tan x\,\sec^3 x+\tfrac{3}{4}\int\sec^3 x\,dx \overset{77}{=} \tfrac{1}{4}\tan x\,\sec^3 x+\tfrac{3}{4}\left(\tfrac{1}{2}\tan x\,\sec x+\tfrac{1}{2}\int\sec x\,dx\right)$

$\overset{14}{=} \tfrac{1}{4}\tan x\,\sec^3 x+\tfrac{3}{8}\tan x\,\sec x+\tfrac{3}{8}\ln\left|\sec x+\tan x\right|+C$

24. $\int x^3\arcsin(x^2)\,dx = \int u\arcsin u\,\left(\tfrac{1}{2}du\right)$ $\quad\left[\begin{matrix}u = x^2,\\ du = 2x\,dx\end{matrix}\right]$

$$\overset{90}{=} \frac{1}{2}\left[\frac{2u^2-1}{4}\arcsin u+\frac{u\sqrt{1-u^2}}{4}\right]+C = \frac{2x^4-1}{8}\arcsin(x^2)+\frac{x^2\sqrt{1-x^4}}{8}+C$$

25. Let $u = \ln x$ and $a = 2$. Then $du = dx/x$ and

$$\int\frac{\sqrt{4+(\ln x)^2}}{x}\,dx = \int\sqrt{a^2+u^2}\,du \overset{21}{=} \frac{u}{2}\sqrt{a^2+u^2}+\frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right)+C$$
$$= \tfrac{1}{2}(\ln x)\sqrt{4+(\ln x)^2}+2\ln\left[\ln x+\sqrt{4+(\ln x)^2}\right]+C$$

26. $\int x^4e^{-x}\,dx \overset{97}{=} -x^4e^{-x}+4\int x^3e^{-x}\,dx \overset{97}{=} -x^4e^{-x}+4\left(-x^3e^{-x}+3\int x^2e^{-x}\,dx\right)$

$\overset{97}{=} -(x^4+4x^3)e^{-x}+12\left(-x^2e^{-x}+2\int xe^{-x}\,dx\right)$

$\overset{96}{=} -(x^4+4x^3+12x^2)e^{-x}+24\left[(-x-1)e^{-x}\right]+C = -(x^4+4x^3+12x^2+24x+24)e^{-x}+C$

So $\int_0^1 x^4e^{-x}\,dx = \left[-(x^4+4x^3+12x^2+24x+24)e^{-x}\right]_0^1 = -(1+4+12+24+24)e^{-1}+24e^0 = 24-65e^{-1}$.

27. $\displaystyle\int\frac{\cos^{-1}(x^{-2})}{x^3}\,dx = -\frac{1}{2}\int\cos^{-1}u\,du$ $\quad\left[\begin{matrix}u = x^{-2},\\ du = -2x^{-3}\,dx\end{matrix}\right]$

$$\overset{88}{=} -\tfrac{1}{2}\left(u\cos^{-1}u-\sqrt{1-u^2}\right)+C = -\tfrac{1}{2}x^{-2}\cos^{-1}(x^{-2})+\tfrac{1}{2}\sqrt{1-x^{-4}}+C$$

28. $\displaystyle\int \frac{dx}{\sqrt{1-e^{2x}}} = \int \frac{1}{\sqrt{1-u^2}}\left(\frac{du}{u}\right) \quad \begin{bmatrix} u = e^x, \\ du = e^x\,dx,\ dx = du/u \end{bmatrix}$

$$\overset{35}{=} -\frac{1}{1}\ln\left|\frac{1+\sqrt{1-u^2}}{u}\right| + C = -\ln\left|\frac{1+\sqrt{1-e^{2x}}}{e^x}\right| + C = -\ln\left(\frac{1+\sqrt{1-e^{2x}}}{e^x}\right) + C$$

29. Let $u = e^x$. Then $x = \ln u$, $dx = du/u$, so

$$\int \sqrt{e^{2x}-1}\,dx = \int \frac{\sqrt{u^2-1}}{u}\,du \overset{41}{=} \sqrt{u^2-1} - \cos^{-1}(1/u) + C = \sqrt{e^{2x}-1} - \cos^{-1}(e^{-x}) + C.$$

30. Let $u = \alpha t - 3$ and assume that $\alpha \neq 0$. Then $du = \alpha\,dt$ and

$$\int e^t \sin(\alpha t - 3)\,dt = \frac{1}{\alpha}\int e^{(u+3)/\alpha}\sin u\,du = \frac{1}{\alpha}e^{3/\alpha}\int e^{(1/\alpha)u}\sin u\,du$$

$$\overset{98}{=} \frac{1}{\alpha}e^{3/\alpha}\frac{e^{(1/\alpha)u}}{(1/\alpha)^2+1^2}\left(\frac{1}{\alpha}\sin u - \cos u\right) + C = \frac{1}{\alpha}e^{3/\alpha}e^{(1/\alpha)u}\frac{\alpha^2}{1+\alpha^2}\left(\frac{1}{\alpha}\sin u - \cos u\right) + C$$

$$= \frac{1}{1+\alpha^2}e^{(u+3)/\alpha}(\sin u - \alpha\cos u) + C = \frac{1}{1+\alpha^2}\,e^t\left[\sin(\alpha t-3) - \alpha\cos(\alpha t - 3)\right] + C$$

31. $\displaystyle\int \frac{x^4\,dx}{\sqrt{x^{10}-2}} = \int \frac{x^4\,dx}{\sqrt{(x^5)^2-2}} = \frac{1}{5}\int \frac{du}{\sqrt{u^2-2}} \quad \begin{bmatrix} u = x^5, \\ du = 5x^4\,dx \end{bmatrix}$

$$\overset{43}{=} \tfrac{1}{5}\ln\left|u+\sqrt{u^2-2}\,\right| + C = \tfrac{1}{5}\ln\left|x^5+\sqrt{x^{10}-2}\,\right| + C$$

32. Let $u = \tan\theta$ and $a = 3$. Then $du = \sec^2\theta\,d\theta$ and

$$\int \frac{\sec^2\theta\,\tan^2\theta}{\sqrt{9-\tan^2\theta}}\,d\theta = \int \frac{u^2}{\sqrt{a^2-u^2}}\,du \overset{34}{=} -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$

$$= -\frac{1}{2}\tan\theta\,\sqrt{9-\tan^2\theta} + \frac{9}{2}\sin^{-1}\left(\frac{\tan\theta}{3}\right) + C$$

33. Use disks about the x-axis:

$$V = \int_0^\pi \pi(\sin^2 x)^2\,dx = \pi\int_0^\pi \sin^4 x\,dx \overset{73}{=} \pi\left\{\left[-\tfrac{1}{4}\sin^3 x\,\cos x\right]_0^\pi + \tfrac{3}{4}\int_0^\pi \sin^2 x\,dx\right\}$$

$$\overset{63}{=} \pi\left\{0 + \tfrac{3}{4}\left[\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^\pi\right\} = \pi\left[\tfrac{3}{4}\left(\tfrac{1}{2}\pi - 0\right)\right] = \tfrac{3}{8}\pi^2$$

34. Use shells about the y-axis:

$$V = \int_0^1 2\pi x \arcsin x\,dx \overset{90}{=} 2\pi\left[\frac{2x^2-1}{4}\sin^{-1}x + \frac{x\sqrt{1-x^2}}{4}\right]_0^1 = 2\pi\left[\left(\frac{1}{4}\cdot\frac{\pi}{2}+0\right) - 0\right] = \frac{1}{4}\pi^2$$

35. (a) $\displaystyle\frac{d}{du}\left[\frac{1}{b^3}\left(a+bu-\frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C\right] = \frac{1}{b^3}\left[b + \frac{ba^2}{(a+bu)^2} - \frac{2ab}{(a+bu)}\right]$

$$= \frac{1}{b^3}\left[\frac{b(a+bu)^2 + ba^2 - (a+bu)2ab}{(a+bu)^2}\right]$$

$$= \frac{1}{b^3}\left[\frac{b^3u^2}{(a+bu)^2}\right] = \frac{u^2}{(a+bu)^2}$$

(b) Let $t = a + bu \Rightarrow dt = b\,du$. Note that $u = \dfrac{t-a}{b}$ and $du = \dfrac{1}{b}\,dt$.

$$\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\int \frac{(t-a)^2}{t^2}\,dt = \frac{1}{b^3}\int \frac{t^2 - 2at + a^2}{t^2}\,dt = \frac{1}{b^3}\int\left(1 - \frac{2a}{t} + \frac{a^2}{t^2}\right)dt$$

$$= \frac{1}{b^3}\left(t - 2a\ln|t| - \frac{a^2}{t}\right) + C = \frac{1}{b^3}\left(a + bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$$

36. (a) $\dfrac{d}{du}\left[\dfrac{u}{8}(2u^2 - a^2)\sqrt{a^2-u^2} + \dfrac{a^4}{8}\sin^{-1}\dfrac{u}{a} + C\right]$

$$= \frac{u}{8}(2u^2-a^2)\frac{-u}{\sqrt{a^2-u^2}} + \sqrt{a^2-u^2}\left[\frac{u}{8}(4u) + (2u^2-a^2)\tfrac{1}{8}\right] + \frac{a^4}{8}\frac{1/a}{\sqrt{1-u^2/a^2}}$$

$$= -\frac{u^2(2u^2-a^2)}{8\sqrt{a^2-u^2}} + \sqrt{a^2-u^2}\left[\frac{u^2}{2} + \frac{2u^2-a^2}{8}\right] + \frac{a^4}{8\sqrt{a^2-u^2}}$$

$$= \tfrac{1}{2}(a^2-u^2)^{-1/2}\left[-\frac{u^2}{4}(2u^2-a^2) + u^2(a^2-u^2) + \tfrac{1}{4}(a^2-u^2)(2u^2-a^2) + \frac{a^4}{4}\right]$$

$$= \tfrac{1}{2}(a^2-u^2)^{-1/2}[2u^2a^2 - 2u^4] = \frac{u^2(a^2-u^2)}{\sqrt{a^2-u^2}} = u^2\sqrt{a^2-u^2}$$

(b) Let $u = a\sin\theta \Rightarrow du = a\cos\theta\,d\theta$. Then

$$\int u^2\sqrt{a^2-u^2}\,du = \int a^2\sin^2\theta\,a\sqrt{1-\sin^2\theta}\,a\cos\theta\,d\theta = a^4\int\sin^2\theta\,\cos^2\theta\,d\theta$$

$$= a^4\int\tfrac{1}{2}(1+\cos 2\theta)\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{4}a^4\int(1-\cos^2 2\theta)\,d\theta$$

$$= \tfrac{1}{4}a^4\int\left[1 - \tfrac{1}{2}(1+\cos 4\theta)\right]d\theta = \tfrac{1}{4}a^4\left(\tfrac{1}{2}\theta - \tfrac{1}{8}\sin 4\theta\right) + C$$

$$= \tfrac{1}{4}a^4\left(\tfrac{1}{2}\theta - \tfrac{1}{8}\cdot 2\sin 2\theta\,\cos 2\theta\right) + C = \tfrac{1}{4}a^4\left[\tfrac{1}{2}\theta - \tfrac{1}{2}\sin\theta\,\cos\theta(1-2\sin^2\theta)\right] + C$$

$$= \frac{a^4}{8}\left[\sin^{-1}\frac{u}{a} - \frac{u}{a}\frac{\sqrt{a^2-u^2}}{a}\left(1 - \frac{2u^2}{a^2}\right)\right] + C = \frac{a^4}{8}\left[\sin^{-1}\frac{u}{a} - \frac{u}{a}\frac{\sqrt{a^2-u^2}}{a}\frac{a^2-2u^2}{a^2}\right] + C$$

$$= \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$$

37. Maple and Mathematica both give $\int\sec^4 x\,dx = \frac{2}{3}\tan x + \frac{1}{3}\tan x\,\sec^2 x$, while Derive gives the second term as $\dfrac{\sin x}{3\cos^3 x} = \dfrac{1}{3}\dfrac{\sin x}{\cos x}\dfrac{1}{\cos^2 x} = \dfrac{1}{3}\tan x\,\sec^2 x$. Using Formula 77, we get $\int\sec^4 x\,dx = \frac{1}{3}\tan x\,\sec^2 x + \frac{2}{3}\int\sec^2 x\,dx = \frac{1}{3}\tan x\,\sec^2 x + \frac{2}{3}\tan x + C$.

38. Derive gives $\int\csc^5 x\,dx = \dfrac{3}{8}\ln\left(\tan\left(\dfrac{x}{2}\right)\right) - \cos x\left(\dfrac{3}{8\sin^2 x} + \dfrac{1}{4\sin^4 x}\right)$ and Maple gives $-\dfrac{1}{4}\dfrac{\cos x}{\sin^4 x} - \dfrac{3}{8}\dfrac{\cos x}{\sin^2 x} + \dfrac{3}{8}\ln(\csc x - \cot x)$. Using a half-angle identity for tangent, $\tan\dfrac{x}{2} = \dfrac{1-\cos x}{\sin x}$, we have $\ln\tan\dfrac{x}{2} = \ln\dfrac{1-\cos x}{\sin x} = \ln\left(\dfrac{1}{\sin x} - \dfrac{\cos x}{\sin x}\right) = \ln(\csc x - \cot x)$, so those two answers are equivalent.

Mathematica gives

$$I = -\frac{3}{32}\csc^2\frac{x}{2} - \frac{1}{64}\csc^4\frac{x}{2} - \frac{3}{8}\log\cos\frac{x}{2} + \frac{3}{8}\log\sin\frac{x}{2} + \frac{3}{32}\sec^2\frac{x}{2} + \frac{1}{64}\sec^4\frac{x}{2}$$

$$= \frac{3}{8}\left(\log\sin\frac{x}{2} - \log\cos\frac{x}{2}\right) + \frac{3}{32}\left(\sec^2\frac{x}{2} - \csc^2\frac{x}{2}\right) + \frac{1}{64}\left(\sec^4\frac{x}{2} - \csc^4\frac{x}{2}\right)$$

$$= \frac{3}{8}\log\frac{\sin(x/2)}{\cos(x/2)} + \frac{3}{32}\left[\frac{1}{\cos^2(x/2)} - \frac{1}{\sin^2(x/2)}\right] + \frac{1}{64}\left[\frac{1}{\cos^4(x/2)} - \frac{1}{\sin^4(x/2)}\right]$$

$$= \frac{3}{8}\log\tan\frac{x}{2} + \frac{3}{32}\left[\frac{\sin^2(x/2) - \cos^2(x/2)}{\cos^2(x/2)\,\sin^2(x/2)}\right] + \frac{1}{64}\left[\frac{\sin^4(x/2) - \cos^4(x/2)}{\cos^4(x/2)\,\sin^4(x/2)}\right]$$

Now $$\frac{\sin^2(x/2) - \cos^2(x/2)}{\cos^2(x/2)\,\sin^2(x/2)} = \frac{\dfrac{1-\cos x}{2} - \dfrac{1+\cos x}{2}}{\dfrac{1+\cos x}{2}\cdot\dfrac{1-\cos x}{2}} = \frac{-\dfrac{2\cos x}{2}}{\dfrac{1-\cos^2 x}{4}} = \frac{-4\cos x}{\sin^2 x}$$

and $$\frac{\sin^4(x/2) - \cos^4(x/2)}{\cos^4(x/2)\,\sin^4(x/2)} = \frac{\sin^2(x/2) - \cos^2(x/2)}{\cos^2(x/2)\,\sin^2(x/2)}\,\frac{\sin^2(x/2) + \cos^2(x/2)}{\cos^2(x/2)\,\sin^2(x/2)}$$

$$= \frac{-4\cos x}{\sin^2 x}\,\frac{1}{\dfrac{1+\cos x}{2}\cdot\dfrac{1-\cos x}{2}} = -\frac{4\cos x}{\sin^2 x}\,\frac{4}{1-\cos^2 x} = -\frac{16\cos x}{\sin^4 x}$$

Returning to the expression for I, we get

$$I = \frac{3}{8}\log\tan\frac{x}{2} + \frac{3}{32}\left(\frac{-4\cos x}{\sin^2 x}\right) + \frac{1}{64}\left(\frac{-16\cos x}{\sin^4 x}\right) = \frac{3}{8}\log\tan\frac{x}{2} - \frac{3}{8}\frac{\cos x}{\sin^2 x} - \frac{1}{4}\frac{\cos x}{\sin^4 x},$$

so all are equivalent.

Now use Formula 78 to get

$$\int \csc^5 x\,dx = \frac{-1}{4}\cot x\,\csc^3 x + \frac{3}{4}\int \csc^3 x\,dx = -\frac{1}{4}\frac{\cos x}{\sin x}\frac{1}{\sin^3 x} + \frac{3}{4}\left(\frac{-1}{2}\cot x\csc x + \frac{1}{2}\int \csc x\,dx\right)$$

$$= -\frac{1}{4}\frac{\cos x}{\sin^4 x} - \frac{3}{8}\frac{\cos x}{\sin x}\frac{1}{\sin x} + \frac{3}{8}\int \csc x\,dx = -\frac{1}{4}\frac{\cos x}{\sin^4 x} - \frac{3}{8}\frac{\cos x}{\sin^2 x} + \frac{3}{8}\ln|\csc x - \cot x| + C$$

39. Derive gives $\int x^2\sqrt{x^2+4}\,dx = \frac{1}{4}x(x^2+2)\sqrt{x^2+4} - 2\ln\left(\sqrt{x^2+4}+x\right)$. Maple gives $\frac{1}{4}x(x^2+4)^{3/2} - \frac{1}{2}x\sqrt{x^2+4} - 2\,\mathrm{arcsinh}\left(\frac{1}{2}x\right)$. Applying the command `convert(%,ln);` yields

$$\tfrac{1}{4}x(x^2+4)^{3/2} - \tfrac{1}{2}x\sqrt{x^2+4} - 2\ln\left(\tfrac{1}{2}x + \tfrac{1}{2}\sqrt{x^2+4}\right) = \tfrac{1}{4}x(x^2+4)^{1/2}\left[(x^2+4)-2\right] - 2\ln\left[\left(x+\sqrt{x^2+4}\right)/2\right]$$

$$= \tfrac{1}{4}x(x^2+2)\sqrt{x^2+4} - 2\ln\left(\sqrt{x^2+4}+x\right) + 2\ln 2$$

Mathematica gives $\frac{1}{4}x(2+x^2)\sqrt{3+x^2} - 2\,\mathrm{arcsinh}(x/2)$. Applying the `TrigToExp` and `Simplify` commands gives $\frac{1}{4}\left[x(2+x^2)\sqrt{4+x^2} - 8\log\left(\frac{1}{2}\left(x+\sqrt{4+x^2}\right)\right)\right] = \frac{1}{4}x(x^2+2)\sqrt{x^2+4} - 2\ln\left(x+\sqrt{4+x^2}\right) + 2\ln 2$, so all are equivalent (without constant).

Now use Formula 22 to get

$$\int x^2\sqrt{2^2+x^2}\,dx = \frac{x}{8}(2^2+2x^2)\sqrt{2^2+x^2} - \frac{2^4}{8}\ln\left(x+\sqrt{2^2+x^2}\right) + C$$

$$= \frac{x}{8}(2)(2+x^2)\sqrt{4+x^2} - 2\ln\left(x+\sqrt{4+x^2}\right) + C$$

$$= \tfrac{1}{4}x(x^2+2)\sqrt{x^2+4} - 2\ln\left(\sqrt{x^2+4}+x\right) + C$$

40. Derive gives $\displaystyle\int \frac{dx}{e^x(3e^x+2)} = -\frac{e^{-x}}{2} + \frac{3\ln(3e^x+2)}{4} - \frac{3x}{4}$, Maple gives $\displaystyle\frac{3}{4}\ln(3e^x+2) - \frac{1}{2e^x} - \frac{3}{4}\ln(e^x)$, and Mathematica gives

$$-\frac{e^{-x}}{2} + \frac{3}{4}\log(3+2e^{-x}) = -\frac{e^{-x}}{2} + \frac{3}{4}\log\left(\frac{3e^x+2}{e^x}\right) = -\frac{e^{-x}}{2} + \frac{3}{4}\frac{\ln(3e^x+2)}{\ln e^x} = -\frac{e^{-x}}{2} + \frac{3}{4}\ln(3e^x+2) - \frac{3}{4}x,$$

so all are equivalent. Now let $u = e^x$, so $du = e^x\,dx$ and $dx = du/u$. Then

$$\int \frac{1}{e^x(3e^x+2)}\,dx = \int \frac{1}{u(3u+2)}\,\frac{du}{u} = \int \frac{1}{u^2(2+3u)}\,du \overset{50}{=} -\frac{1}{2u} + \frac{3}{2^2}\ln\left|\frac{2+3u}{u}\right| + C$$
$$= -\frac{1}{2e^x} + \frac{3}{4}\ln(2+3e^x) - \frac{3}{4}\ln e^x + C = -\frac{1}{2e^x} + \frac{3}{4}\ln(3e^x+2) - \frac{3}{4}x + C$$

41. Derive and Maple give $\displaystyle\int \cos^4 x\,dx = \frac{\sin x\,\cos^3 x}{4} + \frac{3\sin x\,\cos x}{8} + \frac{3x}{8}$, while Mathematica gives

$$\frac{3x}{8} + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) = \frac{3x}{8} + \frac{1}{4}(2\sin x\,\cos x) + \frac{1}{32}(2\sin 2x\,\cos 2x)$$
$$= \frac{3x}{8} + \frac{1}{2}\sin x\,\cos x + \frac{1}{16}[2\sin x\,\cos x\,(2\cos^2 x - 1)]$$
$$= \frac{3x}{8} + \frac{1}{2}\sin x\,\cos x + \frac{1}{4}\sin x\,\cos^3 x - \frac{1}{8}\sin x\,\cos x,$$

so all are equivalent.

Using tables,

$$\int \cos^4 x\,dx \overset{74}{=} \tfrac{1}{4}\cos^3 x\,\sin x + \tfrac{3}{4}\int \cos^2 x\,dx \overset{64}{=} \tfrac{1}{4}\cos^3 x\,\sin x + \tfrac{3}{4}\left(\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right) + C$$
$$= \tfrac{1}{4}\cos^3 x\,\sin x + \tfrac{3}{8}x + \tfrac{3}{16}(2\sin x\,\cos x) + C = \tfrac{1}{4}\cos^3 x\,\sin x + \tfrac{3}{8}x + \tfrac{3}{8}\sin x\,\cos x + C$$

42. Derive gives $\displaystyle\int x^2\sqrt{1-x^2}\,dx = \frac{\arcsin x}{8} + \frac{x\sqrt{1-x^2}(2x^2-1)}{8}$, Maple gives

$$-\frac{x}{4}(1-x^2)^{3/2} + \frac{x}{8}\sqrt{1-x^2} + \frac{1}{8}\arcsin x = \frac{x}{8}(1-x^2)^{1/2}[-2(1-x^2)+1] + \frac{1}{8}\arcsin x$$
$$= \frac{x}{8}(1-x^2)^{1/2}(2x^2-1) + \frac{1}{8}\arcsin x,$$

and Mathematica gives $\frac{1}{8}\left(x\sqrt{1-x^2}(-1+2x^2) + \arcsin x\right)$, so all are equivalent.

Now use Formula 31 to get

$$\int x^2\sqrt{1-x^2}\,dx = \frac{x}{8}(2x^2-1)\sqrt{1-x^2} + \frac{1}{8}\sin^{-1}x + C$$

43. Maple gives $\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \frac{1}{2}\ln(1+\tan^2 x)$, Mathematica gives $\int \tan^5 x\,dx = \frac{1}{4}[-1-2\cos(2x)]\sec^4 x - \ln(\cos x)$, and Derive gives $\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln(\cos x)$. These expressions are equivalent, and none includes absolute value bars or a constant of integration. Note that Mathematica's and Derive's expressions suggest that the integral is undefined where $\cos x < 0$, which is not the case. Using Formula 75, $\int \tan^5 x\,dx = \frac{1}{5-1}\tan^{5-1}x - \int \tan^{5-2}x\,dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\,dx$. Using Formula 69, $\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C$, so $\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C$.

44. Derive, Maple, and Mathematica all give $\int \frac{1}{\sqrt{1+\sqrt[3]{x}}}\,dx = \frac{2}{5}\sqrt{\sqrt[3]{x}+1}\left(3\sqrt[3]{x^2}-4\sqrt[3]{x}+8\right)$. [Maple adds a constant of $-\frac{16}{5}$.] We'll change the form of the integral by letting $u=\sqrt[3]{x}$, so that $u^3=x$ and $3u^2\,du=dx$. Then

$$\int \frac{1}{\sqrt{1+\sqrt[3]{x}}}\,dx = \int \frac{3u^2\,du}{\sqrt{1+u}} \overset{56}{=} 3\left[\frac{2}{15(1)^3}\left(8(1)^2+3(1)^2u^2-4(1)(1)u\right)\sqrt{1+u}\right]+C$$

$$= \tfrac{2}{5}(8+3u^2-4u)\sqrt{1+u}+C = \tfrac{2}{5}\left(8+3\sqrt[3]{x^2}-4\sqrt[3]{x}\right)\sqrt{1+\sqrt[3]{x}}+C$$

45. (a) $F(x)=\int f(x)\,dx = \int \frac{1}{x\sqrt{1-x^2}}\,dx \overset{35}{=} -\frac{1}{1}\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|+C = -\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|+C.$

f has domain $\{x \mid x\neq 0,\, 1-x^2>0\} = \{x \mid x\neq 0,\ |x|<1\} = (-1,0)\cup(0,1)$. F has the same domain.

(b) Derive gives $F(x)=\ln\left(\sqrt{1-x^2}-1\right)-\ln x$ and Mathematica gives $F(x)=\ln x-\ln\left(1+\sqrt{1-x^2}\right)$. Both are correct if you take absolute values of the logarithm arguments, and both would then have the same domain. Maple gives $F(x)=-\operatorname{arctanh}\left(1/\sqrt{1-x^2}\right)$. This function has domain

$\left\{x \mid |x|<1, -1<1/\sqrt{1-x^2}<1\right\} = \left\{x \mid |x|<1, 1/\sqrt{1-x^2}<1\right\} = \left\{x \mid |x|<1, \sqrt{1-x^2}>1\right\} = \emptyset$,

the empty set! If we apply the command `convert(%,ln);` to Maple's answer, we get

$-\frac{1}{2}\ln\left(\frac{1}{\sqrt{1-x^2}}+1\right)+\frac{1}{2}\ln\left(1-\frac{1}{\sqrt{1-x^2}}\right)$, which has the same domain, $\emptyset$.

46. None of Maple, Mathematica and Derive is able to evaluate $\int(1+\ln x)\sqrt{1+(x\ln x)^2}\,dx$. However, if we let $u=x\ln x$, then $du=(1+\ln x)\,dx$ and the integral is simply $\int\sqrt{1+u^2}\,du$, which any CAS can evaluate. The antiderivative is $\frac{1}{2}\ln\left(x\ln x+\sqrt{1+(x\ln x)^2}\right)+\frac{1}{2}x\ln x\sqrt{1+(x\ln x)^2}+C$.

DISCOVERY PROJECT Patterns in Integrals

1. (a) The CAS results are listed. Note that the absolute value symbols are missing, as is the familiar "$+C$".

(i) $\int \frac{1}{(x+2)(x+3)}\,dx = \ln(x+2)-\ln(x+3)$ (ii) $\int \frac{1}{(x+1)(x+5)}\,dx = \frac{\ln(x+1)}{4}-\frac{\ln(x+5)}{4}$

(iii) $\int \frac{1}{(x+2)(x-5)}\,dx = \frac{\ln(x-5)}{7}-\frac{\ln(x+2)}{7}$ (iv) $\int \frac{1}{(x+2)^2}\,dx = -\frac{1}{x+2}$

(b) If $a\neq b$, it appears that $\ln(x+a)$ is divided by $b-a$ and $\ln(x+b)$ is divided by $a-b$, so we guess that $\int \frac{1}{(x+a)(x+b)}\,dx = \frac{\ln(x+a)}{b-a}+\frac{\ln(x+b)}{a-b}+C$. If $a=b$, as in part (a)(iv), it appears that $\int \frac{1}{(x+a)^2}\,dx = -\frac{1}{x+a}+C$.

(c) The CAS verifies our guesses. Now $\frac{1}{(x+a)(x+b)} = \frac{A}{x+a}+\frac{B}{x+b} \quad\Rightarrow\quad 1=A(x+b)+B(x+a)$.

Setting $x=-b$ gives $B=1/(a-b)$ and setting $x=-a$ gives $A=1/(b-a)$. So

$$\int \frac{1}{(x+a)(x+b)}\,dx = \int\left[\frac{1/(b-a)}{x+a}+\frac{1/(a-b)}{x+b}\right]dx = \frac{\ln|x+a|}{b-a}+\frac{\ln|x+b|}{a-b}+C$$

[continued]

and our guess for $a \neq b$ is correct. If $a = b$, then $\dfrac{1}{(x+a)(x+b)} = \dfrac{1}{(x+a)^2} = (x+a)^{-2}$. Letting $u = x + a \quad \Rightarrow$ $du = dx$, we have $\int (x+a)^{-2}\,dx = \int u^{-2}\,du = -\dfrac{1}{u} + C = -\dfrac{1}{x+a} + C$, and our guess for $a = b$ is also correct.

2. (a) (i) $\int \sin x\,\cos 2x\,dx = \dfrac{\cos x}{2} - \dfrac{\cos 3x}{6}$ (ii) $\int \sin 3x\,\cos 7x\,dx = \dfrac{\cos 4x}{8} - \dfrac{\cos 10x}{20}$

(iii) $\int \sin 8x\,\cos 3x\,dx = -\dfrac{\cos 11x}{22} - \dfrac{\cos 5x}{10}$

(b) Looking at the sums and differences of a and b in part (a), we guess that

$$\int \sin ax\,\cos bx\,dx = \frac{\cos((a-b)x)}{2(b-a)} - \frac{\cos((a+b)x)}{2(a+b)} + C$$

Note that $\cos((a-b)x) = \cos((b-a)x)$.

(c) The CAS verifies our guess. Again, we can prove that the guess is correct by differentiating:

$$\begin{aligned}\frac{d}{dx}\left[\frac{\cos((a-b)x)}{2(b-a)} - \frac{\cos((a+b)x)}{2(a+b)}\right] &= \frac{1}{2(b-a)}\,[-\sin((a-b)x)](a-b) - \frac{1}{2(a+b)}\,[-\sin((a+b)x)](a+b) \\ &= \tfrac{1}{2}\sin(ax-bx) + \tfrac{1}{2}\sin(ax+bx) \\ &= \tfrac{1}{2}(\sin ax\,\cos bx - \cos ax\,\sin bx) + \tfrac{1}{2}(\sin ax\,\cos bx + \cos ax\,\sin bx) \\ &= \sin ax\,\cos bx\end{aligned}$$

Our formula is valid for $a \neq b$.

3. (a) (i) $\int \ln x\,dx = x\ln x - x$ (ii) $\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$

(iii) $\int x^2\ln x\,dx = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3$ (iv) $\int x^3\ln x\,dx = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4$

(v) $\int x^7\ln x\,dx = \frac{1}{8}x^8\ln x - \frac{1}{64}x^8$

(b) We guess that $\int x^n\ln x\,dx = \dfrac{1}{n+1}x^{n+1}\ln x - \dfrac{1}{(n+1)^2}x^{n+1}$.

(c) Let $u = \ln x$, $dv = x^n\,dx \quad \Rightarrow \quad du = \dfrac{dx}{x}$, $v = \dfrac{1}{n+1}x^{n+1}$. Then

$$\int x^n\ln x\,dx = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\int x^n\,dx = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\cdot\frac{1}{n+1}x^{n+1},$$

which verifies our guess. We must have $n + 1 \neq 0 \quad \Leftrightarrow \quad n \neq -1$.

4. (a) (i) $\int xe^x\,dx = e^x(x-1)$ (ii) $\int x^2e^x\,dx = e^x(x^2 - 2x + 2)$

(iii) $\int x^3e^x\,dx = e^x(x^3 - 3x^2 + 6x - 6)$ (iv) $\int x^4e^x\,dx = e^x(x^4 - 4x^3 + 12x^2 - 24x + 24)$

(v) $\int x^5e^x\,dx = e^x\left(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120\right)$

(b) Notice from part (a) that we can write

$$\int x^4e^x\,dx = e^x(x^4 - 4x^3 + 4\cdot 3x^2 - 4\cdot 3\cdot 2x + 4\cdot 3\cdot 2\cdot 1)$$

and $$\int x^5e^x\,dx = e^x(x^5 - 5x^4 + 5\cdot 4x^3 - 5\cdot 4\cdot 3x^2 + 5\cdot 4\cdot 3\cdot 2x - 5\cdot 4\cdot 3\cdot 2\cdot 1)$$

So we guess that

$$\begin{aligned}\int x^6 e^x\,dx &= e^x(x^6 - 6x^5 + 6\cdot 5x^4 - 6\cdot 5\cdot 4x^3 + 6\cdot 5\cdot 4\cdot 3x^2 - 6\cdot 5\cdot 4\cdot 3\cdot 2x + 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)\\ &= e^x(x^6 - 6x^5 + 30x^4 - 120x^3 + 360x^2 - 720x + 720)\end{aligned}$$

The CAS verifies our guess.

(c) From the results in part (a), as well as our prediction in part (b), we speculate that

$$\int x^n e^x\,dx = e^x\left[x^n - nx^{n-1} + n(n-1)x^{n-2} - n(n-1)(n-2)x^{n-3} + \cdots \pm n!x \mp n!\right] = e^x \sum_{i=0}^{n} (-1)^{n-i} \frac{n!}{i!}\,x^i.$$

(We have reversed the order of the polynomial's terms.)

(d) Let S_n be the statement that $\int x^n e^x\,dx = e^x \sum\limits_{i=0}^{n} (-1)^{n-i} \dfrac{n!}{i!}\,x^i$.

S_1 is true by part (a)(i). Suppose S_k is true for some k, and consider S_{k+1}. Integrating by parts with $u = x^{k+1}$, $dv = e^x\,dx \quad\Rightarrow\quad du = (k+1)x^k\,dx$, $v = e^x$, we get

$$\begin{aligned}\int x^{k+1}e^x\,dx &= x^{k+1}e^x - (k+1)\int x^k e^x\,dx = x^{k+1}e^x - (k+1)\left[e^x \sum_{i=0}^{k} (-1)^{k-i}\frac{k!}{i!}\,x^i\right]\\ &= e^x\left[x^{k+1} - (k+1)\sum_{i=0}^{k}(-1)^{k-i}\frac{k!}{i!}\,x^i\right] = e^x\left[x^{k+1} + \sum_{i=0}^{k}(-1)^{k-i+1}\frac{(k+1)k!}{i!}\,x^i\right]\\ &= e^x \sum_{i=0}^{k+1}(-1)^{(k+1)-i}\frac{(k+1)!}{i!}\,x^i\end{aligned}$$

This verifies S_n for $n = k+1$. Thus, by mathematical induction, S_n is true for all n, where n is a positive integer.

7.7 Approximate Integration

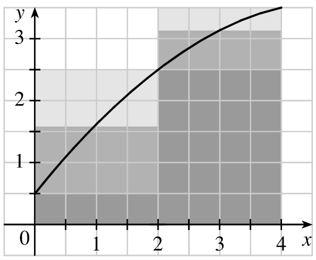
1. (a) $\Delta x = (b-a)/n = (4-0)/2 = 2$

$L_2 = \sum\limits_{i=1}^{2} f(x_{i-1})\,\Delta x = f(x_0)\cdot 2 + f(x_1)\cdot 2 = 2\,[f(0) + f(2)] = 2(0.5 + 2.5) = 6$

$R_2 = \sum\limits_{i=1}^{2} f(x_i)\,\Delta x = f(x_1)\cdot 2 + f(x_2)\cdot 2 = 2\,[f(2) + f(4)] = 2(2.5 + 3.5) = 12$

$M_2 = \sum\limits_{i=1}^{2} f(\overline{x}_i)\Delta x = f(\overline{x}_1)\cdot 2 + f(\overline{x}_2)\cdot 2 = 2\,[f(1) + f(3)] \approx 2(1.6 + 3.2) = 9.6$

(b)

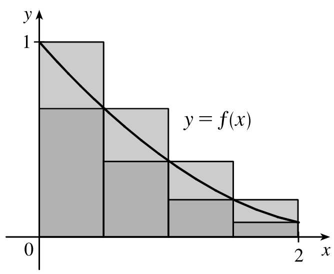


L_2 is an underestimate, since the area under the small rectangles is less than the area under the curve, and R_2 is an overestimate, since the area under the large rectangles is greater than the area under the curve. It appears that M_2 is an overestimate, though it is fairly close to I. See the solution to Exercise 47 for a proof of the fact that if f is concave down on $[a, b]$, then the Midpoint Rule is an overestimate of $\int_a^b f(x)\,dx$.

(c) $T_2 = \left(\frac{1}{2}\,\Delta x\right)[f(x_0) + 2f(x_1) + f(x_2)] = \frac{2}{2}[f(0) + 2f(2) + f(4)] = 0.5 + 2(2.5) + 3.5 = 9.$

This approximation is an underestimate, since the graph is concave down. Thus, $T_2 = 9 < I$. See the solution to Exercise 47 for a general proof of this conclusion.

(d) For any n, we will have $L_n < T_n < I < M_n < R_n$.

2. 

The diagram shows that $L_4 > T_4 > \int_0^2 f(x)\,dx > R_4$, and it appears that M_4 is a bit less than $\int_0^2 f(x)\,dx$. In fact, for any function that is concave upward, it can be shown that $L_n > T_n > \int_0^2 f(x)\,dx > M_n > R_n$.

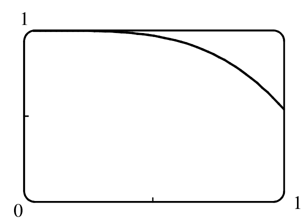
(a) Since $0.9540 > 0.8675 > 0.8632 > 0.7811$, it follows that $L_n = 0.9540$, $T_n = 0.8675$, $M_n = 0.8632$, and $R_n = 0.7811$.

(b) Since $M_n < \int_0^2 f(x)\,dx < T_n$, we have $0.8632 < \int_0^2 f(x)\,dx < 0.8675$.

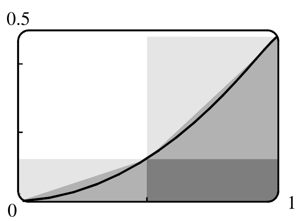
3. $f(x) = \cos\left(x^2\right)$, $\Delta x = \frac{1-0}{4} = \frac{1}{4}$

(a) $T_4 = \frac{1}{4 \cdot 2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{2}{4}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right] \approx 0.895759$

(b) $M_4 = \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right] \approx 0.908907$



The graph shows that f is concave down on $[0, 1]$. So T_4 is an underestimate and M_4 is an overestimate. We can conclude that $0.895759 < \int_0^1 \cos\left(x^2\right)dx < 0.908907$.

4. 

(a) Since f is increasing on $[0, 1]$, L_2 will underestimate I (since the area of the darkest rectangle is less than the area under the curve), and R_2 will overestimate I. Since f is concave upward on $[0, 1]$, M_2 will underestimate I and T_2 will overestimate I (the area under the straight line segments is greater than the area under the curve).

(b) For any n, we will have $L_n < M_n < I < T_n < R_n$.

(c) $L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x = \frac{1}{5}[f(0.0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)] \approx 0.1187$

$R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x = \frac{1}{5}[f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)] \approx 0.2146$

$M_5 = \sum_{i=1}^{5} f(\overline{x}_i)\,\Delta x = \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \approx 0.1622$

$T_5 = \left(\frac{1}{2}\,\Delta x\right)[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)] \approx 0.1666$

From the graph, it appears that the Midpoint Rule gives the best approximation. (This is in fact the case, since $I \approx 0.16371405$.)

5. (a) $f(x) = \dfrac{x}{1+x^2}$, $\quad \Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{10} = \dfrac{1}{5}$

$M_{10} = \frac{1}{5}\left[f\left(\frac{1}{10}\right) + f\left(\frac{3}{10}\right) + f\left(\frac{5}{10}\right) + \cdots + f\left(\frac{19}{10}\right)\right] \approx 0.806598$

(b) $S_{10} = \frac{1}{5 \cdot 3}\left[f(0) + 4f\left(\frac{1}{5}\right) + 2f\left(\frac{2}{5}\right) + 4f\left(\frac{3}{5}\right) + 2f\left(\frac{4}{5}\right) + \cdots + 4f\left(\frac{9}{5}\right) + f(2)\right] \approx 0.804779$

Actual: $I = \displaystyle\int_0^2 \frac{x}{1+x^2}\,dx = \left[\frac{1}{2}\ln\left|1+x^2\right|\right]_0^2 \qquad [u = 1+x^2,\, du = 2x\,dx]$

$= \frac{1}{2}\ln 5 - \frac{1}{2}\ln 1 = \frac{1}{2}\ln 5 \approx 0.804719$

Errors: $E_M = \text{actual} - M_{10} = I - M_{10} \approx -0.001879$

$E_S = \text{actual} - S_{10} = I - S_{10} \approx -0.000060$

6. (a) $f(x) = x\cos x$, $\quad \Delta x = \dfrac{b-a}{n} = \dfrac{\pi - 0}{4} = \dfrac{\pi}{4}$

$M_4 = \frac{\pi}{4}\left[f\left(\frac{\pi}{8}\right) + f\left(\frac{3\pi}{8}\right) + f\left(\frac{5\pi}{8}\right) + f\left(\frac{7\pi}{8}\right)\right] \approx -1.945744$

(b) $S_4 = \frac{\pi}{4 \cdot 3}\left[f(0) + 4f\left(\frac{\pi}{4}\right) + 2f\left(\frac{2\pi}{4}\right) + 4f\left(\frac{3\pi}{4}\right) + f(\pi)\right] \approx -1.985611$

Actual: $I = \int_0^\pi x\cos x\,dx = \Big[x\sin x + \cos x\Big]_0^\pi \qquad$ [use parts with $u = x$ and $dv = \cos x\,dx$]

$= (0 + (-1)) - (0 + 1) = -2$

Errors: $E_M = \text{actual} - M_4 = I - M_4 \approx -0.054256$

$E_S = \text{actual} - S_4 = I - S_4 \approx -0.014389$

7. $f(x) = \sqrt{x^3 - 1}$, $\Delta x = \dfrac{b-a}{n} = \dfrac{2-1}{10} = \dfrac{1}{10}$

(a) $T_{10} = \frac{1}{10 \cdot 2}[f(1) + 2f(1.1) + 2f(1.2) + 2f(1.3) + 2f(1.4) + 2f(1.5)$
$\quad + 2f(1.6) + 2f(1.7) + 2f(1.8) + 2f(1.9) + f(2)]$

≈ 1.506361

(b) $M_{10} = \frac{1}{10}[f(1.05) + f(1.15) + f(1.25) + f(1.35) + f(1.45) + f(1.55) + f(1.65) + f(1.75) + f(1.85) + f(1.95)]$
≈ 1.518362

(c) $S_{10} = \frac{1}{10 \cdot 3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4)$
$\quad + 4f(1.5) + 2f(1.6) + 4f(1.7) + 2f(1.8) + 4f(1.9) + f(2)]$

≈ 1.511519

8. $f(x) = \dfrac{1}{1+x^6}$, $\Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{8} = \dfrac{1}{4}$

(a) $T_8 = \frac{1}{4 \cdot 2}[f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + 2f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2)] \approx 1.040756$

(b) $M_8 = \frac{1}{4}[f(0.125) + f(0.375) + f(0.625) + f(0.875) + f(1.125) + f(1.375) + f(1.625) + f(1.875)] \approx 1.041109$

(c) $S_8 = \frac{1}{4 \cdot 3}[f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + 2f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)] \approx 1.042172$

9. $f(x) = \dfrac{e^x}{1+x^2}$, $\Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{10} = \dfrac{1}{5}$

(a) $T_{10} = \frac{1}{5 \cdot 2}[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + 2f(1)$
$\quad + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)]$

≈ 2.660833

(b) $M_{10} = \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9) + f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$
≈ 2.664377

(c) $S_{10} = \frac{1}{5 \cdot 3}[f(0) + 4f(0.2) + 2f(0.4) + 4f(0.6) + 2f(0.8)$
$+ 4f(1) + 2f(1.2) + 4f(1.4) + 2f(1.6) + 4f(1.8) + f(2)] \approx 2.663244$

10. $f(x) = \sqrt[3]{1 + \cos x}$, $\Delta x = \frac{\pi/2 - 0}{4} = \frac{\pi}{8}$

(a) $T_4 = \frac{\pi}{8 \cdot 2}\left[f(0) + 2f\left(\frac{\pi}{8}\right) + 2f\left(\frac{2\pi}{8}\right) + 2f\left(\frac{3\pi}{8}\right) + f\left(\frac{\pi}{2}\right)\right] \approx 1.838967$

(b) $M_4 = \frac{\pi}{8}\left[f\left(\frac{\pi}{16}\right) + f\left(\frac{3\pi}{16}\right) + f\left(\frac{5\pi}{16}\right) + f\left(\frac{7\pi}{16}\right)\right] \approx 1.845390$

(c) $S_4 = \frac{\pi}{8 \cdot 3}\left[f(0) + 4f\left(\frac{\pi}{8}\right) + 2f\left(\frac{2\pi}{8}\right) + 4f\left(\frac{3\pi}{8}\right) + f\left(\frac{\pi}{2}\right)\right] \approx 1.843245$

11. $f(x) = x^3 \sin x$, $\Delta x = \frac{4 - 0}{8} = \frac{1}{2}$

(a) $T_8 = \frac{1}{2 \cdot 2}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + 2f\left(\frac{5}{2}\right) + 2f(3) + 2f\left(\frac{7}{2}\right) + f(4)\right] \approx -7.276910$

(b) $M_8 = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right)\right] \approx -4.818251$

(c) $S_8 = \frac{1}{2 \cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4)\right] \approx -5.605350$

12. $f(x) = e^{1/x}$, $\Delta x = \frac{3 - 1}{8} = \frac{1}{4}$

(a) $T_8 = \frac{1}{4 \cdot 2}\left[f(1) + 2f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{7}{4}\right) + 2f(2) + 2f\left(\frac{9}{4}\right) + 2f\left(\frac{5}{2}\right) + 2f\left(\frac{11}{4}\right) + f(3)\right] \approx 3.534934$

(b) $M_8 = \frac{1}{4}\left[f\left(\frac{9}{8}\right) + f\left(\frac{11}{8}\right) + f\left(\frac{13}{8}\right) + f\left(\frac{15}{8}\right) + f\left(\frac{17}{8}\right) + f\left(\frac{19}{8}\right) + f\left(\frac{21}{8}\right) + f\left(\frac{23}{8}\right)\right] \approx 3.515248$

(c) $S_8 = \frac{1}{4 \cdot 3}\left[f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{7}{4}\right) + 2f(2) + 4f\left(\frac{9}{4}\right) + 2f\left(\frac{5}{2}\right) + 4f\left(\frac{11}{4}\right) + f(3)\right] \approx 3.522375$

13. $f(y) = \sqrt{y}\cos y$, $\Delta y = \frac{4 - 0}{8} = \frac{1}{2}$

(a) $T_8 = \frac{1}{2 \cdot 2}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + 2f\left(\frac{5}{2}\right) + 2f(3) + 2f\left(\frac{7}{2}\right) + f(4)\right] \approx -2.364034$

(b) $M_8 = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right)\right] \approx -2.310690$

(c) $S_8 = \frac{1}{2 \cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4)\right] \approx -2.346520$

14. $f(t) = \dfrac{1}{\ln t}$, $\Delta t = \dfrac{3 - 2}{10} = \dfrac{1}{10}$

(a) $T_{10} = \frac{1}{10 \cdot 2}\{f(2) + 2[f(2.1) + f(2.2) + \cdots + f(2.9)] + f(3)\} \approx 1.119061$

(b) $M_{10} = \frac{1}{10}[f(2.05) + f(2.15) + \cdots + f(2.85) + f(2.95)] \approx 1.118107$

(c) $S_{10} = \frac{1}{10 \cdot 3}[f(2) + 4f(2.1) + 2f(2.2) + 4f(2.3) + 2f(2.4) + 4f(2.5) + 2f(2.6)$
$+ 4f(2.7) + 2f(2.8) + 4f(2.9) + f(3)] \approx 1.118428$

15. $f(x) = \dfrac{x^2}{1 + x^4}$, $\Delta x = \dfrac{1 - 0}{10} = \dfrac{1}{10}$

(a) $T_{10} = \frac{1}{10 \cdot 2}\{f(0) + 2[f(0.1 + f(0.2) + \cdots + f(0.9)] + f(1)\} \approx 0.243747$

(b) $M_{10} = \frac{1}{10}[f(0.05) + f(0.15) + \cdots + f(0.85) + f(0.95)] \approx 0.243748$

(c) $S_{10} = \frac{1}{10 \cdot 3}[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + 2f(0.4) + 4f(0.5) + 2f(0.6)$
$+ 4f(0.7) + 2f(0.8) + 4f(0.9) + f(1)] \approx 0.243751$

Note: $\int_0^1 f(x)\,dx \approx 0.24374775$. This is a rare case where the Trapezoidal and Midpoint Rules give better approximations than Simpson's Rule.

16. $f(t) = \dfrac{\sin t}{t}$, $\Delta t = \dfrac{3-1}{4} = \dfrac{1}{2}$

(a) $T_4 = \frac{1}{2 \cdot 2}[f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3)] \approx 0.901645$

(b) $M_4 = \frac{1}{2}[f(1.25) + f(1.75) + f(2.25) + f(2.75)] \approx 0.903031$

(c) $S_4 = \frac{1}{2 \cdot 3}[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)] \approx 0.902558$

17. $f(x) = \ln(1 + e^x)$, $\Delta x = \frac{4-0}{8} = \frac{1}{2}$

(a) $T_8 = \frac{1}{2 \cdot 2}\{f(0) + 2[f(0.5) + f(1) + \cdots + f(3) + f(3.5)] + f(4)\} \approx 8.814278$

(b) $M_8 = \frac{1}{2}[f(0.25) + f(0.75) + \cdots + f(3.25) + f(3.75)] \approx 8.799212$

(c) $S_8 = \frac{1}{2 \cdot 3}[f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4)] \approx 8.804229$

18. $f(x) = \sqrt{x + x^3}$, $\Delta x = \frac{1-0}{10} = \frac{1}{10}$

(a) $T_{10} = \frac{1}{2 \cdot 2}\{f(0) + 2[f(0.1) + f(0.2) + \cdots + f(0.8) + f(0.9)] + f(1)\} \approx 0.787092$

(b) $M_{10} = \frac{1}{2}[f(0.05) + f(0.15) + \cdots + f(0.85) + f(0.95)] \approx 0.793821$

(c) $S_{10} = \frac{1}{2 \cdot 3}[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + 2f(0.4) + 4f(0.5) + 2f(0.6)$
$\qquad + 4f(0.7) + 2f(0.8) + 4f(0.9) + f(1)]$
≈ 0.789915

19. $f(x) = \cos(x^2)$, $\Delta x = \frac{1-0}{8} = \frac{1}{8}$

(a) $T_8 = \frac{1}{8 \cdot 2}\left\{f(0) + 2\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right)\right] + f(1)\right\} \approx 0.902333$

$M_8 = \frac{1}{8}\left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + f\left(\frac{5}{16}\right) + \cdots + f\left(\frac{15}{16}\right)\right] = 0.905620$

(b) $f(x) = \cos(x^2)$, $f'(x) = -2x\sin(x^2)$, $f''(x) = -2\sin(x^2) - 4x^2\cos(x^2)$. For $0 \le x \le 1$, sin and cos are positive, so $|f''(x)| = 2\sin(x^2) + 4x^2\cos(x^2) \le 2 \cdot 1 + 4 \cdot 1 \cdot 1 = 6$ since $\sin(x^2) \le 1$ and $\cos\left(x^2\right) \le 1$ for all x, and $x^2 \le 1$ for $0 \le x \le 1$. So for $n = 8$, we take $K = 6$, $a = 0$, and $b = 1$ in Theorem 3, to get $|E_T| \le 6 \cdot 1^3/(12 \cdot 8^2) = \frac{1}{128} = 0.0078125$ and $|E_M| \le \frac{1}{256} = 0.00390625$. [A better estimate is obtained by noting from a graph of f'' that $|f''(x)| \le 4$ for $0 \le x \le 1$.]

(c) Take $K = 6$ [as in part (b)] in Theorem 3. $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 0.0001 \quad \Leftrightarrow \quad \dfrac{6(1-0)^3}{12n^2} \le 10^{-4} \quad \Leftrightarrow$

$\dfrac{1}{2n^2} \le \dfrac{1}{10^4} \quad \Leftrightarrow \quad 2n^2 \ge 10^4 \quad \Leftrightarrow \quad n^2 \ge 5000 \quad \Leftrightarrow \quad n \ge 71$. Take $n = 71$ for T_n. For E_M, again take $K = 6$ in Theorem 3 to get $|E_M| \le 10^{-4} \quad \Leftrightarrow \quad 4n^2 \ge 10^4 \quad \Leftrightarrow \quad n^2 \ge 2500 \quad \Leftrightarrow \quad n \ge 50$. Take $n = 50$ for M_n.

20. $f(x) = e^{1/x}$, $\Delta x = \frac{2-1}{10} = \frac{1}{10}$

(a) $T_{10} = \frac{1}{10 \cdot 2}[f(1) + 2f(1.1) + 2f(1.2) + \cdots + 2f(1.9) + f(2)] \approx 2.021976$

$M_{10} = \frac{1}{10}[f(1.05) + f(1.15) + f(1.25) + \cdots + f(1.95)] \approx 2.019102$

(b) $f(x) = e^{1/x}$, $f'(x) = -\frac{1}{x^2}e^{1/x}$, $f''(x) = \frac{2x+1}{x^4}\,e^{1/x}$. Now f'' is decreasing on $[1,2]$, so let $x = 1$ to take $K = 3e$.

$|E_T| \le \frac{3e(2-1)^3}{12(10)^2} = \frac{e}{400} \approx 0.006796$. $|E_M| \le \frac{|E_T|}{2} = \frac{e}{800} \approx 0.003398$.

(c) Take $K = 3e$ [as in part (b)] in Theorem 3. $|E_T| \le \frac{K(b-a)^3}{12n^2} \le 0.0001 \Leftrightarrow \frac{3e(2-1)^3}{12n^2} \le 10^{-4} \Leftrightarrow$

$\frac{e}{4n^2} \le \frac{1}{10^4} \Leftrightarrow n^2 \ge \frac{10^4 e}{4} \Leftrightarrow n \ge 83$. Take $n = 83$ for T_n. For E_M, again take $K = 3e$ in Theorem 3 to get

$|E_M| \le 10^{-4} \Leftrightarrow n^2 \ge \frac{10^4 e}{8} \Leftrightarrow n \ge 59$. Take $n = 59$ for M_n.

21. $f(x) = \sin x$, $\Delta x = \frac{\pi - 0}{10} = \frac{\pi}{10}$

(a) $T_{10} = \frac{\pi}{10 \cdot 2}\left[f(0) + 2f\left(\frac{\pi}{10}\right) + 2f\left(\frac{2\pi}{10}\right) + \cdots + 2f\left(\frac{9\pi}{10}\right) + f(\pi)\right] \approx 1.983524$

$M_{10} = \frac{\pi}{10}\left[f\left(\frac{\pi}{20}\right) + f\left(\frac{3\pi}{20}\right) + f\left(\frac{5\pi}{20}\right) + \cdots + f\left(\frac{19\pi}{20}\right)\right] \approx 2.008248$

$S_{10} = \frac{\pi}{10 \cdot 3}\left[f(0) + 4f\left(\frac{\pi}{10}\right) + 2f\left(\frac{2\pi}{10}\right) + 4f\left(\frac{3\pi}{10}\right) + \cdots + 4f\left(\frac{9\pi}{10}\right) + f(\pi)\right] \approx 2.000110$

Since $I = \int_0^\pi \sin x\,dx = \left[-\cos x\right]_0^\pi = 1 - (-1) = 2$, $E_T = I - T_{10} \approx 0.016476$, $E_M = I - M_{10} \approx -0.008248$, and $E_S = I - S_{10} \approx -0.000110$.

(b) $f(x) = \sin x \Rightarrow \left|f^{(n)}(x)\right| \le 1$, so take $K = 1$ for all error estimates.

$|E_T| \le \frac{K(b-a)^3}{12n^2} = \frac{1(\pi - 0)^3}{12(10)^2} = \frac{\pi^3}{1200} \approx 0.025839$. $|E_M| \le \frac{|E_T|}{2} = \frac{\pi^3}{2400} \approx 0.012919$.

$|E_S| \le \frac{K(b-a)^5}{180n^4} = \frac{1(\pi - 0)^5}{180(10)^4} = \frac{\pi^5}{1{,}800{,}000} \approx 0.000170$.

The actual error is about 64% of the error estimate in all three cases.

(c) $|E_T| \le 0.00001 \Leftrightarrow \frac{\pi^3}{12n^2} \le \frac{1}{10^5} \Leftrightarrow n^2 \ge \frac{10^5\pi^3}{12} \Rightarrow n \ge 508.3$. Take $n = 509$ for T_n.

$|E_M| \le 0.00001 \Leftrightarrow \frac{\pi^3}{24n^2} \le \frac{1}{10^5} \Leftrightarrow n^2 \ge \frac{10^5\pi^3}{24} \Rightarrow n \ge 359.4$. Take $n = 360$ for M_n.

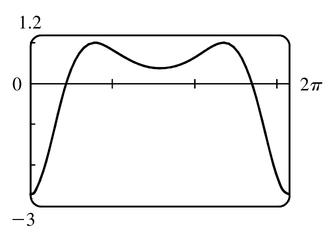
$|E_S| \le 0.00001 \Leftrightarrow \frac{\pi^5}{180n^4} \le \frac{1}{10^5} \Leftrightarrow n^4 \ge \frac{10^5\pi^5}{180} \Rightarrow n \ge 20.3$.

Take $n = 22$ for S_n (since n must be even).

22. From Example 7(b), we take $K = 76e$ to get $|E_S| \le \frac{76e(1)^5}{180n^4} \le 0.00001 \Rightarrow n^4 \ge \frac{76e}{180(0.00001)} \Rightarrow n \ge 18.4$.

Take $n = 20$ (since n must be even).

23. (a) Using a CAS, we differentiate $f(x) = e^{\cos x}$ twice, and find that $f''(x) = e^{\cos x}(\sin^2 x - \cos x)$. From the graph, we see that the maximum value of $|f''(x)|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f''(0) = -e$, we can use $K = e$ or $K = 2.8$.



(b) A CAS gives $M_{10} \approx 7.954926518$. (In Maple, use `Student[Calculus1][RiemannSum]` or `Student[Calculus1][ApproximateInt]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K = e$, we get $|E_M| \le \dfrac{e(2\pi - 0)^3}{24 \cdot 10^2} \approx 0.280945995$.

With $K = 2.8$, we get $|E_M| \le \dfrac{2.8(2\pi - 0)^3}{24 \cdot 10^2} = 0.\,289391916$.

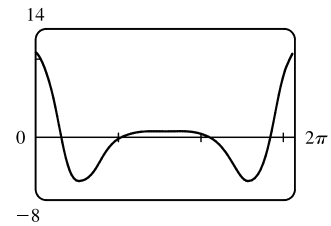
(d) A CAS gives $I \approx 7.954926521$.

(e) The actual error is only about 3×10^{-9}, much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph

$f^{(4)}(x) = e^{\cos x}(\sin^4 x - 6\sin^2 x\, \cos x + 3 - 7\sin^2 x + \cos x)$.

From the graph, we see that the maximum value of $\left|f^{(4)}(x)\right|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f^{(4)}(0) = 4e$, we can use $K = 4e$ or $K = 10.9$.



(g) A CAS gives $S_{10} \approx 7.953789422$. (In Maple, use `Student[Calculus1][ApproximateInt]`.)

(h) Using Theorem 4 with $K = 4e$, we get $|E_S| \le \dfrac{4e(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059153618$.

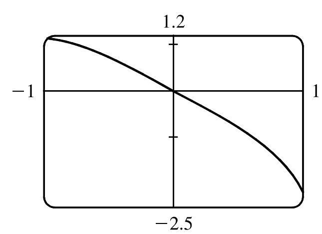
With $K = 10.9$, we get $|E_S| \le \dfrac{10.9(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059299814$.

(i) The actual error is about $7.954926521 - 7.953789422 \approx 0.00114$. This is quite a bit smaller than the estimate in part (h), though the difference is not nearly as great as it was in the case of the Midpoint Rule.

(j) To ensure that $|E_S| \le 0.0001$, we use Theorem 4: $|E_S| \le \dfrac{4e(2\pi)^5}{180 \cdot n^4} \le 0.0001 \quad\Rightarrow\quad \dfrac{4e(2\pi)^5}{180 \cdot 0.0001} \le n^4 \quad\Rightarrow$ $n^4 \ge 5{,}915{,}362 \quad\Leftrightarrow\quad n \ge 49.3$. So we must take $n \ge 50$ to ensure that $|I - S_n| \le 0.0001$. ($K = 10.9$ leads to the same value of n.)

24. (a) Using the CAS, we differentiate $f(x) = \sqrt{4 - x^3}$ twice, and find that $f''(x) = -\dfrac{9x^4}{4(4 - x^3)^{3/2}} - \dfrac{3x}{(4 - x^3)^{1/2}}$.

From the graph, we see that $|f''(x)| < 2.2$ on $[-1, 1]$.



(b) A CAS gives $M_{10} \approx 3.995804152$. (In Maple, use `Student[Calculus1][RiemannSum]` or `Student[Calculus1][ApproximateInt]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K = 2.2$, we get $|E_M| \le \dfrac{2.2\,[1 - (-1)]^3}{24 \cdot 10^2} \approx 0.00733$.

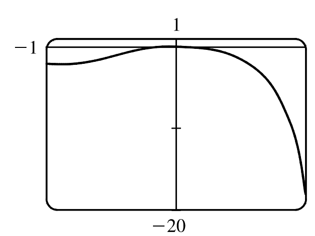
(d) A CAS gives $I \approx 3.995487677$.

(e) The actual error is about -0.0003165, much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph

$f^{(4)}(x) = \dfrac{9}{16}\dfrac{x^2(x^6 - 224x^3 - 1280)}{(4 - x^3)^{7/2}}$.

From the graph, we see that $\left|f^{(4)}(x)\right| < 18.1$ on $[-1, 1]$.



(g) A CAS gives $S_{10} \approx 3.995449790$. (In Maple, use `Student[Calculus1][ApproximateInt]`.)

(h) Using Theorem 4 with $K = 18.1$, we get $|E_S| \le \dfrac{18.1\,[1-(-1)]^5}{180 \cdot 10^4} \approx 0.000322$.

(i) The actual error is about $3.995487677 - 3.995449790 \approx 0.0000379$. This is quite a bit smaller than the estimate in part (h).

(j) To ensure that $|E_S| \le 0.0001$, we use Theorem 4: $|E_S| \le \dfrac{18.1(2)^5}{180 \cdot n^4} \le 0.0001 \quad\Rightarrow\quad \dfrac{18.1(2)^5}{180 \cdot 0.0001} \le n^4 \quad\Rightarrow$ $n^4 \ge 32{,}178 \quad \Rightarrow n \ge 13.4$. So we must take $n \ge 14$ to ensure that $|I - S_n| \le 0.0001$.

25. $I = \int_0^1 xe^x\,dx = [(x-1)e^x]_0^1$ [parts or Formula 96] $= 0 - (-1) = 1$, $f(x) = xe^x$, $\Delta x = 1/n$

$n = 5$:

$$L_5 = \tfrac{1}{5}[f(0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)] \approx 0.742943$$
$$R_5 = \tfrac{1}{5}[f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)] \approx 1.286599$$
$$T_5 = \tfrac{1}{5 \cdot 2}[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)] \approx 1.014771$$
$$M_5 = \tfrac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \approx 0.992621$$

$$E_L = I - L_5 \approx 1 - 0.742943 = 0.257057$$
$$E_R \approx 1 - 1.286599 = -0.286599$$
$$E_T \approx 1 - 1.014771 = -0.014771$$
$$E_M \approx 1 - 0.992621 = 0.007379$$

$n = 10$:

$$L_{10} = \tfrac{1}{10}[f(0) + f(0.1) + f(0.2) + \cdots + f(0.9)] \approx 0.867782$$
$$R_{10} = \tfrac{1}{10}[f(0.1) + f(0.2) + \cdots + f(0.9) + f(1)] \approx 1.139610$$
$$T_{10} = \tfrac{1}{10 \cdot 2}\{f(0) + 2[f(0.1) + f(0.2) + \cdots + f(0.9)] + f(1)\} \approx 1.003696$$
$$M_{10} = \tfrac{1}{10}[f(0.05) + f(0.15) + \cdots + f(0.85) + f(0.95)] \approx 0.998152$$

$$E_L = I - L_{10} \approx 1 - 0.867782 = 0.132218$$
$$E_R \approx 1 - 1.139610 = -0.139610$$
$$E_T \approx 1 - 1.003696 = -0.003696$$
$$E_M \approx 1 - 0.998152 = 0.001848$$

$n = 20$:

$$L_{20} = \tfrac{1}{20}[f(0) + f(0.05) + f(0.10) + \cdots + f(0.95)] \approx 0.932967$$
$$R_{20} = \tfrac{1}{20}[f(0.05) + f(0.10) + \cdots + f(0.95) + f(1)] \approx 1.068881$$
$$T_{20} = \tfrac{1}{20 \cdot 2}\{f(0) + 2[f(0.05) + f(0.10) + \cdots + f(0.95)] + f(1)\} \approx 1.000924$$
$$M_{20} = \tfrac{1}{20}[f(0.025) + f(0.075) + f(0.125) + \cdots + f(0.975)] \approx 0.999538$$

$$E_L = I - L_{20} \approx 1 - 0.932967 = 0.067033$$
$$E_R \approx 1 - 1.068881 = -0.068881$$
$$E_T \approx 1 - 1.000924 = -0.000924$$
$$E_M \approx 1 - 0.999538 = 0.000462$$

n	L_n	R_n	T_n	M_n
5	0.742943	1.286599	1.014771	0.992621
10	0.867782	1.139610	1.003696	0.998152
20	0.932967	1.068881	1.000924	0.999538

n	E_L	E_R	E_T	E_M
5	0.257057	−0.286599	−0.014771	0.007379
10	0.132218	−0.139610	−0.003696	0.001848
20	0.067033	−0.068881	−0.000924	0.000462

Observations:

1. E_L and E_R are always opposite in sign, as are E_T and E_M.
2. As n is doubled, E_L and E_R are decreased by about a factor of 2, and E_T and E_M are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of n increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

26. $I = \int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2} - (-1) = \frac{1}{2}, f(x) = \frac{1}{x^2}, \Delta x = \frac{1}{n}$

$n = 5$: $L_5 = \frac{1}{5}[f(1) + f(1.2) + f(1.4) + f(1.6) + f(1.8)] \approx 0.580783$

$R_5 = \frac{1}{5}[f(1.2) + f(1.4) + f(1.6) + f(1.8) + f(2)] \approx 0.430783$

$T_5 = \frac{1}{5 \cdot 2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)] \approx 0.505783$

$M_5 = \frac{1}{5}[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)] \approx 0.497127$

$E_L = I - L_5 \approx \frac{1}{2} - 0.580783 = -0.080783$

$E_R \approx \frac{1}{2} - 0.430783 = 0.069217$

$E_T \approx \frac{1}{2} - 0.505783 = -0.005783$

$E_M \approx \frac{1}{2} - 0.497127 = 0.002873$

$n = 10$: $L_{10} = \frac{1}{10}[f(1) + f(1.1) + f(1.2) + \cdots + f(1.9)] \approx 0.538955$

$R_{10} = \frac{1}{10}[f(1.1) + f(1.2) + \cdots + f(1.9) + f(2)] \approx 0.463955$

$T_{10} = \frac{1}{10 \cdot 2}\{f(1) + 2[f(1.1) + f(1.2) + \cdots + f(1.9)] + f(2)\} \approx 0.501455$

$M_{10} = \frac{1}{10}[f(1.05) + f(1.15) + \cdots + f(1.85) + f(1.95)] \approx 0.499274$

$E_L = I - L_{10} \approx \frac{1}{2} - 0.538955 = -0.038955$

$E_R \approx \frac{1}{2} - 0.463955 = 0.036049$

$E_T \approx \frac{1}{2} - 0.501455 = -0.001455$

$E_M \approx \frac{1}{2} - 0.499274 = 0.000726$

$n = 20$: $L_{20} = \frac{1}{20}[f(1) + f(1.05) + f(1.10) + \cdots + f(1.95)] \approx 0.519114$

$R_{20} = \frac{1}{20}[f(1.05) + f(1.10) + \cdots + f(1.95) + f(2)] \approx 0.481614$

$T_{20} = \frac{1}{20 \cdot 2}\{f(1) + 2[f(1.05) + f(1.10) + \cdots + f(1.95)] + f(2)\} \approx 0.500364$

$M_{20} = \frac{1}{20}[f(1.025) + f(1.075) + f(1.125) + \cdots + f(1.975)] \approx 0.499818$

$$\begin{aligned} E_L &= I - L_{20} \approx \tfrac{1}{2} - 0.519114 = -0.019114 \\ E_R &\approx \tfrac{1}{2} - 0.481614 = 0.018386 \\ E_T &\approx \tfrac{1}{2} - 0.500364 = -0.000364 \\ E_M &\approx \tfrac{1}{2} - 0.499818 = 0.000182 \end{aligned}$$

n	L_n	R_n	T_n	M_n
5	0.580783	0.430783	0.505783	0.497127
10	0.538955	0.463955	0.501455	0.499274
20	0.519114	0.481614	0.500364	0.499818

n	E_L	E_R	E_T	E_M
5	−0.080783	0.069217	−0.005783	0.002873
10	−0.038955	0.036049	−0.001455	0.000726
20	−0.019114	0.018386	−0.000364	0.000182

Observations:

1. E_L and E_R are always opposite in sign, as are E_T and E_M.
2. As n is doubled, E_L and E_R are decreased by about a factor of 2, and E_T and E_M are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of n increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

27. $I = \int_0^2 x^4\,dx = \left[\frac{1}{5}x^5\right]_0^2 = \frac{32}{5} - 0 = 6.4$, $f(x) = x^4$, $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

$n = 6$:

$$\begin{aligned} T_6 &= \tfrac{2}{6 \cdot 2}\left\{f(0) + 2\left[f\left(\tfrac{1}{3}\right) + f\left(\tfrac{2}{3}\right) + f\left(\tfrac{3}{3}\right) + f\left(\tfrac{4}{3}\right) + f\left(\tfrac{5}{3}\right)\right] + f(2)\right\} \approx 6.695473 \\ M_6 &= \tfrac{2}{6}\left[f\left(\tfrac{1}{6}\right) + f\left(\tfrac{3}{6}\right) + f\left(\tfrac{5}{6}\right) + f\left(\tfrac{7}{6}\right) + f\left(\tfrac{9}{6}\right) + f\left(\tfrac{11}{6}\right)\right] \approx 6.252572 \\ S_6 &= \tfrac{2}{6 \cdot 3}\left[f(0) + 4f\left(\tfrac{1}{3}\right) + 2f\left(\tfrac{2}{3}\right) + 4f\left(\tfrac{3}{3}\right) + 2f\left(\tfrac{4}{3}\right) + 4f\left(\tfrac{5}{3}\right) + f(2)\right] \approx 6.403292 \\ E_T &= I - T_6 \approx 6.4 - 6.695473 = -0.295473 \\ E_M &\approx 6.4 - 6.252572 = 0.147428 \\ E_S &\approx 6.4 - 6.403292 = -0.003292 \end{aligned}$$

$n = 12$:

$$\begin{aligned} T_{12} &= \tfrac{2}{12 \cdot 2}\left\{f(0) + 2\left[f\left(\tfrac{1}{6}\right) + f\left(\tfrac{2}{6}\right) + f\left(\tfrac{3}{6}\right) + \cdots + f\left(\tfrac{11}{6}\right)\right] + f(2)\right\} \approx 6.474023 \\ M_6 &= \tfrac{2}{12}\left[f\left(\tfrac{1}{12}\right) + f\left(\tfrac{3}{12}\right) + f\left(\tfrac{5}{12}\right) + \cdots + f\left(\tfrac{23}{12}\right)\right] \approx 6.363008 \\ S_6 &= \tfrac{2}{12 \cdot 3}\left[f(0) + 4f\left(\tfrac{1}{6}\right) + 2f\left(\tfrac{2}{6}\right) + 4f\left(\tfrac{3}{6}\right) + 2f\left(\tfrac{4}{6}\right) + \cdots + 4f\left(\tfrac{11}{6}\right) + f(2)\right] \approx 6.400206 \\ E_T &= I - T_{12} \approx 6.4 - 6.474023 = -0.074023 \\ E_M &\approx 6.4 - 6.363008 = 0.036992 \\ E_S &\approx 6.4 - 6.400206 = -0.000206 \end{aligned}$$

n	T_n	M_n	S_n
6	6.695473	6.252572	6.403292
12	6.474023	6.363008	6.400206

n	E_T	E_M	E_S
6	−0.295473	0.147428	−0.003292
12	−0.074023	0.036992	−0.000206

Observations:

1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.
2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and E_S seems to decrease by a factor of about 16 as n is doubled.

28. $I = \int_1^4 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_1^4 = 4 - 2 = 2$, $f(x) = \frac{1}{\sqrt{x}}$, $\Delta x = \frac{4-1}{n} = \frac{3}{n}$

$n = 6$: $T_6 = \frac{3}{6\cdot 2}\{f(1) + 2[f(\frac{3}{2}) + f(\frac{4}{2}) + f(\frac{5}{2}) + f(\frac{6}{2}) + f(\frac{7}{2})] + f(4)\} \approx 2.008966$

$M_6 = \frac{3}{6}[f(\frac{5}{4}) + f(\frac{7}{4}) + f(\frac{9}{4}) + f(\frac{11}{4}) + f(\frac{13}{4}) + f(\frac{15}{4})] \approx 1.995572$

$S_6 = \frac{3}{6\cdot 3}[f(1) + 4f(\frac{3}{2}) + 2f(\frac{4}{2}) + 4f(\frac{5}{2}) + 2f(\frac{6}{2}) + 4f(\frac{7}{2}) + f(4)] \approx 2.000469$

$E_T = I - T_6 \approx 2 - 2.008966 = -0.008966$,

$E_M \approx 2 - 1.995572 = 0.004428$,

$E_S \approx 2 - 2.000469 = -0.000469$

$n = 12$: $T_{12} = \frac{3}{12\cdot 2}\{f(1) + 2[f(\frac{5}{4}) + f(\frac{6}{4}) + f(\frac{7}{4}) + \cdots + f(\frac{15}{4})] + f(4)\} \approx 2.002269$

$M_{12} = \frac{3}{12}[f(\frac{9}{8}) + f(\frac{11}{8}) + f(\frac{13}{8}) + \cdots + f(\frac{31}{8})] \approx 1.998869$

$S_{12} = \frac{3}{12\cdot 3}[f(1) + 4f(\frac{5}{4}) + 2f(\frac{6}{4}) + 4f(\frac{7}{4}) + 2f(\frac{8}{4}) + \cdots + 4f(\frac{15}{4}) + f(4)] \approx 2.000036$

$E_T = I - T_{12} \approx 2 - 2.002269 = -0.002269$

$E_M \approx 2 - 1.998869 = 0.001131$

$E_S \approx 2 - 2.000036 = -0.000036$

n	T_n	M_n	S_n
6	2.008966	1.995572	2.000469
12	2.002269	1.998869	2.000036

n	E_T	E_M	E_S
6	−0.008966	0.004428	−0.000469
12	−0.002269	0.001131	−0.000036

Observations:

1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.
2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and E_S seems to decrease by a factor of about 16 as n is doubled.

29. (a) $\Delta x = (b-a)/n = (6-0)/6 = 1$

$T_6 = \frac{1}{2}[f(0) + 2f(1) + 2f(2) + 2f(3) + 2f(4) + 2f(5) + f(6)]$

$\approx \frac{1}{2}[2 + 2(1) + 2(3) + 2(5) + 2(4) + 2(3) + 4] = \frac{1}{2}(38) = 19$

(b) $M_6 = 1[f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)] \approx 1.3 + 1.5 + 4.6 + 4.7 + 3.3 + 3.2 = 18.6$

(c) $S_6 = \frac{1}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + 2f(4) + 4f(5) + f(6)]$

$\approx \frac{1}{3}[2 + 4(1) + 2(3) + 4(5) + 2(4) + 4(3) + 4] = \frac{1}{3}(56) = 18.\overline{6}$

30. If x = distance from left end of pool and $w = w(x)$ = width at x, then Simpson's Rule with $n = 8$ and $\Delta x = 2$ gives

Area $= \int_0^{16} w\,dx \approx \frac{2}{3}[0 + 4(6.2) + 2(7.2) + 4(6.8) + 2(5.6) + 4(5.0) + 2(4.8) + 4(4.8) + 0] \approx 84\text{ m}^2$.

31. (a) $\int_1^5 f(x)\,dx \approx M_4 = \frac{5-1}{4}[f(1.5) + f(2.5) + f(3.5) + f(4.5)] = 1(2.9 + 3.6 + 4.0 + 3.9) = 14.4$

(b) $-2 \le f''(x) \le 3 \;\Rightarrow\; |f''(x)| \le 3 \;\Rightarrow\; K = 3$, since $|f''(x)| \le K$. The error estimate for the Midpoint Rule is

$|E_M| \le \frac{K(b-a)^3}{24n^2} = \frac{3(5-1)^3}{24(4)^2} = \frac{1}{2}$.

32. (a) $\int_0^{1.6} g(x)\,dx \approx S_8 = \frac{1.6-0}{8\cdot 3}[g(0)+4g(0.2)+2g(0.4)+4g(0.6)+2g(0.8)+4g(1.0)+2g(1.2)+4g(1.4)+g(1.6)]$

$= \frac{1}{15}[12.1+4(11.6)+2(11.3)+4(11.1)+2(11.7)+4(12.2)+2(12.6)+4(13.0)+13.2]$

$= \frac{1}{15}(288.1) = \frac{2881}{150} \approx 19.2$

(b) $-5 \le g^{(4)}(x) \le 2 \quad\Rightarrow\quad \left|g^{(4)}(x)\right| \le 5 \quad\Rightarrow\quad K=5$, since $\left|g^{(4)}(x)\right| \le K$. The error estimate for Simpson's Rule is

$$|E_S| \le \frac{K(b-a)^5}{180n^4} = \frac{5(1.6-0)^5}{180(8)^4} = \frac{2}{28{,}125} = 7.\overline{1}\times 10^{-5}.$$

33. We use Simpson's Rule with $n=12$ and $\Delta t = \frac{24-0}{12} = 2$.

$$S_{12} = \tfrac{2}{3}[T(0)+4T(2)+2T(4)+4T(6)+2T(8)+4T(10)+2T(12) + 4T(14)+2T(16)+4T(18)+2T(20)+4T(22)+T(24)]$$

$$\approx \tfrac{2}{3}[66.6+4(65.4)+2(64.4)+4(61.7)+2(67.3)+4(72.1)+2(74.9) + 4(77.4)+2(79.1)+4(75.4)+2(75.6)+4(71.4)+67.5] = \tfrac{2}{3}(2550.3) = 1700.2.$$

Thus, $\int_0^{24} T(t)\,dt \approx S_{12}$ and $T_{\text{ave}} = \frac{1}{24-0}\int_0^{24} T(t)\,dt \approx 70.84°\text{F}$.

34. We use Simpson's Rule with $n=10$ and $\Delta x = \frac{1}{2}$:

$$\text{distance} = \int_0^5 v(t)\,dt \approx S_{10} = \tfrac{1}{2\cdot 3}[f(0)+4f(0.5)+2f(1)+\cdots+4f(4.5)+f(5)]$$

$$= \tfrac{1}{6}[0+4(4.67)+2(7.34)+4(8.86)+2(9.73)+4(10.22) + 2(10.51)+4(10.67)+2(10.76)+4(10.81)+10.81]$$

$$= \tfrac{1}{6}(268.41) = 44.735 \text{ m}$$

35. By the Net Change Theorem, the increase in velocity is equal to $\int_0^6 a(t)\,dt$. We use Simpson's Rule with $n=6$ and $\Delta t = (6-0)/6 = 1$ to estimate this integral:

$$\int_0^6 a(t)\,dt \approx S_6 = \tfrac{1}{3}[a(0)+4a(1)+2a(2)+4a(3)+2a(4)+4a(5)+a(6)]$$

$$\approx \tfrac{1}{3}[0+4(0.5)+2(4.1)+4(9.8)+2(12.9)+4(9.5)+0] = \tfrac{1}{3}(113.2) = 37.7\overline{3}\text{ ft/s}$$

36. By the Net Change Theorem, the total amount of water that leaked out during the first six hours is equal to $\int_0^6 r(t)\,dt$. We use Simpson's Rule with $n=6$ and $\Delta t = \frac{6-0}{6} = 1$ to estimate this integral:

$$\int_0^6 r(t)\,dt \approx S_6 = \tfrac{1}{3}[r(0)+4r(1)+2r(2)+4r(3)+2r(4)+4r(5)+r(6)]$$

$$\approx \tfrac{1}{3}[4+4(3)+2(2.4)+4(1.9)+2(1.4)+4(1.1)+1] = \tfrac{1}{3}(36.6) = 12.2 \text{ liters}$$

37. By the Net Change Theorem, the energy used is equal to $\int_0^6 P(t)\,dt$. We use Simpson's Rule with $n=12$ and $\Delta t = \frac{6-0}{12} = \frac{1}{2}$ to estimate this integral:

$$\int_0^6 P(t)\,dt \approx S_{12} = \tfrac{1/2}{3}[P(0)+4P(0.5)+2P(1)+4P(1.5)+2P(2)+4P(2.5)+2P(3) + 4P(3.5)+2P(4)+4P(4.5)+2P(5)+4P(5.5)+P(6)]$$

$$= \tfrac{1}{6}[1814+4(1735)+2(1686)+4(1646)+2(1637)+4(1609)+2(1604) + 4(1611)+2(1621)+4(1666)+2(1745)+4(1886)+2052]$$

$$= \tfrac{1}{6}(61{,}064) = 10{,}177.\overline{3}\text{ megawatt-hours}$$

38. By the Net Change Theorem, the total amount of data transmitted is equal to $\int_0^8 D(t)\,dt \times 3600$ [since $D(t)$ is measured in megabits per second and t is in hours]. We use Simpson's Rule with $n = 8$ and $\Delta t = (8-0)/8 = 1$ to estimate this integral:

$$\begin{aligned}\int_0^8 D(t)\,dt \approx S_8 &= \tfrac{1}{3}[D(0) + 4D(1) + 2D(2) + 4D(3) + 2D(4) + 4D(5) + 2D(6) + 4D(7) + D(8)] \\ &\approx \tfrac{1}{3}[0.35 + 4(0.32) + 2(0.41) + 4(0.50) + 2(0.51) + 4(0.56) + 2(0.56) + 4(0.83) + 0.88] \\ &= \tfrac{1}{3}(13.03) = 4.34\overline{3}\end{aligned}$$

Now multiply by 3600 to obtain 15,636 megabits.

39. (a) Let $y = f(x)$ denote the curve. Using disks, $V = \int_2^{10} \pi[f(x)]^2\,dx = \pi\int_2^{10} g(x)\,dx = \pi I_1$.

Now use Simpson's Rule to approximate I_1:

$$\begin{aligned}I_1 \approx S_8 &= \tfrac{10-2}{3(8)}[g(2) + 4g(3) + 2g(4) + 4g(5) + 2g(6) + 4g(7) + g(8)] \\ &\approx \tfrac{1}{3}[0^2 + 4(1.5)^2 + 2(1.9)^2 + 4(2.2)^2 + 2(3.0)^2 + 4(3.8)^2 + 2(4.0)^2 + 4(3.1)^2 + 0^2] \\ &= \tfrac{1}{3}(181.78)\end{aligned}$$

Thus, $V \approx \pi \cdot \frac{1}{3}(181.78) \approx 190.4$ or 190 cubic units.

(b) Using cylindrical shells, $V = \int_2^{10} 2\pi x f(x)\,dx = 2\pi\int_2^{10} x f(x)\,dx = 2\pi I_1$.

Now use Simpson's Rule to approximate I_1:

$$\begin{aligned}I_1 \approx S_8 = \tfrac{10-2}{3(8)}&[2f(2) + 4\cdot 3f(3) + 2\cdot 4f(4) + 4\cdot 5f(5) + 2\cdot 6f(6) \\ &+ 4\cdot 7f(7) + 2\cdot 8f(8) + 4\cdot 9f(9) + 10f(10)] \\ \approx \tfrac{1}{3}&[2(0) + 12(1.5) + 8(1.9) + 20(2.2) + 12(3.0) + 28(3.8) + 16(4.0) + 36(3.1) + 10(0)] \\ = \tfrac{1}{3}&(395.2)\end{aligned}$$

Thus, $V \approx 2\pi \cdot \frac{1}{3}(395.2) \approx 827.7$ or 828 cubic units.

40. $$\begin{aligned}\text{Work} = \int_0^{18} f(x)\,dx \approx S_6 &= \tfrac{18-0}{6\cdot 3}[f(0) + 4f(3) + 2f(6) + 4f(9) + 2f(12) + 4f(15) + f(18)] \\ &= 1\cdot[9.8 + 4(9.1) + 2(8.5) + 4(8.0) + 2(7.7) + 4(7.5) + 7.4] = 148 \text{ joules}\end{aligned}$$

41. The curve is $y = f(x) = 1/(1 + e^{-x})$. Using disks, $V = \int_0^{10} \pi[f(x)]^2\,dx = \pi\int_0^{10} g(x)\,dx = \pi I_1$. Now use Simpson's Rule to approximate I_1:

$$\begin{aligned}I_1 \approx S_{10} &= \tfrac{10-0}{10\cdot 3}[g(0) + 4g(1) + 2g(2) + 4g(3) + 2g(4) + 4g(5) + 2g(6) + 4g(7) + 2g(8) + 4g(9) + g(10)] \\ &\approx 8.80825\end{aligned}$$

Thus, $V \approx \pi I_1 \approx 27.7$ or 28 cubic units.

42. Using Simpson's Rule with $n = 10$, $\Delta x = \frac{\pi/2}{10}$, $L = 1$, $\theta_0 = \frac{42\pi}{180}$ radians, $g = 9.8\text{ m/s}^2$, $k^2 = \sin^2\left(\frac{1}{2}\theta_0\right)$, and $f(x) = 1/\sqrt{1 - k^2\sin^2 x}$, we get

$$\begin{aligned}T &= 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2 x}} \approx 4\sqrt{\frac{L}{g}}\,S_{10} \\ &= 4\sqrt{\tfrac{1}{9.8}}\left(\tfrac{\pi/2}{10\cdot 3}\right)\left[f(0) + 4f\left(\tfrac{\pi}{20}\right) + 2f\left(\tfrac{2\pi}{20}\right) + \cdots + 4f\left(\tfrac{9\pi}{20}\right) + f\left(\tfrac{\pi}{2}\right)\right] \approx 2.07665\end{aligned}$$

43. $I(\theta) = \dfrac{N^2 \sin^2 k}{k^2}$, where $k = \dfrac{\pi N d \sin\theta}{\lambda}$, $N = 10{,}000$, $d = 10^{-4}$, and $\lambda = 632.8 \times 10^{-9}$. So $I(\theta) = \dfrac{(10^4)^2 \sin^2 k}{k^2}$,

where $k = \dfrac{\pi(10^4)(10^{-4})\,\sin\theta}{632.8 \times 10^{-9}}$. Now $n = 10$ and $\Delta\theta = \dfrac{10^{-6} - (-10^{-6})}{10} = 2 \times 10^{-7}$, so

$M_{10} = 2 \times 10^{-7}[I(-0.0000009) + I(-0.0000007) + \cdots + I(0.0000009)] \approx 59.4$.

44. $f(x) = \cos(\pi x)$, $\Delta x = \frac{20-0}{10} = 2 \;\;\Rightarrow$

$$T_{10} = \tfrac{2}{2}\{f(0) + 2[f(2) + f(4) + \cdots + f(18)] + f(20)\} = 1[\cos 0 + 2(\cos 2\pi + \cos 4\pi + \cdots + \cos 18\pi) + \cos 20\pi]$$
$$= 1 + 2(1+1+1+1+1+1+1+1+1) + 1 = 20$$

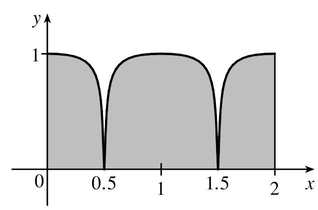
The actual value is $\int_0^{20} \cos(\pi x)\,dx = \frac{1}{\pi}\big[\sin \pi x\big]_0^{20} = \frac{1}{\pi}(\sin 20\pi - \sin 0) = 0$. The discrepancy is due to the fact that the function is sampled only at points of the form $2n$, where its value is $f(2n) = \cos(2n\pi) = 1$.

45. Consider the function f whose graph is shown. The area $\int_0^2 f(x)\,dx$ is close to 2. The Trapezoidal Rule gives

$T_2 = \frac{2-0}{2\cdot 2}\,[f(0) + 2f(1) + f(2)] = \frac{1}{2}\,[1 + 2\cdot 1 + 1] = 2$.

The Midpoint Rule gives $M_2 = \frac{2-0}{2}\,[f(0.5) + f(1.5)] = 1[0+0] = 0$,

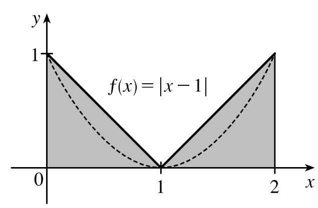
so the Trapezoidal Rule is more accurate.



46. Consider the function $f(x) = |x - 1|$, $0 \le x \le 2$. The area $\int_0^2 f(x)\,dx$ is exactly 1. So is the right endpoint approximation:

$R_2 = f(1)\,\Delta x + f(2)\,\Delta x = 0\cdot 1 + 1\cdot 1 = 1$. But Simpson's Rule approximates f with the parabola $y = (x-1)^2$, shown dashed, and

$S_2 = \dfrac{\Delta x}{3}\,[f(0) + 4f(1) + f(2)] = \dfrac{1}{3}\,[1 + 4\cdot 0 + 1] = \dfrac{2}{3}$.



47. Since the Trapezoidal and Midpoint approximations on the interval $[a,b]$ are the sums of the Trapezoidal and Midpoint approximations on the subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, we can focus our attention on one such interval. The condition $f''(x) < 0$ for $a \le x \le b$ means that the graph of f is concave down as in Figure 5. In that figure, T_n is the area of the trapezoid $AQRD$, $\int_a^b f(x)\,dx$ is the area of the region $AQPRD$, and M_n is the area of the trapezoid $ABCD$, so $T_n < \int_a^b f(x)\,dx < M_n$. In general, the condition $f'' < 0$ implies that the graph of f on $[a,b]$ lies above the chord joining the points $(a, f(a))$ and $(b, f(b))$. Thus, $\int_a^b f(x)\,dx > T_n$. Since M_n is the area under a tangent to the graph, and since $f'' < 0$ implies that the tangent lies above the graph, we also have $M_n > \int_a^b f(x)\,dx$. Thus, $T_n < \int_a^b f(x)\,dx < M_n$.

48. Let f be a polynomial of degree ≤ 3; say $f(x) = Ax^3 + Bx^2 + Cx + D$. It will suffice to show that Simpson's estimate is exact when there are two subintervals ($n = 2$), because for a larger even number of subintervals the sum of exact estimates is exact. As in the derivation of Simpson's Rule, we can assume that $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. Then Simpson's approximation is

$$\int_{-h}^{h} f(x)\,dx \approx \tfrac{1}{3}h[f(-h)+4f(0)+f(h)] = \tfrac{1}{3}h\big[\big(-Ah^3+Bh^2-Ch+D\big)+4D+\big(Ah^3+Bh^2+Ch+D\big)\big]$$
$$= \tfrac{1}{3}h[2Bh^2+6D] = \tfrac{2}{3}Bh^3+2Dh$$

The exact value of the integral is

$$\int_{-h}^{h}(Ax^3+Bx^2+Cx+D)\,dx = 2\int_0^h (Bx^2+D)\,dx \qquad \text{[by Theorem 5.5.7(a) and (b)]}$$
$$= 2\big[\tfrac{1}{3}Bx^3+Dx\big]_0^h = \tfrac{2}{3}Bh^3+2Dh$$

Thus, Simpson's Rule is exact.

49. $T_n = \frac{1}{2}\,\Delta x\,[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)]$ and

$M_n = \Delta x\,[f(\overline{x}_1)+f(\overline{x}_2)+\cdots+f(\overline{x}_{n-1})+f(\overline{x}_n)]$, where $\overline{x}_i = \frac{1}{2}(x_{i-1}+x_i)$. Now

$T_{2n} = \frac{1}{2}\big(\frac{1}{2}\Delta x\big)[f(x_0)+2f(\overline{x}_1)+2f(x_1)+2f(\overline{x}_2)+2f(x_2)+\cdots+2f(\overline{x}_{n-1})+2f(x_{n-1})+2f(\overline{x}_n)+f(x_n)]$ so

$$\tfrac{1}{2}(T_n+M_n) = \tfrac{1}{2}T_n+\tfrac{1}{2}M_n$$
$$= \tfrac{1}{4}\Delta x[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)]+\tfrac{1}{4}\Delta x[2f(\overline{x}_1)+2f(\overline{x}_2)+\cdots+2f(\overline{x}_{n-1})+2f(\overline{x}_n)]$$
$$= T_{2n}$$

50. $T_n = \dfrac{\Delta x}{2}\left[f(x_0)+2\sum\limits_{i=1}^{n-1} f(x_i)+f(x_n)\right]$ and $M_n = \Delta x\sum\limits_{i=1}^{n} f\!\left(x_i-\dfrac{\Delta x}{2}\right)$, so

$$\tfrac{1}{3}T_n+\tfrac{2}{3}M_n = \tfrac{1}{3}(T_n+2M_n) = \frac{\Delta x}{3\cdot 2}\left[f(x_0)+2\sum_{i=1}^{n-1} f(x_i)+f(x_n)+4\sum_{i=1}^{n} f\!\left(x_i-\frac{\Delta x}{2}\right)\right]$$

where $\Delta x = \dfrac{b-a}{n}$. Let $\delta x = \dfrac{b-a}{2n}$. Then $\Delta x = 2\delta x$, so

$$\tfrac{1}{3}T_n+\tfrac{2}{3}M_n = \frac{\delta x}{3}\left[f(x_0)+2\sum_{i=1}^{n-1} f(x_i)+f(x_n)+4\sum_{i=1}^{n} f(x_i-\delta x)\right]$$
$$= \tfrac{1}{3}\delta x[f(x_0)+4f(x_1-\delta x)+2f(x_1)+4f(x_2-\delta x)$$
$$+2f(x_2)+\cdots+2f(x_{n-1})+4f(x_n-\delta x)+f(x_n)]$$

Since $x_0, x_1-\delta x, x_1, x_2-\delta x, x_2, \ldots, x_{n-1}, x_n-\delta x, x_n$ are the subinterval endpoints for S_{2n}, and since $\delta x = \dfrac{b-a}{2n}$ is the width of the subintervals for S_{2n}, the last expression for $\frac{1}{3}T_n+\frac{2}{3}M_n$ is the usual expression for S_{2n}. Therefore, $\frac{1}{3}T_n+\frac{2}{3}M_n = S_{2n}$.

7.8 Improper Integrals

1. (a) Since $y = \dfrac{x}{x-1}$ has an infinite discontinuity at $x=1$, $\displaystyle\int_1^2 \frac{x}{x-1}\,dx$ is a Type 2 improper integral.

(b) Since $\displaystyle\int_0^\infty \frac{1}{1+x^3}\,dx$ has an infinite interval of integration, it is an improper integral of Type 1.

(c) Since $\displaystyle\int_{-\infty}^\infty x^2e^{-x^2}\,dx$ has an infinite interval of integration, it is an improper integral of Type 1.

(d) Since $y=\cot x$ has an infinite discontinuity at $x=0$, $\int_0^{\pi/4}\cot x\,dx$ is a Type 2 improper integral.

2. (a) Since $y = \tan x$ is defined and continuous on $\left[0, \frac{\pi}{4}\right]$, $\int_0^{\pi/4} \tan x\,dx$ is proper.

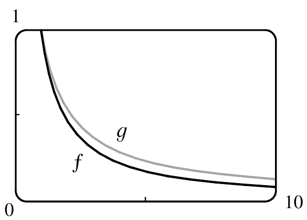
(b) Since $y = \tan x$ has an infinite discontinuity at $x = \frac{\pi}{2}$, $\int_0^{\pi} \tan x\,dx$ is a Type 2 improper integral.

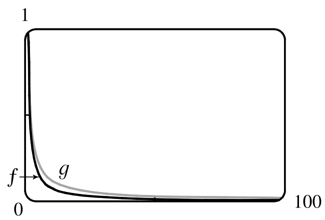
(c) Since $y = \dfrac{1}{x^2 - x - 2} = \dfrac{1}{(x-2)(x+1)}$ has an infinite discontinuity at $x = -1$, $\displaystyle\int_{-1}^{1} \frac{dx}{x^2 - x - 2}$ is a Type 2 improper integral.

(d) Since $\int_0^{\infty} e^{-x^3}\,dx$ has an infinite interval of integration, it is an improper integral of Type 1.

3. The area under the graph of $y = 1/x^3 = x^{-3}$ between $x = 1$ and $x = t$ is $A(t) = \int_1^t x^{-3}\,dx = \left[-\frac{1}{2}x^{-2}\right]_1^t = -\frac{1}{2}t^{-2} - \left(-\frac{1}{2}\right) = \frac{1}{2} - 1/\left(2t^2\right)$. So the area for $1 \le x \le 10$ is $A(10) = 0.5 - 0.005 = 0.495$, the area for $1 \le x \le 100$ is $A(100) = 0.5 - 0.00005 = 0.49995$, and the area for $1 \le x \le 1000$ is $A(1000) = 0.5 - 0.0000005 = 0.4999995$. The total area under the curve for $x \ge 1$ is $\lim\limits_{t\to\infty} A(t) = \lim\limits_{t\to\infty} \left[\frac{1}{2} - 1/(2t^2)\right] = \frac{1}{2}$.

4. (a)





(b) The area under the graph of f from $x = 1$ to $x = t$ is

$$F(t) = \int_1^t f(x)\,dx = \int_1^t x^{-1.1}\,dx = \left[-\frac{1}{0.1}x^{-0.1}\right]_1^t$$
$$= -10(t^{-0.1} - 1) = 10(1 - t^{-0.1})$$

and the area under the graph of g is

$$G(t) = \int_1^t g(x)\,dx = \int_1^t x^{-0.9}\,dx = \left[\frac{1}{0.1}x^{0.1}\right]_1^t = 10(t^{0.1} - 1).$$

t	$F(t)$	$G(t)$
10	2.06	2.59
100	3.69	5.85
10^4	6.02	15.12
10^6	7.49	29.81
10^{10}	9	90
10^{20}	9.9	990

(c) The total area under the graph of f is $\lim\limits_{t\to\infty} F(t) = \lim\limits_{t\to\infty} 10(1 - t^{-0.1}) = 10$.

The total area under the graph of g does not exist, since $\lim\limits_{t\to\infty} G(t) = \lim\limits_{t\to\infty} 10(t^{0.1} - 1) = \infty$.

5. $$\int_3^{\infty} \frac{1}{(x-2)^{3/2}}\,dx = \lim_{t\to\infty} \int_3^t (x-2)^{-3/2}\,dx = \lim_{t\to\infty} \left[-2\,(x-2)^{-1/2}\right]_3^t \qquad [u = x - 2,\, du = dx]$$
$$= \lim_{t\to\infty} \left(\frac{-2}{\sqrt{t-2}} + \frac{2}{\sqrt{1}}\right) = 0 + 2 = 2. \qquad \text{Convergent}$$

6. $$\int_0^{\infty} \frac{1}{\sqrt[4]{1+x}}\,dx = \lim_{t\to\infty} \int_0^t (1+x)^{-1/4}\,dx = \lim_{t\to\infty} \left[\tfrac{4}{3}(1+x)^{3/4}\right]_0^t \qquad [u = 1 + x,\, du = dx]$$
$$= \lim_{t\to\infty} \left[\tfrac{4}{3}(1+t)^{3/4} - \tfrac{4}{3}\right] = \infty. \qquad \text{Divergent}$$

7. $\displaystyle\int_{-\infty}^{0} \frac{1}{3-4x}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{3-4x}\,dx = \lim_{t\to-\infty}\left[-\tfrac{1}{4}\ln|3-4x|\right]_t^0 = \lim_{t\to-\infty}\left[-\tfrac{1}{4}\ln 3 + \tfrac{1}{4}\ln|3-4t|\right] = \infty.$

Divergent

8. $\displaystyle\int_1^{\infty} \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{4(2x+1)^2}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{4(2t+1)^2} + \frac{1}{36}\right] = 0 + \frac{1}{36}.$

Convergent

9. $\int_2^{\infty} e^{-5p}\,dp = \lim_{t\to\infty}\int_2^t e^{-5p}\,dp = \lim_{t\to\infty}\left[-\tfrac{1}{5}e^{-5p}\right]_2^t = \lim_{t\to\infty}\left(-\tfrac{1}{5}e^{-5t} + \tfrac{1}{5}e^{-10}\right) = 0 + \tfrac{1}{5}e^{-10} = \tfrac{1}{5}e^{-10}.$ Convergent

10. $\displaystyle\int_{-\infty}^{0} 2^r\,dr = \lim_{t\to-\infty}\int_t^0 2^r\,dr = \lim_{t\to-\infty}\left[\frac{2^r}{\ln 2}\right]_t^0 = \lim_{t\to-\infty}\left(\frac{1}{\ln 2} - \frac{2^t}{\ln 2}\right) = \frac{1}{\ln 2} - 0 = \frac{1}{\ln 2}.$ Convergent

11. $\displaystyle\int_0^{\infty} \frac{x^2}{\sqrt{1+x^3}}\,dx = \lim_{t\to\infty}\int_0^t \frac{x^2}{\sqrt{1+x^3}}\,dx = \lim_{t\to\infty}\left[\frac{2}{3}\sqrt{1+x^3}\right]_0^t = \lim_{t\to\infty}\left(\tfrac{2}{3}\sqrt{1+t^3} - \tfrac{2}{3}\right) = \infty.$ Divergent

12. $I = \int_{-\infty}^{\infty}(y^3 - 3y^2)\,dy = I_1 + I_2 = \int_{-\infty}^{0}(y^3 - 3y^2)\,dy + \int_0^{\infty}(y^3 - 3y^2)\,dy$, but

$I_1 = \lim_{t\to-\infty}\left[\tfrac{1}{4}y^4 - y^3\right]_t^0 = \lim_{t\to-\infty}(t^3 - \tfrac{1}{4}t^4) = -\infty$. Since I_1 is divergent, I is divergent,

and there is no need to evaluate I_2. Divergent

13. $\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_0^{\infty} xe^{-x^2}\,dx.$

$\int_{-\infty}^{0} xe^{-x^2}\,dx = \lim_{t\to-\infty}\left(-\tfrac{1}{2}\right)\left[e^{-x^2}\right]_t^0 = \lim_{t\to-\infty}\left(-\tfrac{1}{2}\right)\left(1 - e^{-t^2}\right) = -\tfrac{1}{2}\cdot 1 = -\tfrac{1}{2}$, and

$\int_0^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty}\left(-\tfrac{1}{2}\right)\left[e^{-x^2}\right]_0^t = \lim_{t\to\infty}\left(-\tfrac{1}{2}\right)\left(e^{-t^2} - 1\right) = -\tfrac{1}{2}\cdot(-1) = \tfrac{1}{2}.$

Therefore, $\int_{-\infty}^{\infty} xe^{-x^2}\,dx = -\tfrac{1}{2} + \tfrac{1}{2} = 0.$ Convergent

14. $\displaystyle\int_1^{\infty} \frac{e^{-1/x}}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{e^{-1/x}}{x^2}\,dx = \lim_{t\to\infty}\left[e^{-1/x}\right]_1^t = \lim_{t\to\infty}(e^{-1/t} - e^{-1}) = 1 - \frac{1}{e}.$ Convergent

15. $\int_0^{\infty}\sin^2\alpha\,d\alpha = \lim_{t\to\infty}\int_0^t \tfrac{1}{2}(1 - \cos 2\alpha)\,d\alpha = \lim_{t\to\infty}\left[\tfrac{1}{2}\left(\alpha - \tfrac{1}{2}\sin 2\alpha\right)\right]_0^t = \lim_{t\to\infty}\left[\tfrac{1}{2}\left(t - \tfrac{1}{2}\sin 2t\right) - 0\right] = \infty.$

Divergent

16. $\int_0^{\infty}\sin\theta\, e^{\cos\theta}\,d\theta = \lim_{t\to\infty}\int_0^t \sin\theta\, e^{\cos\theta}\,d\theta = \lim_{t\to\infty}\left[-e^{\cos\theta}\right]_0^t = \lim_{t\to\infty}(-e^{\cos t} + e)$

This limit does not exist since $\cos t$ oscillates in value between -1 and 1, so $e^{\cos t}$ oscillates in value between e^{-1} and e^1. Divergent

17. $\displaystyle\int_1^{\infty} \frac{1}{x^2+x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x(x+1)}\,dx = \lim_{t\to\infty}\int_1^t\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$ [partial fractions]

$$= \lim_{t\to\infty}\left[\ln|x| - \ln|x+1|\right]_1^t = \lim_{t\to\infty}\left[\ln\left|\frac{x}{x+1}\right|\right]_1^t = \lim_{t\to\infty}\left(\ln\frac{t}{t+1} - \ln\frac{1}{2}\right) = 0 - \ln\frac{1}{2} = \ln 2.$$

Convergent

18. $\displaystyle\int_2^\infty \frac{dv}{v^2+2v-3} = \lim_{t\to\infty}\int_2^t \frac{dv}{(v+3)(v-1)} = \lim_{t\to\infty}\int_2^t\left(\frac{-\frac14}{v+3}+\frac{\frac14}{v-1}\right)dv = \lim_{t\to\infty}\left[-\tfrac14\ln|v+3|+\tfrac14\ln|v-1|\right]_2^t$

$\displaystyle = \frac14\lim_{t\to\infty}\left[\ln\frac{v-1}{v+3}\right]_2^t = \frac14\lim_{t\to\infty}\left(\ln\frac{t-1}{t+3}-\ln\frac15\right) = \frac14(0+\ln 5) = \frac14\ln 5.$ Convergent

19. $\int_{-\infty}^0 ze^{2z}\,dz = \lim\limits_{t\to-\infty}\int_t^0 ze^{2z}\,dz = \lim\limits_{t\to-\infty}\left[\tfrac12 ze^{2z}-\tfrac14 e^{2z}\right]_t^0$ $\left[\begin{array}{c}\text{integration by parts with}\\ u=z,\ dv=e^{2z}\,dz\end{array}\right]$

$= \lim\limits_{t\to-\infty}\left[\left(0-\tfrac14\right)-\left(\tfrac12 te^{2t}-\tfrac14 e^{2t}\right)\right] = -\tfrac14-0+0$ [by l'Hospital's Rule] $= -\tfrac14.$ Convergent

20. $\int_2^\infty ye^{-3y}\,dy = \lim\limits_{t\to\infty}\int_2^t ye^{-3y}\,dy = \lim\limits_{t\to\infty}\left[-\tfrac13 ye^{-3y}-\tfrac19 e^{-3y}\right]_2^t$ $\left[\begin{array}{c}\text{integration by parts with}\\ u=y,\ dv=e^{-3y}\,dy\end{array}\right]$

$= \lim\limits_{t\to\infty}\left[\left(-\tfrac13 te^{-3t}-\tfrac19 e^{-3t}\right)-\left(-\tfrac23 e^{-6}-\tfrac19 e^{-6}\right)\right] = 0-0+\tfrac79 e^{-6}$ [by l'Hospital's Rule] $= \tfrac79 e^{-6}.$

Convergent

21. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t$ $\left[\begin{array}{c}\text{by substitution with}\\ u=\ln x,\ du=dx/x\end{array}\right]$ $\displaystyle = \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty.$ Divergent

22. $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x}-\frac1x\right]_1^t$ $\left[\begin{array}{c}\text{integration by parts with}\\ u=\ln x,\ dv=(1/x^2)\,dx\end{array}\right]$

$\displaystyle = \lim_{t\to\infty}\left(-\frac{\ln t}{t}-\frac1t+1\right) \overset{\text{H}}{=} \lim_{t\to\infty}\left(-\frac{1/t}{1}\right)-\lim_{t\to\infty}\frac1t+\lim_{t\to\infty}1 = 0-0+1 = 1.$ Convergent

23. $\displaystyle\int_{-\infty}^0 \frac{z}{z^4+4}\,dz = \lim_{t\to-\infty}\int_t^0 \frac{z}{z^4+4}\,dz = \lim_{t\to-\infty}\frac12\left[\frac12\tan^{-1}\left(\frac{z^2}{2}\right)\right]_t^0$ $\left[\begin{array}{c}u=z^2,\\ du=2z\,dz\end{array}\right]$

$\displaystyle = \lim_{t\to-\infty}\left[0-\frac14\tan^{-1}\left(\frac{t^2}{2}\right)\right] = -\frac14\left(\frac{\pi}{2}\right) = -\frac{\pi}{8}.$ Convergent

24. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty}\int_e^t \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{\ln x}\right]_e^t$ $\left[\begin{array}{c}u=\ln x,\\ du=(1/x)\,dx\end{array}\right]$

$\displaystyle = \lim_{t\to\infty}\left(-\frac{1}{\ln t}+1\right) = 0+1 = 1.$ Convergent

25. $\displaystyle\int_0^\infty e^{-\sqrt{y}}\,dy = \lim_{t\to\infty}\int_0^t e^{-\sqrt{y}}\,dy = \lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-x}\,(2x\,dx)$ $\left[\begin{array}{c}x=\sqrt{y},\\ dx=1/(2\sqrt{y}\,)\,dy\end{array}\right]$

$\displaystyle = \lim_{t\to\infty}\left\{\left[-2xe^{-x}\right]_0^{\sqrt{t}}+\int_0^{\sqrt{t}} 2e^{-x}\,dx\right\}$ $\left[\begin{array}{cc}u=2x, & dv=e^{-x}\,dx\\ du=2\,dx, & v=-e^{-x}\end{array}\right]$

$\displaystyle = \lim_{t\to\infty}\left(-2\sqrt{t}\,e^{-\sqrt{t}}+\left[-2e^{-x}\right]_0^{\sqrt{t}}\right) = \lim_{t\to\infty}\left(\frac{-2\sqrt{t}}{e^{\sqrt{t}}}-\frac{2}{e^{\sqrt{t}}}+2\right) = 0-0+2 = 2.$

Convergent

Note: $\displaystyle\lim_{t\to\infty}\frac{\sqrt{t}}{e^{\sqrt{t}}} \overset{\text{H}}{=} \lim_{t\to\infty}\frac{2\sqrt{t}}{2\sqrt{t}\,e^{\sqrt{t}}} = \lim_{t\to\infty}\frac{1}{e^{\sqrt{t}}} = 0$

26. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}+x\sqrt{x}}\,dx = \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to\infty}\int_1^{\sqrt{t}} \frac{1}{u(1+u^2)}\,(2u\,du)$ $\quad\left[\begin{array}{l} u=\sqrt{x}, \\ du = 1/(2\sqrt{x}\,)\,dx \end{array}\right]$

$$= \lim_{t\to\infty}\int_1^{\sqrt{t}} \frac{2}{1+u^2}\,du = \lim_{t\to\infty}\left[2\tan^{-1}u\right]_1^{\sqrt{t}} = \lim_{t\to\infty} 2(\tan^{-1}\sqrt{t}-\tan^{-1}1)$$

$$= 2\left(\tfrac{\pi}{2}-\tfrac{\pi}{4}\right) = \tfrac{\pi}{2}.$$ Convergent

27. $\displaystyle\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\Big[\ln|x|\Big]_t^1 = \lim_{t\to 0^+}(-\ln t) = \infty.$ Divergent

28. $\displaystyle\int_0^5 \frac{1}{\sqrt[3]{5-x}}\,dx = \lim_{t\to 5^-}\int_0^t (5-x)^{-1/3}\,dx = \lim_{t\to 5^-}\left[-\tfrac{3}{2}(5-x)^{2/3}\right]_0^t = \lim_{t\to 5^-}\left\{-\tfrac{3}{2}[(5-t)^{2/3}-5^{2/3}]\right\}$

$$= \tfrac{3}{2}5^{2/3}.$$ Convergent

29. $\displaystyle\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}} = \lim_{t\to -2^+}\int_t^{14}(x+2)^{-1/4}\,dx = \lim_{t\to -2^+}\left[\frac{4}{3}(x+2)^{3/4}\right]_t^{14} = \frac{4}{3}\lim_{t\to -2^+}\left[16^{3/4}-(t+2)^{3/4}\right]$

$$= \tfrac{4}{3}(8-0) = \tfrac{32}{3}.$$ Convergent

30. $\displaystyle\int_{-1}^2 \frac{x}{(x+1)^2}\,dx = \lim_{t\to -1^+}\int_t^2 \frac{x}{(x+1)^2}\,dx = \lim_{t\to -1^+}\int_t^2\left[\frac{1}{x+1}-\frac{1}{(x+1)^2}\right]dx$ [partial fractions]

$$= \lim_{t\to -1^+}\left[\ln|x+1|+\frac{1}{x+1}\right]_t^2 = \lim_{t\to -1^+}\left[\ln 3+\frac{1}{3}-\left(\ln(t+1)+\frac{1}{t+1}\right)\right] = -\infty.$$ Divergent

Note: To justify the last step, $\displaystyle\lim_{t\to -1^+}\left[\ln(t+1)+\frac{1}{t+1}\right] = \lim_{x\to 0^+}\left(\ln x+\frac{1}{x}\right)$ $\left[\begin{array}{c}\text{substitute} \\ x \text{ for } t+1\end{array}\right]$ $\displaystyle = \lim_{x\to 0^+}\frac{x\ln x+1}{x} = \infty$

since $\displaystyle\lim_{x\to 0^+}(x\ln x) = \lim_{x\to 0^+}\frac{\ln x}{1/x} \overset{\text{H}}{=} \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0.$

31. $\displaystyle\int_{-2}^3 \frac{dx}{x^4} = \int_{-2}^0 \frac{dx}{x^4}+\int_0^3 \frac{dx}{x^4}$, but $\displaystyle\int_{-2}^0 \frac{dx}{x^4} = \lim_{t\to 0^-}\left[-\frac{x^{-3}}{3}\right]_{-2}^t = \lim_{t\to 0^-}\left[-\frac{1}{3t^3}-\frac{1}{24}\right] = \infty.$ Divergent

32. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\left[\sin^{-1}x\right]_0^t = \lim_{t\to 1^-}\sin^{-1}t = \frac{\pi}{2}.$ Convergent

33. There is an infinite discontinuity at $x=1$. $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 (x-1)^{-1/3}\,dx+\int_1^9 (x-1)^{-1/3}\,dx.$

Here $\int_0^1 (x-1)^{-1/3}\,dx = \lim_{t\to 1^-}\int_0^t (x-1)^{-1/3}\,dx = \lim_{t\to 1^-}\left[\frac{3}{2}(x-1)^{2/3}\right]_0^t = \lim_{t\to 1^-}\left[\frac{3}{2}(t-1)^{2/3}-\frac{3}{2}\right] = -\frac{3}{2}$

and $\int_1^9 (x-1)^{-1/3}\,dx = \lim_{t\to 1^+}\int_t^9 (x-1)^{-1/3}\,dx = \lim_{t\to 1^+}\left[\frac{3}{2}(x-1)^{2/3}\right]_t^9 = \lim_{t\to 1^+}\left[6-\frac{3}{2}(t-1)^{2/3}\right] = 6$. Thus,

$\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = -\frac{3}{2}+6 = \frac{9}{2}.$ Convergent

34. There is an infinite discontinuity at $w=2$.

$\displaystyle\int_0^2 \frac{w}{w-2}\,dw = \lim_{t\to 2^-}\int_0^t\left(1+\frac{2}{w-2}\right)dw = \lim_{t\to 2^-}\Big[w+2\ln|w-2|\Big]_0^t = \lim_{t\to 2^-}(t+2\ln|t-2|-2\ln 2) = -\infty$, so

$\displaystyle\int_0^2 \frac{w}{w-2}\,dw$ diverges, and hence, $\displaystyle\int_0^5 \frac{w}{w-2}\,dw$ diverges. Divergent

35. $\int_0^{\pi/2} \tan^2\theta\,d\theta = \lim\limits_{t\to(\pi/2)^-} \int_0^t \tan^2\theta\,d\theta = \lim\limits_{t\to(\pi/2)^-} \int_0^t (\sec^2\theta - 1)\,d\theta = \lim\limits_{t\to(\pi/2)^-} \Big[\tan\theta - \theta\Big]_0^t$

$= \lim\limits_{t\to(\pi/2)^-} (\tan t - t) = \infty$ since $\tan t \to \infty$ as $t \to \frac{\pi}{2}^-$. Divergent

36. $\displaystyle\int_0^4 \frac{dx}{x^2 - x - 2} = \int_0^4 \frac{dx}{(x-2)(x+1)} = \int_0^2 \frac{dx}{(x-2)(x+1)} + \int_2^4 \frac{dx}{(x-2)(x+1)}$

Considering only $\displaystyle\int_0^2 \frac{dx}{(x-2)(x+1)}$ and using partial fractions, we have

$$\int_0^2 \frac{dx}{(x-2)(x+1)} = \lim_{t\to 2^-} \int_0^t \left(\frac{\frac{1}{3}}{x-2} - \frac{\frac{1}{3}}{x+1}\right) dx = \lim_{t\to 2^-} \Big[\tfrac{1}{3}\ln|x-2| - \tfrac{1}{3}\ln|x+1|\Big]_0^t$$

$$= \lim_{t\to 2^-} \Big[\tfrac{1}{3}\ln|t-2| - \tfrac{1}{3}\ln|t+1| - \tfrac{1}{3}\ln 2 + 0\Big] = -\infty \text{ since } \ln|t-2| \to -\infty \text{ as } t \to 2^-.$$

Thus, $\displaystyle\int_0^2 \frac{dx}{x^2-x-2}$ is divergent, and hence, $\displaystyle\int_0^4 \frac{dx}{x^2-x-2}$ is divergent as well.

37. $\int_0^1 r\ln r\,dr = \lim\limits_{t\to 0^+} \int_t^1 r\ln r\,dr = \lim\limits_{t\to 0^+} \Big[\frac{1}{2}r^2\ln r - \frac{1}{4}r^2\Big]_t^1$ $\quad\begin{bmatrix} u = \ln r, & dv = r\,dr \\ du = (1/r)\,dr, & v = \frac{1}{2}r^2 \end{bmatrix}$

$= \lim\limits_{t\to 0^+} \Big[\big(0 - \tfrac{1}{4}\big) - \big(\tfrac{1}{2}t^2\ln t - \tfrac{1}{4}t^2\big)\Big] = -\tfrac{1}{4} - 0 = -\tfrac{1}{4}$

since $\lim\limits_{t\to 0^+} t^2\ln t = \lim\limits_{t\to 0^+} \dfrac{\ln t}{1/t^2} \overset{\text{H}}{=} \lim\limits_{t\to 0^+} \dfrac{1/t}{-2/t^3} = \lim\limits_{t\to 0^+} (-\tfrac{1}{2}t^2) = 0.$ Convergent

38. $\displaystyle\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{\sin\theta}}\,d\theta = \lim_{t\to 0^+} \int_t^{\pi/2} \frac{\cos\theta}{\sqrt{\sin\theta}}\,d\theta = \lim_{t\to 0^+} \Big[2\sqrt{\sin\theta}\Big]_t^{\pi/2}$ $\quad\begin{bmatrix} u = \sin\theta, \\ du = \cos\theta\,d\theta \end{bmatrix}$

$= \lim\limits_{t\to 0^+} (2 - 2\sqrt{\sin t}\,) = 2 - 0 = 2.$ Convergent

39. $\displaystyle\int_{-1}^0 \frac{e^{1/x}}{x^3}\,dx = \lim_{t\to 0^-} \int_{-1}^t \frac{1}{x}e^{1/x}\cdot\frac{1}{x^2}\,dx = \lim_{t\to 0^-} \int_{-1}^{1/t} ue^u\,(-du)$ $\quad\begin{bmatrix} u = 1/x, \\ du = -dx/x^2 \end{bmatrix}$

$= \lim\limits_{t\to 0^-} \Big[(u-1)e^u\Big]_{1/t}^{-1}$ $\quad\begin{bmatrix}\text{use parts} \\ \text{or Formula 96}\end{bmatrix}$ $= \lim\limits_{t\to 0^-} \left[-2e^{-1} - \left(\dfrac{1}{t} - 1\right)e^{1/t}\right]$

$= -\dfrac{2}{e} - \lim\limits_{s\to -\infty} (s-1)e^s \quad [s = 1/t] \quad = -\dfrac{2}{e} - \lim\limits_{s\to -\infty} \dfrac{s-1}{e^{-s}} \overset{\text{H}}{=} -\dfrac{2}{e} - \lim\limits_{s\to -\infty} \dfrac{1}{-e^{-s}}$

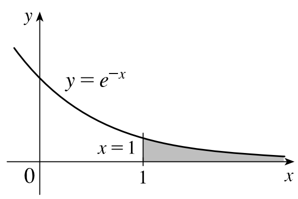
$= -\dfrac{2}{e} - 0 = -\dfrac{2}{e}.$ Convergent

40. $\displaystyle\int_0^1 \frac{e^{1/x}}{x^3}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}e^{1/x}\cdot\frac{1}{x^2}\,dx = \lim_{t\to 0^+} \int_{1/t}^1 ue^u\,(-du)$ $\quad\begin{bmatrix} u = 1/x, \\ du = -dx/x^2 \end{bmatrix}$

$= \lim\limits_{t\to 0^+} \Big[(u-1)e^u\Big]_1^{1/t}$ $\quad\begin{bmatrix}\text{use parts} \\ \text{or Formula 96}\end{bmatrix}$ $= \lim\limits_{t\to 0^+} \left[\left(\dfrac{1}{t} - 1\right)e^{1/t} - 0\right]$

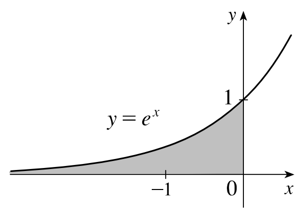
$= \lim\limits_{s\to\infty} (s-1)e^s \quad [s = 1/t] \quad = \infty.$ Divergent

41.



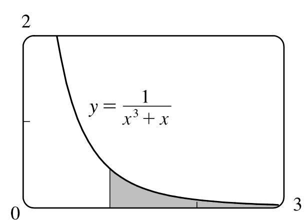
$$\text{Area} = \int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx = \lim_{t\to\infty}\Big[-e^{-x}\Big]_1^t$$
$$= \lim_{t\to\infty}(-e^{-t}+e^{-1}) = 0 + e^{-1} = 1/e$$

42.



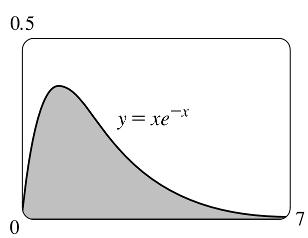
$$\text{Area} = \int_{-\infty}^0 e^{x}\,dx = \lim_{t\to-\infty}\int_t^0 e^{x}\,dx = \lim_{t\to-\infty}\Big[e^{x}\Big]_t^0$$
$$= \lim_{t\to-\infty}(e^0 - e^t) = 1 - 0 = 1$$

43.



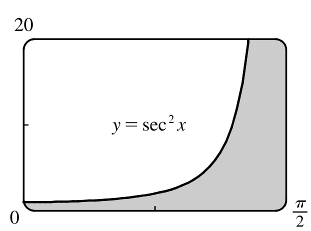
$$\text{Area} = \int_1^\infty \frac{1}{x^3+x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x(x^2+1)}\,dx$$
$$= \lim_{t\to\infty}\int_1^t\left(\frac{1}{x}-\frac{x}{x^2+1}\right)dx \qquad \text{[partial fractions]}$$
$$= \lim_{t\to\infty}\left[\ln|x| - \frac{1}{2}\ln\left|x^2+1\right|\right]_1^t = \lim_{t\to\infty}\left[\ln\frac{x}{\sqrt{x^2+1}}\right]_1^t$$
$$= \lim_{t\to\infty}\left(\ln\frac{t}{\sqrt{t^2+1}} - \ln\frac{1}{\sqrt{2}}\right) = \ln 1 - \ln 2^{-1/2} = \frac{1}{2}\ln 2$$

44.



$$\text{Area} = \int_0^\infty xe^{-x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x}\,dx$$
$$= \lim_{t\to\infty}\Big[-xe^{-x}-e^{-x}\Big]_0^t \qquad \text{[use parts wtih } u = x \text{ and } dv = e^{-x}\,dx\text{]}$$
$$= \lim_{t\to\infty}\big[(-te^{-t}-e^{-t})-(-1)\big]$$
$$= 0 \quad \text{[use l'Hospital's Rule]} \quad - 0 + 1 = 1$$

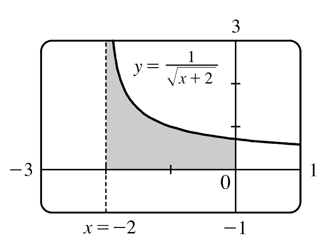
45.



$$\text{Area} = \int_0^{\pi/2}\sec^2 x\,dx = \lim_{t\to(\pi/2)^-}\int_0^t \sec^2 x\,dx = \lim_{t\to(\pi/2)^-}\Big[\tan x\Big]_0^t$$
$$= \lim_{t\to(\pi/2)^-}(\tan t - 0) = \infty$$

Infinite area

46.



$$\text{Area} = \int_{-2}^0 \frac{1}{\sqrt{x+2}}\,dx = \lim_{t\to-2^+}\int_t^0 \frac{1}{\sqrt{x+2}}\,dx = \lim_{t\to-2^+}\big[2\sqrt{x+2}\,\big]_t^0$$
$$= \lim_{t\to-2^+}\left(2\sqrt{2}-2\sqrt{t+2}\right) = 2\sqrt{2} - 0 = 2\sqrt{2}$$

47. (a)

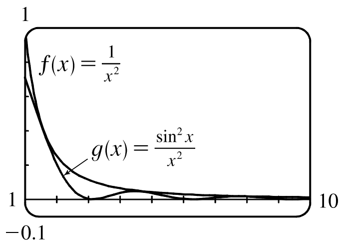
t	$\int_1^t g(x)\,dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1000	0.672957
10,000	0.673407

$g(x) = \dfrac{\sin^2 x}{x^2}$.

It appears that the integral is convergent.

(b) $-1 \le \sin x \le 1 \quad \Rightarrow \quad 0 \le \sin^2 x \le 1 \quad \Rightarrow \quad 0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$. Since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent [Equation 2 with $p = 2 > 1$], $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent by the Comparison Theorem.

(c)



Since $\int_1^\infty f(x)\,dx$ is finite and the area under $g(x)$ is less than the area under $f(x)$ on any interval $[1, t]$, $\int_1^\infty g(x)\,dx$ must be finite; that is, the integral is convergent.

48. (a)

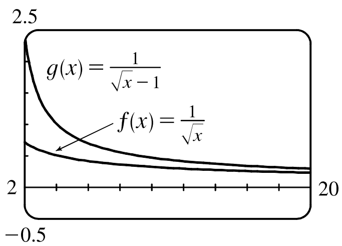
t	$\int_2^t g(x)dx$
5	3.830327
10	6.801200
100	23.328769
1000	69.023361
10,000	208.124560

$g(x) = \dfrac{1}{\sqrt{x} - 1}$.

It appears that the integral is divergent.

(b) For $x \ge 2$, $\sqrt{x} > \sqrt{x} - 1 \quad \Rightarrow \quad \dfrac{1}{\sqrt{x}} < \dfrac{1}{\sqrt{x} - 1}$. Since $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}}\,dx$ is divergent [Equation 2 with $p = \frac{1}{2} \le 1$], $\displaystyle\int_2^\infty \frac{1}{\sqrt{x} - 1}\,dx$ is divergent by the Comparison Theorem.

(c)



Since $\int_2^\infty f(x)\,dx$ is infinite and the area under $g(x)$ is greater than the area under $f(x)$ on any interval $[2, t]$, $\int_2^\infty g(x)\,dx$ must be infinite; that is, the integral is divergent.

49. For $x > 0$, $\dfrac{x}{x^3 + 1} < \dfrac{x}{x^3} = \dfrac{1}{x^2}$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by Equation 2 with $p = 2 > 1$, so $\displaystyle\int_1^\infty \frac{x}{x^3 + 1}\,dx$ is convergent by the Comparison Theorem. $\displaystyle\int_0^1 \frac{x}{x^3 + 1}\,dx$ is a constant, so $\displaystyle\int_0^\infty \frac{x}{x^3 + 1}\,dx = \int_0^1 \frac{x}{x^3 + 1}\,dx + \int_1^\infty \frac{x}{x^3 + 1}\,dx$ is also convergent.

50. For $x \geq 1$, $\dfrac{1+\sin^2 x}{\sqrt{x}} \geq \dfrac{1}{\sqrt{x}}$. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is divergent by Equation 2 with $p = \frac{1}{2} \leq 1$, so $\displaystyle\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,dx$ is divergent by the Comparison Theorem.

51. For $x > 1$, $f(x) = \dfrac{x+1}{\sqrt{x^4 - x}} > \dfrac{x+1}{\sqrt{x^4}} > \dfrac{x}{x^2} = \dfrac{1}{x}$, so $\displaystyle\int_2^\infty f(x)\,dx$ diverges by comparison with $\displaystyle\int_2^\infty \frac{1}{x}\,dx$, which diverges by Equation 2 with $p = 1 \leq 1$. Thus, $\int_1^\infty f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^\infty f(x)\,dx$ also diverges.

52. For $x \geq 0$, $\arctan x < \dfrac{\pi}{2} < 2$, so $\dfrac{\arctan x}{2+e^x} < \dfrac{2}{2+e^x} < \dfrac{2}{e^x} = 2e^{-x}$. Now

$I = \displaystyle\int_0^\infty 2e^{-x}\,dx = \lim_{t\to\infty}\int_0^t 2e^{-x}\,dx = \lim_{t\to\infty}\left[-2e^{-x}\right]_0^t = \lim_{t\to\infty}\left(-\frac{2}{e^t} + 2\right) = 2$, so I is convergent, and by comparison,

$\displaystyle\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$ is convergent.

53. For $0 < x \leq 1$, $\dfrac{\sec^2 x}{x\sqrt{x}} > \dfrac{1}{x^{3/2}}$. Now

$I = \displaystyle\int_0^1 x^{-3/2}\,dx = \lim_{t\to 0^+}\int_t^1 x^{-3/2}\,dx = \lim_{t\to 0^+}\left[-2x^{-1/2}\right]_t^1 = \lim_{t\to 0^+}\left(-2 + \frac{2}{\sqrt{t}}\right) = \infty$, so I is divergent, and by

comparison, $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}$ is divergent.

54. For $0 < x \leq 1$, $\dfrac{\sin^2 x}{\sqrt{x}} \leq \dfrac{1}{\sqrt{x}}$. Now

$I = \displaystyle\int_0^\pi \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^\pi x^{-1/2}\,dx = \lim_{t\to 0^+}\left[2x^{1/2}\right]_t^\pi = \lim_{t\to 0^+}\left(2\pi - 2\sqrt{t}\right) = 2\pi - 0 = 2\pi$, so I is convergent, and by

comparison, $\displaystyle\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$ is convergent.

55. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(1+x)} = \int_0^1 \frac{dx}{\sqrt{x}(1+x)} + \int_1^\infty \frac{dx}{\sqrt{x}(1+x)} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{\sqrt{x}(1+x)} + \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}(1+x)}$. Now

$\displaystyle\int \frac{dx}{\sqrt{x}(1+x)} = \int \frac{2u\,du}{u(1+u^2)}\quad \begin{bmatrix} u = \sqrt{x},\ x = u^2, \\ dx = 2u\,du \end{bmatrix} \quad = 2\int \frac{du}{1+u^2} = 2\tan^{-1} u + C = 2\tan^{-1}\sqrt{x} + C$, so

$$\begin{aligned}\int_0^\infty \frac{dx}{\sqrt{x}(1+x)} &= \lim_{t\to 0^+}\left[2\tan^{-1}\sqrt{x}\right]_t^1 + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{x}\right]_1^t \\ &= \lim_{t\to 0^+}\left[2\left(\tfrac{\pi}{4}\right) - 2\tan^{-1}\sqrt{t}\right] + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{t} - 2\left(\tfrac{\pi}{4}\right)\right] = \tfrac{\pi}{2} - 0 + 2\left(\tfrac{\pi}{2}\right) - \tfrac{\pi}{2} = \pi.\end{aligned}$$

56. $\displaystyle\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \int_2^3 \frac{dx}{x\sqrt{x^2-4}} + \int_3^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+}\int_t^3 \frac{dx}{x\sqrt{x^2-4}} + \lim_{t\to\infty}\int_3^t \frac{dx}{x\sqrt{x^2-4}}$. Now

$\displaystyle\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{2\sec\theta\,\tan\theta\,d\theta}{2\sec\theta\,2\tan\theta}\quad \begin{bmatrix} x = 2\sec\theta, \text{ where} \\ 0 \leq \theta < \pi/2 \text{ or } \pi \leq \theta < 3\pi/2 \end{bmatrix} \quad = \tfrac{1}{2}\theta + C = \tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right) + C$, so

$\displaystyle\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+}\left[\tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right)\right]_t^3 + \lim_{t\to\infty}\left[\tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right)\right]_3^t = \tfrac{1}{2}\sec^{-1}\left(\tfrac{3}{2}\right) - 0 + \tfrac{1}{2}\left(\tfrac{\pi}{2}\right) - \tfrac{1}{2}\sec^{-1}\left(\tfrac{3}{2}\right) = \tfrac{\pi}{4}$.

57. If $p = 1$, then $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x} = \lim_{t\to 0^+} [\ln x]_t^1 = \infty.$ Divergent

If $p \neq 1$, then $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^p}$ [note that the integral is not improper if $p < 0$]

$$= \lim_{t\to 0^+} \left[\frac{x^{-p+1}}{-p+1}\right]_t^1 = \lim_{t\to 0^+} \frac{1}{1-p}\left[1 - \frac{1}{t^{p-1}}\right]$$

If $p > 1$, then $p - 1 > 0$, so $\dfrac{1}{t^{p-1}} \to \infty$ as $t \to 0^+$, and the integral diverges.

If $p < 1$, then $p - 1 < 0$, so $\dfrac{1}{t^{p-1}} \to 0$ as $t \to 0^+$ and $\displaystyle\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}\left[\lim_{t\to 0^+} \left(1 - t^{1-p}\right)\right] = \frac{1}{1-p}.$

Thus, the integral converges if and only if $p < 1$, and in that case its value is $\dfrac{1}{1-p}$.

58. Let $u = \ln x$. Then $du = dx/x \;\Rightarrow\; \displaystyle\int_e^\infty \frac{dx}{x(\ln x)^p} = \int_1^\infty \frac{du}{u^p}$. By Example 4, this converges to $\dfrac{1}{p-1}$ if $p > 1$ and diverges otherwise.

59. First suppose $p = -1$. Then

$\displaystyle\int_0^1 x^p \ln x\,dx = \int_0^1 \frac{\ln x}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{\ln x}{x}\,dx = \lim_{t\to 0^+} \left[\tfrac{1}{2}(\ln x)^2\right]_t^1 = -\tfrac{1}{2} \lim_{t\to 0^+} (\ln t)^2 = -\infty$, so the integral diverges. Now suppose $p \neq -1$. Then integration by parts gives

$\displaystyle\int x^p \ln x\,dx = \frac{x^{p+1}}{p+1}\ln x - \int \frac{x^p}{p+1}\,dx = \frac{x^{p+1}}{p+1}\ln x - \frac{x^{p+1}}{(p+1)^2} + C$. If $p < -1$, then $p + 1 < 0$, so

$$\int_0^1 x^p \ln x\,dx = \lim_{t\to 0^+} \left[\frac{x^{p+1}}{p+1}\ln x - \frac{x^{p+1}}{(p+1)^2}\right]_t^1 = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right) \lim_{t\to 0^+} \left[t^{p+1}\left(\ln t - \frac{1}{p+1}\right)\right] = \infty.$$

If $p > -1$, then $p + 1 > 0$ and

$$\int_0^1 x^p \ln x\,dx = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right) \lim_{t\to 0^+} \frac{\ln t - 1/(p+1)}{t^{-(p+1)}} \overset{\text{H}}{=} \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right) \lim_{t\to 0^+} \frac{1/t}{-(p+1)t^{-(p+2)}}$$

$$= \frac{-1}{(p+1)^2} + \frac{1}{(p+1)^2} \lim_{t\to 0^+} t^{p+1} = \frac{-1}{(p+1)^2}$$

Thus, the integral converges to $-\dfrac{1}{(p+1)^2}$ if $p > -1$ and diverges otherwise.

60. (a) $\boldsymbol{n = 0}$: $\displaystyle\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty} \int_0^t e^{-x}\,dx = \lim_{t\to\infty} \left[-e^{-x}\right]_0^t = \lim_{t\to\infty} \left[-e^{-t} + 1\right] = 0 + 1 = 1$

$\boldsymbol{n = 1}$: $\displaystyle\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty} \int_0^t xe^{-x}\,dx$. To evaluate $\displaystyle\int xe^{-x}\,dx$, we'll use integration by parts with $u = x$, $dv = e^{-x}\,dx \;\Rightarrow\; du = dx$, $v = -e^{-x}$.

So $\displaystyle\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} - e^{-x} + C = (-x - 1)e^{-x} + C$ and

$$\lim_{t\to\infty}\int_0^t xe^{-x}\,dx = \lim_{t\to\infty}\left[(-x-1)e^{-x}\right]_0^t = \lim_{t\to\infty}\left[(-t-1)e^{-t}+1\right] = \lim_{t\to\infty}\left[-te^{-t}-e^{-t}+1\right]$$

$$= 0 - 0 + 1 \quad \text{[use l'Hospital's Rule]} \quad = 1$$

$\boldsymbol{n=2}$: $\displaystyle\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx$. To evaluate $\displaystyle\int x^2e^{-x}\,dx$, we could use integration by parts again or Formula 97. Thus,

$$\lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx = \lim_{t\to\infty}\left[-x^2e^{-x}\right]_0^t + 2\lim_{t\to\infty}\int_0^t xe^{-x}\,dx$$

$$= 0 + 0 + 2(1) \quad \text{[use l'Hospital's Rule and the result for } n=1\text{]} \quad = 2$$

$\boldsymbol{n=3}$: $\displaystyle\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t x^3e^{-x}\,dx \overset{97}{=} \lim_{t\to\infty}\left[-x^3e^{-x}\right]_0^t + 3\lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx$

$$= 0 + 0 + 3(2) \quad \text{[use l'Hospital's Rule and the result for } n=2\text{]} \quad = 6$$

(b) For $n = 1$, 2, and 3, we have $\int_0^\infty x^n e^{-x}\,dx = 1$, 2, and 6. The values for the integral are equal to the factorials for n, so we guess $\int_0^\infty x^n e^{-x}\,dx = n!$.

(c) Suppose that $\int_0^\infty x^k e^{-x}\,dx = k!$ for some positive integer k. Then $\int_0^\infty x^{k+1}e^{-x}\,dx = \lim\limits_{t\to\infty}\int_0^t x^{k+1}e^{-x}\,dx$.

To evaluate $\int x^{k+1}e^{-x}\,dx$, we use parts with $u = x^{k+1}$, $dv = e^{-x}\,dx \;\Rightarrow\; du = (k+1)x^k\,dx$, $v = -e^{-x}$.

So $\int x^{k+1}e^{-x}dx = -x^{k+1}e^{-x} - \int -(k+1)x^ke^{-x}\,dx = -x^{k+1}e^{-x} + (k+1)\int x^ke^{-x}\,dx$ and

$$\lim_{t\to\infty}\int_0^t x^{k+1}e^{-x}\,dx = \lim_{t\to\infty}\left[-x^{k+1}e^{-x}\right]_0^t + (k+1)\lim_{t\to\infty}\int_0^t x^ke^{-x}\,dx$$

$$= \lim_{t\to\infty}\left[-t^{k+1}e^{-t}+0\right] + (k+1)k! = 0 + 0 + (k+1)! = (k+1)!,$$

so the formula holds for $k+1$. By induction, the formula holds for all positive integers. (Since $0! = 1$, the formula holds for $n = 0$, too.)

61. (a) $I = \int_{-\infty}^\infty x\,dx = \int_{-\infty}^0 x\,dx + \int_0^\infty x\,dx$, and $\int_0^\infty x\,dx = \lim\limits_{t\to\infty}\int_0^t x\,dx = \lim\limits_{t\to\infty}\left[\frac12x^2\right]_0^t = \lim\limits_{t\to\infty}\left[\frac12t^2-0\right] = \infty$, so I is divergent.

(b) $\int_{-t}^t x\,dx = \left[\frac12x^2\right]_{-t}^t = \frac12t^2 - \frac12t^2 = 0$, so $\lim\limits_{t\to\infty}\int_{-t}^t x\,dx = 0$. Therefore, $\int_{-\infty}^\infty x\,dx \neq \lim\limits_{t\to\infty}\int_{-t}^t x\,dx$.

62. Let $k = \dfrac{M}{2RT}$ so that $\overline{v} = \dfrac{4}{\sqrt{\pi}}k^{3/2}\displaystyle\int_0^\infty v^3e^{-kv^2}\,dv$. Let I denote the integral and use parts to integrate I. Let $\alpha = v^2$, $d\beta = ve^{-kv^2}\,dv \;\Rightarrow\; d\alpha = 2v\,dv$, $\beta = -\dfrac{1}{2k}e^{-kv^2}$:

$$I = \lim_{t\to\infty}\left[-\frac{1}{2k}v^2e^{-kv^2}\right]_0^t + \frac1k\int_0^\infty ve^{-kv^2}\,dv_0^t = -\frac{1}{2k}\lim_{t\to\infty}\left(t^2e^{-kt^2}\right) + \frac1k\lim_{t\to\infty}\left[-\frac{1}{2k}e^{-kv^2}\right]$$

$$\overset{\text{H}}{=} -\frac{1}{2k}\cdot 0 - \frac{1}{2k^2}(0-1) = \frac{1}{2k^2}$$

Thus, $\overline{v} = \dfrac{4}{\sqrt{\pi}}k^{3/2}\cdot\dfrac{1}{2k^2} = \dfrac{2}{(k\pi)^{1/2}} = \dfrac{2}{[\pi M/(2RT)]^{1/2}} = \dfrac{2\sqrt2\sqrt{RT}}{\sqrt{\pi M}} = \sqrt{\dfrac{8RT}{\pi M}}$.

63. Volume $= \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \pi \lim_{t\to\infty} \int_1^t \frac{dx}{x^2} = \pi \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = \pi \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = \pi < \infty.$

64. Work $= \int_R^\infty \frac{GMm}{r^2}\,dr = \lim_{t\to\infty} \int_R^t \frac{GMm}{r^2}\,dr = \lim_{t\to\infty} GMm\left[\frac{-1}{r}\right]_R^t = GMm \lim_{t\to\infty}\left(\frac{-1}{t} + \frac{1}{R}\right) = \frac{GMm}{R}$, where

M = mass of the earth $= 5.98 \times 10^{24}$ kg, m = mass of satellite $= 10^3$ kg, R = radius of the earth $= 6.37 \times 10^6$ m, and

G = gravitational constant $= 6.67 \times 10^{-11}$ N·m²/kg.

Therefore, Work $= \dfrac{6.67 \times 10^{-11} \cdot 5.98 \times 10^{24} \cdot 10^3}{6.37 \times 10^6} \approx 6.26 \times 10^{10}$ J.

65. Work $= \int_R^\infty F\,dr = \lim_{t\to\infty} \int_R^t \frac{GmM}{r^2}\,dr = \lim_{t\to\infty} GmM\left(\frac{1}{R} - \frac{1}{t}\right) = \frac{GmM}{R}$. The initial kinetic energy provides the work,

so $\frac{1}{2}mv_0^2 = \frac{GmM}{R} \;\Rightarrow\; v_0 = \sqrt{\frac{2GM}{R}}$.

66. $y(s) = \int_s^R \frac{2r}{\sqrt{r^2 - s^2}}\,x(r)\,dr$ and $x(r) = \frac{1}{2}(R - r)^2 \;\Rightarrow$

$$\begin{aligned} y(s) &= \lim_{t\to s^+} \int_t^R \frac{r(R-r)^2}{\sqrt{r^2 - s^2}}\,dr = \lim_{t\to s^+} \int_t^R \frac{r^3 - 2Rr^2 + R^2 r}{\sqrt{r^2 - s^2}}\,dr \\ &= \lim_{t\to s^+} \left[\int_t^R \frac{r^3\,dr}{\sqrt{r^2 - s^2}} - 2R\int_t^R \frac{r^2\,dr}{\sqrt{r^2 - s^2}} + R^2 \int_t^R \frac{r\,dr}{\sqrt{r^2 - s^2}}\right] = \lim_{t\to s^+}\left(I_1 - 2RI_2 + R^2 I_3\right) = L \end{aligned}$$

For I_1: Let $u = \sqrt{r^2 - s^2} \;\Rightarrow\; u^2 = r^2 - s^2$, $r^2 = u^2 + s^2$, $2r\,dr = 2u\,du$, so, omitting limits and constant of integration,

$$\begin{aligned} I_1 &= \int \frac{(u^2 + s^2)u}{u}\,du = \int (u^2 + s^2)\,du = \tfrac{1}{3}u^3 + s^2 u = \tfrac{1}{3}u(u^2 + 3s^2) \\ &= \tfrac{1}{3}\sqrt{r^2 - s^2}\,(r^2 - s^2 + 3s^2) = \tfrac{1}{3}\sqrt{r^2 - s^2}\,(r^2 + 2s^2) \end{aligned}$$

For I_2: Using Formula 44, $I_2 = \frac{r}{2}\sqrt{r^2 - s^2} + \frac{s^2}{2}\ln\left|r + \sqrt{r^2 - s^2}\right|$.

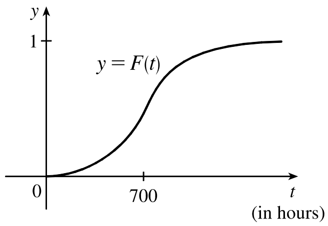
For I_3: Let $u = r^2 - s^2 \;\Rightarrow\; du = 2r\,dr$. Then $I_3 = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\cdot 2\sqrt{u} = \sqrt{r^2 - s^2}$.

Thus,

$$\begin{aligned} L &= \lim_{t\to s^+}\left[\tfrac{1}{3}\sqrt{r^2 - s^2}\,(r^2 + 2s^2) - 2R\left(\frac{r}{2}\sqrt{r^2 - s^2} + \frac{s^2}{2}\ln\left|r + \sqrt{r^2 - s^2}\right|\right) + R^2\sqrt{r^2 - s^2}\right]_t^R \\ &= \lim_{t\to s^+}\left[\tfrac{1}{3}\sqrt{R^2 - s^2}(R^2 + 2s^2) - 2R\left(\frac{R}{2}\sqrt{R^2 - s^2} + \frac{s^2}{2}\ln\left|R + \sqrt{R^2 - s^2}\right|\right) + R^2\sqrt{R^2 - s^2}\right] \\ &\qquad - \lim_{t\to s^+}\left[\tfrac{1}{3}\sqrt{t^2 - s^2}\,(t^2 + 2s^2) - 2R\left(\frac{t}{2}\sqrt{t^2 - s^2} + \frac{s^2}{2}\ln\left|t + \sqrt{t^2 - s^2}\right|\right) + R^2\sqrt{t^2 - s^2}\right] \\ &= \left[\tfrac{1}{3}\sqrt{R^2 - s^2}\,(R^2 + 2s^2) - Rs^2\ln\left|R + \sqrt{R^2 - s^2}\right|\right] - \left[-Rs^2\ln|s|\right] \\ &= \tfrac{1}{3}\sqrt{R^2 - s^2}\,(R^2 + 2s^2) - Rs^2\ln\left(\frac{R + \sqrt{R^2 - s^2}}{s}\right) \end{aligned}$$

67. We would expect a small percentage of bulbs to burn out in the first few hundred hours, most of the bulbs to burn out after close to 700 hours, and a few overachievers to burn on and on.

(a)



(b) $r(t) = F'(t)$ is the rate at which the fraction $F(t)$ of burnt-out bulbs increases as t increases. This could be interpreted as a fractional burnout rate.

(c) $\int_0^\infty r(t)\,dt = \lim\limits_{x\to\infty} F(x) = 1$, since all of the bulbs will eventually burn out.

68. $I = \displaystyle\int_0^\infty te^{kt}\,dt = \lim_{s\to\infty}\left[\frac{1}{k^2}(kt-1)\,e^{kt}\right]_0^s$ [Formula 96, or parts] $= \displaystyle\lim_{s\to\infty}\left[\left(\frac{1}{k}se^{ks} - \frac{1}{k^2}e^{ks}\right) - \left(-\frac{1}{k^2}\right)\right]$.

Since $k < 0$ the first two terms approach 0 (you can verify that the first term does so with l'Hospital's Rule), so the limit is equal to $1/k^2$. Thus, $M = -kI = -k(1/k^2) = -1/k = -1/(-0.000121) \approx 8264.5$ years.

69.
$$\gamma = \int_0^\infty \frac{cN(1-e^{-kt})}{k}\,e^{-\lambda t}\,dt = \frac{cN}{k}\lim_{x\to\infty}\int_0^x \left[e^{-\lambda t} - e^{(-k-\lambda)t}\right]dt$$
$$= \frac{cN}{k}\lim_{x\to\infty}\left[\frac{1}{-\lambda}\,e^{-\lambda t} - \frac{1}{-k-\lambda}\,e^{(-k-\lambda)t}\right]_0^x = \frac{cN}{k}\lim_{x\to\infty}\left[\frac{1}{-\lambda e^{\lambda x}} + \frac{1}{(k+\lambda)e^{(k+\lambda)x}} - \left(\frac{1}{-\lambda} + \frac{1}{k+\lambda}\right)\right]$$
$$= \frac{cN}{k}\left(\frac{1}{\lambda} - \frac{1}{k+\lambda}\right) = \frac{cN}{k}\left(\frac{k+\lambda-\lambda}{\lambda(k+\lambda)}\right) = \frac{cN}{\lambda(k+\lambda)}$$

70.
$$\int_0^\infty u(t)\,dt = \lim_{x\to\infty}\int_0^x \frac{r}{V}\,C_0\,e^{-rt/V}\,dt = \frac{r}{V}\,C_0\lim_{x\to\infty}\left[\frac{e^{-rt/V}}{-r/V}\right]_0^x = \frac{r}{V}\,C_0\left(-\frac{V}{r}\right)\lim_{x\to\infty}\left(e^{-rx/V} - 1\right)$$
$$= -C_0(0-1) = C_0.$$

$\int_0^\infty u(t)\,dt$ represents the total amount of urea removed from the blood if dialysis is continued indefinitely. The fact that $\int_0^\infty u(t)\,dt = C_0$ means that, in the limit, as $t \to \infty$, all the urea in the blood at time $t = 0$ is removed. The calculation says nothing about how rapidly that limit is approached.

71. $I = \displaystyle\int_a^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\int_a^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty}\left[\tan^{-1}x\right]_a^t = \lim_{t\to\infty}\left(\tan^{-1}t - \tan^{-1}a\right) = \tfrac{\pi}{2} - \tan^{-1}a.$

$I < 0.001 \;\Rightarrow\; \tfrac{\pi}{2} - \tan^{-1}a < 0.001 \;\Rightarrow\; \tan^{-1}a > \tfrac{\pi}{2} - 0.001 \;\Rightarrow\; a > \tan\left(\tfrac{\pi}{2} - 0.001\right) \approx 1000.$

72. $f(x) = e^{-x^2}$ and $\Delta x = \frac{4-0}{8} = \frac{1}{2}$.

$\int_0^4 f(x)\,dx \approx S_8 = \frac{1}{2\cdot 3}[f(0) + 4f(0.5) + 2f(1) + \cdots + 2f(3) + 4f(3.5) + f(4)] \approx \frac{1}{6}(5.31717808) \approx 0.8862$

Now $x > 4 \;\Rightarrow\; -x\cdot x < -x\cdot 4 \;\Rightarrow\; e^{-x^2} < e^{-4x} \;\Rightarrow\; \int_4^\infty e^{-x^2}\,dx < \int_4^\infty e^{-4x}\,dx.$

$\int_4^\infty e^{-4x}\,dx = \lim\limits_{t\to\infty}\left[-\frac{1}{4}e^{-4x}\right]_4^t = -\frac{1}{4}\left(0 - e^{-16}\right) = 1/(4e^{16}) \approx 0.0000000281 < 0.0000001$, as desired.

73. (a) $F(s) = \displaystyle\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = \lim_{n\to\infty}\left[-\frac{e^{-st}}{s}\right]_0^n = \lim_{n\to\infty}\left(\frac{e^{-sn}}{-s} + \frac{1}{s}\right)$. This converges to $\dfrac{1}{s}$ only if $s > 0$.

Therefore $F(s) = \dfrac{1}{s}$ with domain $\{s \mid s > 0\}$.

(b) $F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty e^t e^{-st}\,dt = \lim_{n\to\infty} \int_0^n e^{t(1-s)}\,dt = \lim_{n\to\infty} \left[\frac{1}{1-s}e^{t(1-s)}\right]_0^n$

$= \lim_{n\to\infty} \left(\frac{e^{(1-s)n}}{1-s} - \frac{1}{1-s}\right)$

This converges only if $1 - s < 0 \quad\Rightarrow\quad s > 1$, in which case $F(s) = \dfrac{1}{s-1}$ with domain $\{s \mid s > 1\}$.

(c) $F(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{n\to\infty} \int_0^n te^{-st}\,dt$. Use integration by parts: let $u = t$, $dv = e^{-st}\,dt \quad\Rightarrow\quad du = dt$,

$v = -\dfrac{e^{-st}}{s}$. Then $F(s) = \lim_{n\to\infty} \left[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_0^n = \lim_{n\to\infty} \left(\frac{-n}{se^{sn}} - \frac{1}{s^2e^{sn}} + 0 + \frac{1}{s^2}\right) = \dfrac{1}{s^2}$ only if $s > 0$.

Therefore, $F(s) = \dfrac{1}{s^2}$ and the domain of F is $\{s \mid s > 0\}$.

74. $0 \le f(t) \le Me^{at} \quad\Rightarrow\quad 0 \le f(t)e^{-st} \le Me^{at}e^{-st}$ for $t \ge 0$. Now use the Comparison Theorem:

$$\int_0^\infty Me^{at}e^{-st}\,dt = \lim_{n\to\infty} M\int_0^n e^{t(a-s)}\,dt = M\cdot\lim_{n\to\infty}\left[\frac{1}{a-s}e^{t(a-s)}\right]_0^n = M\cdot\lim_{n\to\infty}\frac{1}{a-s}\left[e^{n(a-s)} - 1\right]$$

This is convergent only when $a - s < 0 \quad\Rightarrow\quad s > a$. Therefore, by the Comparison Theorem, $F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$ is also convergent for $s > a$.

75. $G(s) = \int_0^\infty f'(t)e^{-st}\,dt$. Integrate by parts with $u = e^{-st}$, $dv = f'(t)\,dt \quad\Rightarrow\quad du = -se^{-st}$, $v = f(t)$:

$$G(s) = \lim_{n\to\infty}\left[f(t)e^{-st}\right]_0^n + s\int_0^\infty f(t)e^{-st}\,dt = \lim_{n\to\infty} f(n)e^{-sn} - f(0) + sF(s)$$

But $0 \le f(t) \le Me^{at} \quad\Rightarrow\quad 0 \le f(t)e^{-st} \le Me^{at}e^{-st}$ and $\lim_{t\to\infty} Me^{t(a-s)} = 0$ for $s > a$. So by the Squeeze Theorem,

$\lim_{t\to\infty} f(t)e^{-st} = 0$ for $s > a \quad\Rightarrow\quad G(s) = 0 - f(0) + sF(s) = sF(s) - f(0)$ for $s > a$.

76. Assume without loss of generality that $a < b$. Then

$$\begin{aligned}
\int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx &= \lim_{t\to-\infty}\int_t^a f(x)\,dx + \lim_{u\to\infty}\int_a^u f(x)\,dx \\
&= \lim_{t\to-\infty}\int_t^a f(x)\,dx + \lim_{u\to\infty}\left[\int_a^b f(x)\,dx + \int_b^u f(x)\,dx\right] \\
&= \lim_{t\to-\infty}\int_t^a f(x)\,dx + \int_a^b f(x)\,dx + \lim_{u\to\infty}\int_b^u f(x)\,dx \\
&= \lim_{t\to-\infty}\left[\int_t^a f(x)\,dx + \int_a^b f(x)\,dx\right] + \int_b^\infty f(x)\,dx \\
&= \lim_{t\to-\infty}\int_t^b f(x)\,dx + \int_b^\infty f(x)\,dx = \int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx
\end{aligned}$$

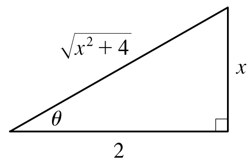
77. We use integration by parts: let $u = x$, $dv = xe^{-x^2}\,dx \quad\Rightarrow\quad du = dx$, $v = -\frac{1}{2}e^{-x^2}$. So

$$\int_0^\infty x^2e^{-x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}xe^{-x^2}\right]_0^t + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx = \lim_{t\to\infty}\left[-\frac{t}{2e^{t^2}}\right] + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

(The limit is 0 by l'Hospital's Rule.)

78. $\int_0^\infty e^{-x^2}\,dx$ is the area under the curve $y = e^{-x^2}$ for $0 \le x < \infty$ and $0 < y \le 1$. Solving $y = e^{-x^2}$ for x, we get $y = e^{-x^2} \;\Rightarrow\; \ln y = -x^2 \;\Rightarrow\; -\ln y = x^2 \;\Rightarrow\; x = \pm\sqrt{-\ln y}$. Since x is positive, choose $x = \sqrt{-\ln y}$, and the area is represented by $\int_0^1 \sqrt{-\ln y}\,dy$. Therefore, each integral represents the same area, so the integrals are equal.

79. For the first part of the integral, let $x = 2\tan\theta \;\Rightarrow\; dx = 2\sec^2\theta\,d\theta$.



$$\int \frac{1}{\sqrt{x^2+4}}\,dx = \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|.$$

From the figure, $\tan\theta = \dfrac{x}{2}$, and $\sec\theta = \dfrac{\sqrt{x^2+4}}{2}$. So

$$\begin{aligned} I = \int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2}\right)dx &= \lim_{t\to\infty}\left[\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| - C\ln|x+2|\right]_0^t \\ &= \lim_{t\to\infty}\left[\ln\frac{\sqrt{t^2+4}+t}{2} - C\ln(t+2) - (\ln 1 - C\ln 2)\right] \\ &= \lim_{t\to\infty}\left[\ln\left(\frac{\sqrt{t^2+4}+t}{2\,(t+2)^C}\right) + \ln 2^C\right] = \ln\left(\lim_{t\to\infty}\frac{t+\sqrt{t^2+4}}{(t+2)^C}\right) + \ln 2^{C-1} \end{aligned}$$

Now $L = \displaystyle\lim_{t\to\infty}\frac{t+\sqrt{t^2+4}}{(t+2)^C} \overset{\text{H}}{=} \lim_{t\to\infty}\frac{1+t/\sqrt{t^2+4}}{C\,(t+2)^{C-1}} = \frac{2}{C\displaystyle\lim_{t\to\infty}(t+2)^{C-1}}$.

If $C < 1$, $L = \infty$ and I diverges.

If $C = 1$, $L = 2$ and I converges to $\ln 2 + \ln 2^0 = \ln 2$.

If $C > 1$, $L = 0$ and I diverges to $-\infty$.

80. $$\begin{aligned} I = \int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right)dx &= \lim_{t\to\infty}\left[\tfrac{1}{2}\ln(x^2+1) - \tfrac{1}{3}C\ln(3x+1)\right]_0^t = \lim_{t\to\infty}\left[\ln(t^2+1)^{1/2} - \ln(3t+1)^{C/3}\right] \\ &= \lim_{t\to\infty}\left(\ln\frac{(t^2+1)^{1/2}}{(3t+1)^{C/3}}\right) = \ln\left(\lim_{t\to\infty}\frac{\sqrt{t^2+1}}{(3t+1)^{C/3}}\right) \end{aligned}$$

For $C \le 0$, the integral diverges. For $C > 0$, we have

$$L = \lim_{t\to\infty}\frac{\sqrt{t^2+1}}{(3t+1)^{C/3}} \overset{\text{H}}{=} \lim_{t\to\infty}\frac{t\,/\sqrt{t^2+1}}{C(3t+1)^{(C/3)-1}} = \frac{1}{C}\lim_{t\to\infty}\frac{1}{(3t+1)^{(C/3)-1}}$$

For $C/3 < 1 \;\Leftrightarrow\; C < 3$, $L = \infty$ and I diverges.

For $C = 3$, $L = \frac{1}{3}$ and $I = \ln\frac{1}{3}$.

For $C > 3$, $L = 0$ and I diverges to $-\infty$.

81. No, $I = \int_0^\infty f(x)\,dx$ must be *divergent*. Since $\displaystyle\lim_{x\to\infty} f(x) = 1$, there must exist an N such that if $x \ge N$, then $f(x) \ge \frac{1}{2}$. Thus, $I = I_1 + I_2 = \int_0^N f(x)\,dx + \int_N^\infty f(x)\,dx$, where I_1 is an ordinary definite integral that has a finite value, and I_2 is improper and diverges by comparison with the divergent integral $\int_N^\infty \frac{1}{2}\,dx$.

82. As in Exercise 55, we let $I = \displaystyle\int_0^\infty \frac{x^a}{1+x^b}\,dx = I_1 + I_2$, where $I_1 = \displaystyle\int_0^1 \frac{x^a}{1+x^b}\,dx$ and $I_2 = \displaystyle\int_1^\infty \frac{x^a}{1+x^b}\,dx$. We will show that I_1 converges for $a > -1$ and I_2 converges for $b > a + 1$, so that I converges when $a > -1$ and $b > a + 1$.

[continued]

I_1 is improper only when $a < 0$. When $0 \le x \le 1$, we have $\dfrac{1}{1+x^b} \le 1 \quad \Rightarrow \quad \dfrac{1}{x^{-a}(1+x^b)} \le \dfrac{1}{x^{-a}}$. The integral $\displaystyle\int_0^1 \frac{1}{x^{-a}}\,dx$ converges for $-a < 1$ [or $a > -1$] by Exercise 57, so by the Comparison Theorem, $\displaystyle\int_0^1 \frac{1}{x^{-a}(1+x^b)}\,dx$ converges for $-1 < a < 0$. I_1 is not improper when $a \ge 0$, so it has a finite real value in that case. Therefore, I_1 has a finite real value (converges) when $a > -1$.

I_2 is always improper. When $x \ge 1$, $\dfrac{x^a}{1+x^b} = \dfrac{1}{x^{-a}(1+x^b)} = \dfrac{1}{x^{-a}+x^{b-a}} < \dfrac{1}{x^{b-a}}$. By (2), $\displaystyle\int_1^\infty \frac{1}{x^{b-a}}\,dx$ converges for $b - a > 1$ (or $b > a + 1$), so by the Comparison Theorem, $\displaystyle\int_1^\infty \frac{x^a}{1+x^b}\,dx$ converges for $b > a + 1$.

Thus, I converges if $a > -1$ and $b > a + 1$.

7 Review

TRUE-FALSE QUIZ

1. False. Since the numerator has a higher degree than the denominator, $\dfrac{x(x^2+4)}{x^2-4} = x + \dfrac{8x}{x^2-4} = x + \dfrac{A}{x+2} + \dfrac{B}{x-2}$.

2. True. In fact, $A = -1$, $B = C = 1$.

3. False. It can be put in the form $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-4}$.

4. False. The form is $\dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$.

5. False. This is an improper integral, since the denominator vanishes at $x = 1$.

$$\int_0^4 \frac{x}{x^2-1}\,dx = \int_0^1 \frac{x}{x^2-1}\,dx + \int_1^4 \frac{x}{x^2-1}\,dx \text{ and}$$

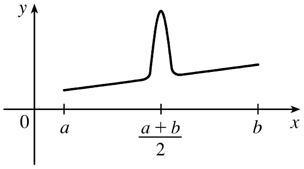
$$\int_0^1 \frac{x}{x^2-1}\,dx = \lim_{t\to1^-}\int_0^t \frac{x}{x^2-1}\,dx = \lim_{t\to1^-}\left[\tfrac{1}{2}\ln\left|x^2-1\right|\right]_0^t = \lim_{t\to1^-}\tfrac{1}{2}\ln\left|t^2-1\right| = \infty$$

So the integral diverges.

6. True by Theorem 7.8.2 with $p = \sqrt{2} > 1$.

7. False. See Exercise 61 in Section 7.8.

8. False. For example, with $n = 1$ the Trapezoidal Rule is much more accurate than the Midpoint Rule for the function in the diagram.



9. (a) True. See the end of Section 7.5.

(b) False. Examples include the functions $f(x) = e^{x^2}$, $g(x) = \sin(x^2)$, and $h(x) = \dfrac{\sin x}{x}$.

10. True. If f is continuous on $[0,\infty)$, then $\int_0^1 f(x)\,dx$ is finite. Since $\int_1^\infty f(x)\,dx$ is finite, so is $\int_0^\infty f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx$.

11. False. If $f(x) = 1/x$, then f is continuous and decreasing on $[1,\infty)$ with $\lim\limits_{x\to\infty} f(x) = 0$, but $\int_1^\infty f(x)\,dx$ is divergent.

12. True.
$$\begin{aligned}\int_a^\infty [f(x)+g(x)]\,dx &= \lim_{t\to\infty}\int_a^t [f(x)+g(x)]\,dx = \lim_{t\to\infty}\left(\int_a^t f(x)\,dx + \int_a^t g(x)\,dx\right)\\ &= \lim_{t\to\infty}\int_a^t f(x)\,dx + \lim_{t\to\infty}\int_a^t g(x)\,dx \quad \left[\begin{matrix}\text{since both limits}\\ \text{in the sum exist}\end{matrix}\right]\\ &= \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx\end{aligned}$$

Since the two integrals are finite, so is their sum.

13. False. Take $f(x) = 1$ for all x and $g(x) = -1$ for all x. Then $\int_a^\infty f(x)\,dx = \infty$ [divergent] and $\int_a^\infty g(x)\,dx = -\infty$ [divergent], but $\int_a^\infty [f(x)+g(x)]\,dx = 0$ [convergent].

14. False. $\int_0^\infty f(x)\,dx$ could converge or diverge. For example, if $g(x) = 1$, then $\int_0^\infty f(x)\,dx$ diverges if $f(x) = 1$ and converges if $f(x) = 0$.

EXERCISES

1.
$$\begin{aligned}\int_1^2 \frac{(x+1)^2}{x}\,dx &= \int_1^2 \frac{x^2+2x+1}{x}\,dx = \int_1^2 \left(x+2+\frac{1}{x}\right)dx = \left[\frac{1}{2}x^2+2x+\ln|x|\right]_1^2\\ &= (2+4+\ln 2) - \left(\tfrac{1}{2}+2+0\right) = \tfrac{7}{2}+\ln 2\end{aligned}$$

2.
$$\begin{aligned}\int_1^2 \frac{x}{(x+1)^2}\,dx &= \int_2^3 \frac{u-1}{u^2}\,du \qquad \left[\begin{matrix}u = x+1,\\ du = dx\end{matrix}\right]\\ &= \int_2^3\left(\frac{1}{u}-\frac{1}{u^2}\right)du = \left[\ln|u|+\frac{1}{u}\right]_2^3 = \left(\ln 3+\frac{1}{3}\right) - \left(\ln 2+\frac{1}{2}\right) = \ln\frac{3}{2}-\frac{1}{6}\end{aligned}$$

3.
$$\begin{aligned}\int \frac{e^{\sin x}}{\sec x}\,dx &= \int \cos x\, e^{\sin x}\,dx = \int e^u\,du \qquad \left[\begin{matrix}u = \sin x,\\ du = \cos x\,dx\end{matrix}\right]\\ &= e^u + C = e^{\sin x} + C\end{aligned}$$

4.
$$\begin{aligned}\int_0^{\pi/6} t\sin 2t\,dt &= \left[-\tfrac{1}{2}t\cos 2t\right]_0^{\pi/6} - \int_0^{\pi/6}\left(-\tfrac{1}{2}\cos 2t\right)dt \qquad \left[\begin{matrix}u = t, & dv = \sin 2t\\ du = dt, & v = -\frac{1}{2}\cos 2t\end{matrix}\right]\\ &= \left(-\tfrac{\pi}{12}\cdot\tfrac{1}{2}\right) - (0) + \left[\tfrac{1}{4}\sin 2t\right]_0^{\pi/6} = -\tfrac{\pi}{24}+\tfrac{1}{8}\sqrt{3}\end{aligned}$$

5.
$$\int \frac{dt}{2t^2+3t+1} = \int \frac{1}{(2t+1)(t+1)}\,dt = \int\left(\frac{2}{2t+1}-\frac{1}{t+1}\right)dt \quad \text{[partial fractions]} \quad = \ln|2t+1| - \ln|t+1| + C$$

6.
$$\begin{aligned}\int_1^2 x^5\ln x\,dx &= \left[\tfrac{1}{6}x^6\ln x\right]_1^2 - \int_1^2 \tfrac{1}{6}x^5\,dx \qquad \left[\begin{matrix}u = \ln x, & dv = x^5\,dx\\ du = \frac{1}{x}\,dx, & v = \frac{1}{6}x^6\end{matrix}\right]\\ &= \tfrac{64}{6}\ln 2 - 0 - \left[\tfrac{1}{36}x^6\right]_1^2 = \tfrac{32}{3}\ln 2 - \left(\tfrac{64}{36}-\tfrac{1}{36}\right) = \tfrac{32}{3}\ln 2 - \tfrac{7}{4}\end{aligned}$$

7.
$$\begin{aligned}\int_0^{\pi/2}\sin^3\theta\,\cos^2\theta\,d\theta &= \int_0^{\pi/2}(1-\cos^2\theta)\cos^2\theta\,\sin\theta\,d\theta = \int_1^0 (1-u^2)u^2\,(-du) \qquad \left[\begin{matrix}u = \cos\theta,\\ du = -\sin\theta\,d\theta\end{matrix}\right]\\ &= \int_0^1 (u^2-u^4)\,du = \left[\tfrac{1}{3}u^3-\tfrac{1}{5}u^5\right]_0^1 = \left(\tfrac{1}{3}-\tfrac{1}{5}\right) - 0 = \tfrac{2}{15}\end{aligned}$$

8. Let $u = \sqrt{e^x - 1}$, so that $u^2 = e^x - 1$, $2u\,du = e^x\,dx$, and $e^x = u^2 + 1$. Then

$$\int \frac{1}{\sqrt{e^x - 1}}\,dx = \int \frac{1}{u}\frac{2u\,du}{u^2+1} = 2\int \frac{1}{u^2+1}\,du = 2\tan^{-1} u + C = 2\tan^{-1}\sqrt{e^x - 1} + C.$$

9. Let $u = \ln t$, $du = dt/t$. Then $\displaystyle\int \frac{\sin(\ln t)}{t}\,dt = \int \sin u\,du = -\cos u + C = -\cos(\ln t) + C.$

10. Let $u = \arctan x$, $du = dx/(1+x^2)$. Then

$$\int_0^1 \frac{\sqrt{\arctan x}}{1+x^2}\,dx = \int_0^{\pi/4} \sqrt{u}\,du = \frac{2}{3}\left[u^{3/2}\right]_0^{\pi/4} = \frac{2}{3}\left[\frac{\pi^{3/2}}{4^{3/2}} - 0\right] = \frac{2}{3}\cdot\frac{1}{8}\pi^{3/2} = \frac{1}{12}\pi^{3/2}.$$

11. Let $x = \sec\theta$. Then

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx = \int_0^{\pi/3} \frac{\tan\theta}{\sec\theta}\sec\theta\,\tan\theta\,d\theta = \int_0^{\pi/3}\tan^2\theta\,d\theta = \int_0^{\pi/3}(\sec^2\theta - 1)\,d\theta = \left[\tan\theta - \theta\right]_0^{\pi/3} = \sqrt{3} - \tfrac{\pi}{3}.$$

12. $\displaystyle\int \frac{e^{2x}}{1+e^{4x}}\,dx = \int \frac{1}{1+u^2}\left(\tfrac{1}{2}\,du\right) \qquad \left[\begin{matrix} u = e^{2x}, \\ du = 2e^{2x}\,dx\end{matrix}\right]$

$$= \tfrac{1}{2}\tan^{-1} u + C = \tfrac{1}{2}\tan^{-1} e^{2x} + C$$

13. Let $w = \sqrt[3]{x}$. Then $w^3 = x$ and $3w^2\,dw = dx$, so $\int e^{\sqrt[3]{x}}\,dx = \int e^w \cdot 3w^2\,dw = 3I$. To evaluate I, let $u = w^2$, $dv = e^w\,dw \;\Rightarrow\; du = 2w\,dw$, $v = e^w$, so $I = \int w^2 e^w\,dw = w^2 e^w - \int 2we^w\,dw$. Now let $U = w$, $dV = e^w\,dw \;\Rightarrow$ $dU = dw$, $V = e^w$. Thus, $I = w^2e^w - 2\left[we^w - \int e^w\,dw\right] = w^2e^w - 2we^w + 2e^w + C_1$, and hence $3I = 3e^w(w^2 - 2w + 2) + C = 3e^{\sqrt[3]{x}}(x^{2/3} - 2x^{1/3} + 2) + C.$

14. $\displaystyle\int \frac{x^2+2}{x+2}\,dx = \int\left(x - 2 + \frac{6}{x+2}\right)dx = \tfrac{1}{2}x^2 - 2x + 6\ln|x+2| + C$

15. $\displaystyle\frac{x-1}{x^2+2x} = \frac{x-1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2} \;\Rightarrow\; x - 1 = A(x+2) + Bx$. Set $x = -2$ to get $-3 = -2B$, so $B = \frac{3}{2}$. Set $x = 0$ to get $-1 = 2A$, so $A = -\frac{1}{2}$. Thus, $\displaystyle\int \frac{x-1}{x^2+2x}\,dx = \int\left(\frac{-\frac{1}{2}}{x} + \frac{\frac{3}{2}}{x+2}\right)dx = -\tfrac{1}{2}\ln|x| + \tfrac{3}{2}\ln|x+2| + C.$

16. $\displaystyle\int \frac{\sec^6\theta}{\tan^2\theta}\,d\theta = \int \frac{(\tan^2\theta+1)^2\sec^2\theta}{\tan^2\theta}\,d\theta \quad \left[\begin{matrix} u = \tan\theta, \\ du = -\sec^2\theta\,d\theta\end{matrix}\right] \quad = \int \frac{(u^2+1)^2}{u^2}\,du = \int \frac{u^4 + 2u^2 + 1}{u^2}\,du$

$$= \int\left(u^2 + 2 + \frac{1}{u^2}\right)du = \frac{u^3}{3} + 2u - \frac{1}{u} + C = \tfrac{1}{3}\tan^3\theta + 2\tan\theta - \cot\theta + C$$

17. $\int x\cosh x\,dx = x\sinh x - \int \sinh x\,dx \qquad \left[\begin{matrix} u = x, & dv = \cosh x\,dx \\ du = dx, & v = \sinh x\end{matrix}\right]$

$$= x\sinh x - \cosh x + C$$

18. $\displaystyle\frac{x^2+8x-3}{x^3+3x^2} = \frac{x^2+8x-3}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3} \;\Rightarrow\; x^2 + 8x - 3 = Ax(x+3) + B(x+3) + Cx^2.$

Taking $x = 0$, we get $-3 = 3B$, so $B = -1$. Taking $x = -3$, we get $-18 = 9C$, so $C = -2$.

Taking $x = 1$, we get $6 = 4A + 4B + C = 4A - 4 - 2$, so $4A = 12$ and $A = 3$. Now

$$\int \frac{x^2 + 8x - 3}{x^3 + 3x^2}\,dx = \int \left(\frac{3}{x} - \frac{1}{x^2} - \frac{2}{x+3} \right) dx = 3\ln|x| + \frac{1}{x} - 2\ln|x+3| + C.$$

19. $$\int \frac{x+1}{9x^2 + 6x + 5}\,dx = \int \frac{x+1}{(9x^2 + 6x + 1) + 4}\,dx = \int \frac{x+1}{(3x+1)^2 + 4}\,dx \qquad \begin{bmatrix} u = 3x+1, \\ du = 3\,dx \end{bmatrix}$$

$$= \int \frac{\left[\frac{1}{3}(u-1)\right] + 1}{u^2 + 4} \left(\frac{1}{3}\,du \right) = \frac{1}{3} \cdot \frac{1}{3} \int \frac{(u-1) + 3}{u^2 + 4}\,du$$

$$= \frac{1}{9} \int \frac{u}{u^2 + 4}\,du + \frac{1}{9} \int \frac{2}{u^2 + 2^2}\,du = \frac{1}{9} \cdot \frac{1}{2} \ln(u^2 + 4) + \frac{2}{9} \cdot \frac{1}{2} \tan^{-1}\left(\frac{1}{2}u \right) + C$$

$$= \tfrac{1}{18} \ln(9x^2 + 6x + 5) + \tfrac{1}{9} \tan^{-1}\left[\tfrac{1}{2}(3x+1)\right] + C$$

20. $$\int \tan^5\theta \sec^3\theta\,d\theta = \int \tan^4\theta \sec^2\theta \sec\theta \tan\theta\,d\theta = \int (\sec^2\theta - 1)^2 \sec^2\theta \sec\theta \tan\theta\,d\theta \qquad \begin{bmatrix} u = \sec\theta, \\ du = \sec\theta \tan\theta\,d\theta \end{bmatrix}$$

$$= \int (u^2 - 1)^2 u^2\,du = \int (u^6 - 2u^4 + u^2)\,du$$

$$= \tfrac{1}{7}u^7 - \tfrac{2}{5}u^5 + \tfrac{1}{3}u^3 + C = \tfrac{1}{7}\sec^7\theta - \tfrac{2}{5}\sec^5\theta + \tfrac{1}{3}\sec^3\theta + C$$

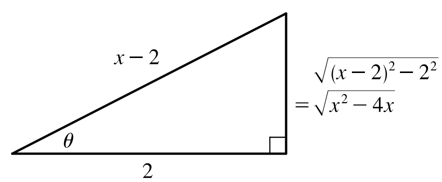
21. $$\int \frac{dx}{\sqrt{x^2 - 4x}} = \int \frac{dx}{\sqrt{(x^2 - 4x + 4) - 4}} = \int \frac{dx}{\sqrt{(x-2)^2 - 2^2}}$$

$$= \int \frac{2\sec\theta \tan\theta\,d\theta}{2\tan\theta} \qquad \begin{bmatrix} x - 2 = 2\sec\theta, \\ dx = 2\sec\theta \tan\theta\,d\theta \end{bmatrix}$$

$$= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C_1$$

$$= \ln\left| \frac{x-2}{2} + \frac{\sqrt{x^2 - 4x}}{2} \right| + C_1$$

$$= \ln\left| x - 2 + \sqrt{x^2 - 4x} \right| + C, \text{ where } C = C_1 - \ln 2$$



22. $$\int \cos\sqrt{t}\,dt = \int 2x \cos x\,dx \qquad \begin{bmatrix} x = \sqrt{t}, \\ x^2 = t, \quad 2x\,dx = dt \end{bmatrix}$$

$$= 2x \sin x - \int 2\sin x\,dx \qquad \begin{bmatrix} u = x, & dv = \cos x\,dx \\ du = dx, & v = \sin x \end{bmatrix}$$

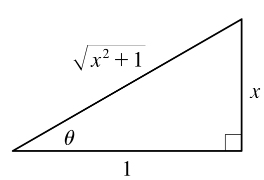
$$= 2x \sin x + 2\cos x + C = 2\sqrt{t} \sin\sqrt{t} + 2\cos\sqrt{t} + C$$

23. Let $x = \tan\theta$, so that $dx = \sec^2\theta\,d\theta$. Then

$$\int \frac{dx}{x\sqrt{x^2 + 1}} = \int \frac{\sec^2\theta\,d\theta}{\tan\theta \sec\theta} = \int \frac{\sec\theta}{\tan\theta}\,d\theta$$

$$= \int \csc\theta\,d\theta = \ln|\csc\theta - \cot\theta| + C$$

$$= \ln\left| \frac{\sqrt{x^2 + 1}}{x} - \frac{1}{x} \right| + C = \ln\left| \frac{\sqrt{x^2 + 1} - 1}{x} \right| + C$$



24. Let $u = \cos x$, $dv = e^x\,dx \;\Rightarrow\; du = -\sin x\,dx$, $v = e^x$: $(*)\; I = \int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx$.

To integrate $\int e^x \sin x\,dx$, let $U = \sin x$, $dV = e^x\,dx \;\Rightarrow\; dU = \cos x\,dx$, $V = e^x$. Then

$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx = e^x \sin x - I$. By substitution in $(*)$, $I = e^x \cos x + e^x \sin x - I \;\Rightarrow$

$2I = e^x(\cos x + \sin x) \;\Rightarrow\; I = \frac{1}{2}e^x(\cos x + \sin x) + C$.

25. $\dfrac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+2} \;\Rightarrow\; 3x^3 - x^2 + 6x - 4 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1)$.

Equating the coefficients gives $A + C = 3$, $B + D = -1$, $2A + C = 6$, and $2B + D = -4 \;\Rightarrow$

$A = 3$, $C = 0$, $B = -3$, and $D = 2$. Now

$$\int \frac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)}\,dx = 3\int \frac{x-1}{x^2+1}\,dx + 2\int \frac{dx}{x^2+2} = \frac{3}{2}\ln(x^2+1) - 3\tan^{-1}x + \sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C.$$

26. $\int x \sin x \cos x\,dx = \int \frac{1}{2}x \sin 2x\,dx \quad \begin{bmatrix} u = \frac{1}{2}x, & dv = \sin 2x\,dx, \\ du = \frac{1}{2}\,dx & v = -\frac{1}{2}\cos 2x \end{bmatrix}$

$$= -\tfrac{1}{4}x\cos 2x + \int \tfrac{1}{4}\cos 2x\,dx = -\tfrac{1}{4}x\cos 2x + \tfrac{1}{8}\sin 2x + C$$

27. $\int_0^{\pi/2} \cos^3 x \,\sin 2x\,dx = \int_0^{\pi/2} \cos^3 x\,(2\sin x\,\cos x)\,dx = \int_0^{\pi/2} 2\cos^4 x\,\sin x\,dx = \left[-\frac{2}{5}\cos^5 x\right]_0^{\pi/2} = \frac{2}{5}$

28. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2 du \;\Rightarrow$

$$\int \frac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1}\,dx = \int \frac{u+1}{u-1}\,3u^2 du = 3\int \left(u^2 + 2u + 2 + \frac{2}{u-1}\right) du$$

$$= u^3 + 3u^2 + 6u + 6\ln|u-1| + C = x + 3x^{2/3} + 6\sqrt[3]{x} + 6\ln|\sqrt[3]{x} - 1| + C$$

29. The integrand is an odd function, so $\displaystyle\int_{-3}^{3} \frac{x}{1+|x|}\,dx = 0$ [by 5.5.7(b)].

30. Let $u = e^{-x}$, $du = -e^{-x}\,dx$. Then

$$\int \frac{dx}{e^x\sqrt{1-e^{-2x}}} = \int \frac{e^{-x}\,dx}{\sqrt{1-(e^{-x})^2}} = \int \frac{-du}{\sqrt{1-u^2}} = -\sin^{-1}u + C = -\sin^{-1}(e^{-x}) + C.$$

31. Let $u = \sqrt{e^x - 1}$. Then $u^2 = e^x - 1$ and $2u\,du = e^x\,dx$. Also, $e^x + 8 = u^2 + 9$. Thus,

$$\int_0^{\ln 10} \frac{e^x\sqrt{e^x-1}}{e^x+8}\,dx = \int_0^3 \frac{u \cdot 2u\,du}{u^2+9} = 2\int_0^3 \frac{u^2}{u^2+9}\,du = 2\int_0^3 \left(1 - \frac{9}{u^2+9}\right) du$$

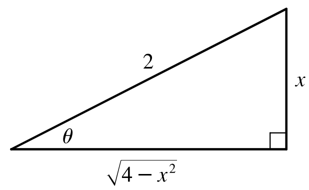
$$= 2\left[u - \frac{9}{3}\tan^{-1}\left(\frac{u}{3}\right)\right]_0^3 = 2\left[(3 - 3\tan^{-1}1) - 0\right] = 2\left(3 - 3\cdot\frac{\pi}{4}\right) = 6 - \frac{3\pi}{2}$$

32. $\displaystyle\int_0^{\pi/4} \frac{x\sin x}{\cos^3 x}\,dx = \int_0^{\pi/4} x\tan x\,\sec^2 x\,dx \quad \begin{bmatrix} u = x, & dv = \tan x\,\sec^2 x\,dx, \\ du = dx & v = \frac{1}{2}\tan^2 x \end{bmatrix}$

$$= \left[\frac{x}{2}\tan^2 x\right]_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4} \tan^2 x\,dx = \frac{\pi}{8}\cdot 1^2 - 0 - \frac{1}{2}\int_0^{\pi/4} (\sec^2 x - 1)\,dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\Big[\tan x - x\Big]_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{1}{2}$$

33. Let $x = 2\sin\theta \;\Rightarrow\; \left(4-x^2\right)^{3/2} = (2\cos\theta)^3$, $dx = 2\cos\theta\,d\theta$, so



$$\int \frac{x^2}{(4-x^2)^{3/2}}\,dx = \int \frac{4\sin^2\theta}{8\cos^3\theta}\,2\cos\theta\,d\theta = \int \tan^2\theta\,d\theta = \int \left(\sec^2\theta - 1\right)d\theta$$

$$= \tan\theta - \theta + C = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C$$

34. Integrate by parts twice, first with $u = (\arcsin x)^2$, $dv = dx$:

$$I = \int (\arcsin x)^2\,dx = x(\arcsin x)^2 - \int 2x\arcsin x\left(\frac{dx}{\sqrt{1-x^2}}\right)$$

Now let $U = \arcsin x$, $dV = \dfrac{x}{\sqrt{1-x^2}}\,dx \;\Rightarrow\; dU = \dfrac{1}{\sqrt{1-x^2}}\,dx$, $V = -\sqrt{1-x^2}$. So

$$I = x(\arcsin x)^2 - 2\left[\arcsin x\left(-\sqrt{1-x^2}\right) + \textstyle\int dx\right] = x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$$

35. $$\int \frac{1}{\sqrt{x+x^{3/2}}}\,dx = \int \frac{dx}{\sqrt{x\left(1+\sqrt{x}\right)}} = \int \frac{dx}{\sqrt{x}\sqrt{1+\sqrt{x}}} \quad \begin{bmatrix} u = 1+\sqrt{x}, \\ du = \dfrac{dx}{2\sqrt{x}} \end{bmatrix} \quad = \int \frac{2\,du}{\sqrt{u}} = \int 2u^{-1/2}\,du$$

$$= 4\sqrt{u} + C = 4\sqrt{1+\sqrt{x}} + C$$

36. $$\int \frac{1-\tan\theta}{1+\tan\theta}\,d\theta = \int \frac{\dfrac{\cos\theta}{\cos\theta} - \dfrac{\sin\theta}{\cos\theta}}{\dfrac{\cos\theta}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta}}\,d\theta = \int \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\,d\theta = \ln|\cos\theta + \sin\theta| + C$$

37. $$\textstyle\int(\cos x + \sin x)^2\cos 2x\,dx = \int\left(\cos^2 x + 2\sin x\cos x + \sin^2 x\right)\cos 2x\,dx = \int(1+\sin 2x)\cos 2x\,dx$$

$$= \textstyle\int \cos 2x\,dx + \frac{1}{2}\int \sin 4x\,dx = \frac{1}{2}\sin 2x - \frac{1}{8}\cos 4x + C$$

Or: $\int(\cos x + \sin x)^2\cos 2x\,dx = \int(\cos x + \sin x)^2(\cos^2 x - \sin^2 x)\,dx$

$$= \textstyle\int(\cos x + \sin x)^3(\cos x - \sin x)\,dx = \frac{1}{4}(\cos x + \sin x)^4 + C_1$$

38. $$\int \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx = \textstyle\int 2^u\,(2\,du) \quad \begin{bmatrix} u = \sqrt{x}, \\ du = 1/(2\sqrt{x})\,dx \end{bmatrix}$$

$$= 2\cdot\frac{2^u}{\ln 2} + C = \frac{2^{\sqrt{x}+1}}{\ln 2} + C$$

39. We'll integrate $I = \displaystyle\int \frac{xe^{2x}}{(1+2x)^2}\,dx$ by parts with $u = xe^{2x}$ and $dv = \dfrac{dx}{(1+2x)^2}$. Then $du = (x\cdot 2e^{2x} + e^{2x}\cdot 1)\,dx$ and $v = -\dfrac{1}{2}\cdot\dfrac{1}{1+2x}$, so

$$I = -\frac{1}{2}\cdot\frac{xe^{2x}}{1+2x} - \int\left[-\frac{1}{2}\cdot\frac{e^{2x}(2x+1)}{1+2x}\right]dx = -\frac{xe^{2x}}{4x+2} + \frac{1}{2}\cdot\frac{1}{2}e^{2x} + C = e^{2x}\left(\frac{1}{4} - \frac{x}{4x+2}\right) + C$$

Thus, $\displaystyle\int_0^{1/2} \frac{xe^{2x}}{(1+2x)^2}\,dx = \left[e^{2x}\left(\frac{1}{4} - \frac{x}{4x+2}\right)\right]_0^{1/2} = e\left(\frac{1}{4} - \frac{1}{8}\right) - 1\left(\frac{1}{4} - 0\right) = \frac{1}{8}e - \frac{1}{4}.$

40. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sqrt{\tan\theta}}{\sin 2\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \frac{\sqrt{\frac{\sin\theta}{\cos\theta}}}{2\sin\theta\cos\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{2}(\sin\theta)^{-1/2}(\cos\theta)^{-3/2}\,d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{2}\left(\frac{\sin\theta}{\cos\theta}\right)^{-1/2}(\cos\theta)^{-2}d\theta$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta\,d\theta = \left[\sqrt{\tan\theta}\right]_{\pi/4}^{\pi/3} = \sqrt{\sqrt{3}} - \sqrt{1} = \sqrt[4]{3} - 1$$

41. $\displaystyle\int_1^\infty \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \tfrac{1}{2}(2x+1)^{-3}\,2\,dx = \lim_{t\to\infty}\left[-\frac{1}{4(2x+1)^2}\right]_1^t$

$$= -\frac{1}{4}\lim_{t\to\infty}\left[\frac{1}{(2t+1)^2} - \frac{1}{9}\right] = -\frac{1}{4}\left(0 - \frac{1}{9}\right) = \frac{1}{36}$$

42. $\displaystyle\int_1^\infty \frac{\ln x}{x^4}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^4}\,dx \qquad \begin{bmatrix} u = \ln x, & dv = dx/x^4, \\ du = dx/x & v = -1/(3x^3) \end{bmatrix}$

$$= \lim_{t\to\infty}\left[-\frac{\ln x}{3x^3}\right]_1^t + \int_1^t \frac{1}{3x^4}\,dx = \lim_{t\to\infty}\left(-\frac{\ln t}{3t^3} + 0 + \left[\frac{-1}{9x^3}\right]_1^t\right) \overset{\text{H}}{=} \lim_{t\to\infty}\left(-\frac{1}{9t^3} + \left[\frac{-1}{9t^3} + \frac{1}{9}\right]\right)$$

$$= 0 + 0 + \tfrac{1}{9} = \tfrac{1}{9}$$

43. $\displaystyle\int \frac{dx}{x\ln x} \quad \begin{bmatrix} u = \ln x, \\ du = dx/x \end{bmatrix} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$, so

$\displaystyle\int_2^\infty \frac{dx}{x\ln x} = \lim_{t\to\infty}\int_2^t \frac{dx}{x\ln x} = \lim_{t\to\infty}\Big[\ln|\ln x|\Big]_2^t = \lim_{t\to\infty}\left[\ln(\ln t) - \ln(\ln 2)\right] = \infty$, so the integral is divergent.

44. Let $u = \sqrt{y-2}$. Then $y = u^2 + 2$ and $dy = 2u\,du$, so

$$\int \frac{y\,dy}{\sqrt{y-2}} = \int \frac{(u^2+2)2u\,du}{u} = 2\int\left(u^2+2\right)du = 2\left[\tfrac{1}{3}u^3 + 2u\right] + C$$

Thus, $\displaystyle\int_2^6 \frac{y\,dy}{\sqrt{y-2}} = \lim_{t\to 2^+}\int_t^6 \frac{y\,dy}{\sqrt{y-2}} = \lim_{t\to 2^+}\left[\tfrac{2}{3}(y-2)^{3/2} + 4\sqrt{y-2}\right]_t^6$

$$= \lim_{t\to 2^+}\left[\tfrac{16}{3} + 8 - \tfrac{2}{3}(t-2)^{3/2} - 4\sqrt{t-2}\right] = \tfrac{40}{3}.$$

45. $\displaystyle\int_0^4 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^4 \frac{\ln x}{\sqrt{x}}\,dx \overset{\star}{=} \lim_{t\to 0^+}\left[2\sqrt{x}\,\ln x - 4\sqrt{x}\right]_t^4$

$$= \lim_{t\to 0^+}\left[(2\cdot 2\ln 4 - 4\cdot 2) - \left(2\sqrt{t}\,\ln t - 4\sqrt{t}\right)\right] \overset{\star\star}{=} (4\ln 4 - 8) - (0 - 0) = 4\ln 4 - 8$$

($\star$) Let $u = \ln x$, $dv = \dfrac{1}{\sqrt{x}}\,dx \;\Rightarrow\; du = \dfrac{1}{x}\,dx$, $v = 2\sqrt{x}$. Then

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\,\ln x - 2\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}\,\ln x - 4\sqrt{x} + C$$

($\star\star$) $\displaystyle\lim_{t\to 0^+}\left(2\sqrt{t}\,\ln t\right) = \lim_{t\to 0^+}\frac{2\ln t}{t^{-1/2}} \overset{\text{H}}{=} \lim_{t\to 0^+}\frac{2/t}{-\frac{1}{2}t^{-3/2}} = \lim_{t\to 0^+}\left(-4\sqrt{t}\right) = 0$

46. Note that $f(x) = 1/(2-3x)$ has an infinite discontinuity at $x = \frac{2}{3}$. Now

$$\int_0^{2/3} \frac{1}{2-3x}\,dx = \lim_{t\to(2/3)^-} \int_0^t \frac{1}{2-3x}\,dx = \lim_{t\to(2/3)^-} \Big[-\tfrac{1}{3}\ln|2-3x|\Big]_0^t = -\tfrac{1}{3}\lim_{t\to(2/3)^-}\Big[\ln|2-3t| - \ln 2\Big] = \infty$$

Since $\int_0^{2/3} \frac{1}{2-3x}\,dx$ diverges, so does $\int_0^1 \frac{1}{2-3x}\,dx$.

47. $\int_0^1 \frac{x-1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 \left(\frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right)dx = \lim_{t\to 0^+} \int_t^1 (x^{1/2} - x^{-1/2})\,dx = \lim_{t\to 0^+}\left[\frac{2}{3}x^{3/2} - 2x^{1/2}\right]_t^1$

$$= \lim_{t\to 0^+}\left[\left(\tfrac{2}{3}-2\right) - \left(\tfrac{2}{3}t^{3/2} - 2t^{1/2}\right)\right] = -\tfrac{4}{3} - 0 = -\tfrac{4}{3}$$

48. $I = \int_{-1}^1 \frac{dx}{x^2-2x} = \int_{-1}^1 \frac{dx}{x(x-2)} = \int_{-1}^0 \frac{dx}{x(x-2)} + \int_0^1 \frac{dx}{x(x-2)} = I_1 + I_2$. Now

$\frac{1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2} \quad\Rightarrow\quad 1 = A(x-2) + Bx$. Set $x = 2$ to get $1 = 2B$, so $B = \frac{1}{2}$. Set $x = 0$ to get $1 = -2A$,

$A = -\frac{1}{2}$. Thus,

$$I_2 = \lim_{t\to 0^+}\int_t^1 \left(\frac{-\frac{1}{2}}{x} + \frac{\frac{1}{2}}{x-2}\right)dx = \lim_{t\to 0^+}\left[-\tfrac{1}{2}\ln|x| + \tfrac{1}{2}\ln|x-2|\right]_t^1 = \lim_{t\to 0^+}\left[(0+0) - \left(-\tfrac{1}{2}\ln t + \tfrac{1}{2}\ln|t-2|\right)\right]$$

$$= -\tfrac{1}{2}\ln 2 + \tfrac{1}{2}\lim_{t\to 0^+}\ln t = -\infty.$$

Since I_2 diverges, I is divergent.

49. Let $u = 2x+1$. Then

$$\int_{-\infty}^{\infty} \frac{dx}{4x^2+4x+5} = \int_{-\infty}^{\infty} \frac{\frac{1}{2}\,du}{u^2+4} = \tfrac{1}{2}\int_{-\infty}^0 \frac{du}{u^2+4} + \tfrac{1}{2}\int_0^{\infty} \frac{du}{u^2+4}$$

$$= \tfrac{1}{2}\lim_{t\to-\infty}\left[\tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}u\right)\right]_t^0 + \tfrac{1}{2}\lim_{t\to\infty}\left[\tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}u\right)\right]_0^t = \tfrac{1}{4}\left[0 - \left(-\tfrac{\pi}{2}\right)\right] + \tfrac{1}{4}\left[\tfrac{\pi}{2} - 0\right] = \tfrac{\pi}{4}.$$

50. $\int_1^{\infty} \frac{\tan^{-1}x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\tan^{-1}x}{x^2}\,dx$. Integrate by parts:

$$\int \frac{\tan^{-1}x}{x^2}\,dx = \frac{-\tan^{-1}x}{x} + \int \frac{1}{x}\frac{dx}{1+x^2} = \frac{-\tan^{-1}x}{x} + \int\left[\frac{1}{x} - \frac{x}{x^2+1}\right]dx$$

$$= \frac{-\tan^{-1}x}{x} + \ln|x| - \tfrac{1}{2}\ln(x^2+1) + C = \frac{-\tan^{-1}x}{x} + \frac{1}{2}\ln\frac{x^2}{x^2+1} + C$$

Thus,

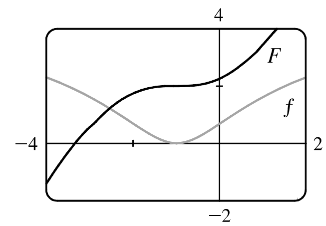
$$\int_1^{\infty} \frac{\tan^{-1}x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\tan^{-1}x}{x} + \frac{1}{2}\ln\frac{x^2}{x^2+1}\right]_1^t = \lim_{t\to\infty}\left[-\frac{\tan^{-1}t}{t} + \frac{1}{2}\ln\frac{t^2}{t^2+1} + \frac{\pi}{4} - \frac{1}{2}\ln\frac{1}{2}\right]$$

$$= 0 + \tfrac{1}{2}\ln 1 + \tfrac{\pi}{4} + \tfrac{1}{2}\ln 2 = \tfrac{\pi}{4} + \tfrac{1}{2}\ln 2$$

51. We first make the substitution $t = x + 1$, so $\ln(x^2 + 2x + 2) = \ln\left[(x+1)^2 + 1\right] = \ln(t^2 + 1)$. Then we use parts with $u = \ln(t^2 + 1)$, $dv = dt$:

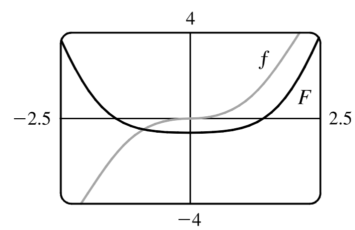
$$\int \ln(t^2+1)\,dt = t\,\ln(t^2+1) - \int \frac{t(2t)\,dt}{t^2+1} = t\,\ln(t^2+1) - 2\int \frac{t^2\,dt}{t^2+1} = t\,\ln(t^2+1) - 2\int \left(1 - \frac{1}{t^2+1}\right) dt$$
$$= t\,\ln(t^2+1) - 2t + 2\arctan t + C$$
$$= (x+1)\,\ln(x^2+2x+2) - 2x + 2\arctan(x+1) + K, \text{ where } K = C - 2$$

[Alternatively, we could have integrated by parts immediately with $u = \ln(x^2 + 2x + 2)$.] Notice from the graph that $f = 0$ where F has a horizontal tangent. Also, F is always increasing, and $f \geq 0$.



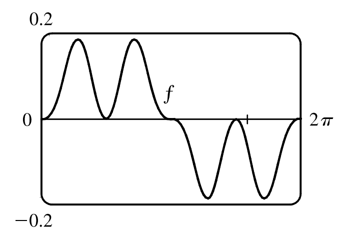
52. Let $u = x^2 + 1$. Then $x^2 = u - 1$ and $x\,dx = \frac{1}{2}\,du$, so

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \int \frac{(u-1)}{\sqrt{u}}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right) du$$
$$= \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right) + C = \tfrac{1}{3}(x^2+1)^{3/2} - (x^2+1)^{1/2} + C$$
$$= \tfrac{1}{3}(x^2+1)^{1/2}\left[(x^2+1) - 3\right] + C = \tfrac{1}{3}\sqrt{x^2+1}\,(x^2-2) + C$$



53. From the graph, it seems as though $\int_0^{2\pi} \cos^2 x\, \sin^3 x\,dx$ is equal to 0.

To evaluate the integral, we write the integral as $I = \int_0^{2\pi} \cos^2 x\,(1 - \cos^2 x)\, \sin x\,dx$ and let $u = \cos x \Rightarrow du = -\sin x\,dx$. Thus, $I = \int_1^1 u^2(1-u^2)(-du) = 0$.

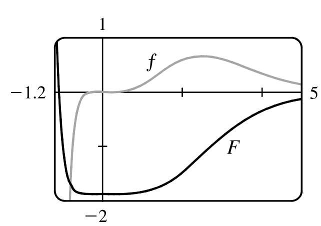


54. (a) To evaluate $\int x^5 e^{-2x}\,dx$ by hand, we would integrate by parts repeatedly, always taking $dv = e^{-2x}$ and starting with $u = x^5$. Each time we would reduce the degree of the x-factor by 1.

(b) To evaluate the integral using tables, we would use Formula 97 (which is proved using integration by parts) until the exponent of x was reduced to 1, and then we would use Formula 96.

(c) $\int x^5 e^{-2x}\,dx = -\frac{1}{8}e^{-2x}\left(4x^5 + 10x^4 + 20x^3 + 30x^2 + 30x + 15\right) + C$

(d)



55. $\int \sqrt{4x^2 - 4x - 3}\,dx = \int \sqrt{(2x-1)^2 - 4}\,dx \quad \begin{bmatrix} u = 2x - 1, \\ du = 2\,dx \end{bmatrix} \quad = \int \sqrt{u^2 - 2^2}\left(\frac{1}{2}\,du\right)$

$$\overset{39}{=} \frac{1}{2}\left(\frac{u}{2}\sqrt{u^2-2^2} - \frac{2^2}{2}\ln\left|u + \sqrt{u^2-2^2}\right|\right) + C = \tfrac{1}{4}u\sqrt{u^2-4} - \ln\left|u + \sqrt{u^2-4}\right| + C$$
$$= \tfrac{1}{4}(2x-1)\sqrt{4x^2-4x-3} - \ln\left|2x - 1 + \sqrt{4x^2-4x-3}\right| + C$$

56. $\int \csc^5 t\,dt \overset{78}{=} -\frac{1}{4}\cot t\,\csc^3 t + \frac{3}{4}\int \csc^3 t\,dt \overset{72}{=} -\frac{1}{4}\cot t\,\csc^3 t + \frac{3}{4}\left[-\frac{1}{2}\csc t\,\cot t + \frac{1}{2}\ln|\csc t - \cot t|\right] + C$

$= -\frac{1}{4}\cot t\,\csc^3 t - \frac{3}{8}\csc t\,\cot t + \frac{3}{8}\ln|\csc t - \cot t| + C$

57. Let $u = \sin x$, so that $du = \cos x\,dx$. Then

$$\int \cos x\,\sqrt{4+\sin^2 x}\,dx = \int \sqrt{2^2+u^2}\,du \overset{21}{=} \frac{u}{2}\sqrt{2^2+u^2} + \frac{2^2}{2}\ln\left(u+\sqrt{2^2+u^2}\right) + C$$

$$= \tfrac{1}{2}\sin x\,\sqrt{4+\sin^2 x} + 2\ln\left(\sin x + \sqrt{4+\sin^2 x}\right) + C$$

58. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$$\int \frac{\cot x\,dx}{\sqrt{1+2\sin x}} = \int \frac{du}{u\sqrt{1+2u}} \overset{\substack{57\text{ with}\\ a=1,\,b=2}}{=} \ln\left|\frac{\sqrt{1+2u}-1}{\sqrt{1+2u}+1}\right| + C = \ln\left|\frac{\sqrt{1+2\sin x}-1}{\sqrt{1+2\sin x}+1}\right| + C$$

59. (a) $\dfrac{d}{du}\left[-\dfrac{1}{u}\sqrt{a^2-u^2} - \sin^{-1}\left(\dfrac{u}{a}\right) + C\right] = \dfrac{1}{u^2}\sqrt{a^2-u^2} + \dfrac{1}{\sqrt{a^2-u^2}} - \dfrac{1}{\sqrt{1-u^2/a^2}}\cdot\dfrac{1}{a}$

$$= \left(a^2-u^2\right)^{-1/2}\left[\frac{1}{u^2}\left(a^2-u^2\right) + 1 - 1\right] = \frac{\sqrt{a^2-u^2}}{u^2}$$

(b) Let $u = a\sin\theta \quad\Rightarrow\quad du = a\cos\theta\,d\theta,\ a^2-u^2 = a^2\left(1-\sin^2\theta\right) = a^2\cos^2\theta$.

$$\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = \int \frac{a^2\cos^2\theta}{a^2\sin^2\theta}\,d\theta = \int \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta = \int(\csc^2\theta - 1)\,d\theta = -\cot\theta - \theta + C$$

$$= -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\left(\frac{u}{a}\right) + C$$

60. Work backward, and use integration by parts with $U = u^{-(n-1)}$ and $dV = (a+bu)^{-1/2}\,du \quad\Rightarrow$

$dU = \dfrac{-(n-1)\,du}{u^n}$ and $V = \dfrac{2}{b}\sqrt{a+bu}$, to get

$$\int \frac{du}{u^{n-1}\sqrt{a+bu}} = \int U\,dV = UV - \int V\,dU = \frac{2\sqrt{a+bu}}{bu^{n-1}} + \frac{2(n-1)}{b}\int \frac{\sqrt{a+bu}}{u^n}\,du$$

$$= \frac{2\sqrt{a+bu}}{bu^{n-1}} + \frac{2(n-1)}{b}\int \frac{a+bu}{u^n\sqrt{a+bu}}\,du$$

$$= \frac{2\sqrt{a+bu}}{bu^{n-1}} + 2(n-1)\int \frac{du}{u^{n-1}\sqrt{a+bu}} + \frac{2a(n-1)}{b}\int \frac{du}{u^n\sqrt{a+bu}}$$

Rearranging the equation gives $\dfrac{2a(n-1)}{b}\displaystyle\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{2\sqrt{a+bu}}{bu^{n-1}} - (2n-3)\int \frac{du}{u^{n-1}\sqrt{a+bu}} \quad\Rightarrow$

$$\int \frac{du}{u^n\sqrt{a+bu}} = \frac{-\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$$

61. For $n \geq 0$, $\int_0^\infty x^n\,dx = \lim\limits_{t\to\infty}\left[x^{n+1}/(n+1)\right]_0^t = \infty$. For $n < 0$, $\int_0^\infty x^n\,dx = \int_0^1 x^n\,dx + \int_1^\infty x^n\,dx$. Both integrals are improper. By (7.8.2), the second integral diverges if $-1 \leq n < 0$. By Exercise 7.8.57, the first integral diverges if $n \leq -1$. Thus, $\int_0^\infty x^n\,dx$ is divergent for all values of n.

62. $I = \int_0^\infty e^{ax}\cos x\,dx = \lim\limits_{t\to\infty}\int_0^t e^{ax}\cos x\,dx \overset{\substack{99\text{ with}\\ b=1}}{=} \lim\limits_{t\to\infty}\left[\dfrac{e^{ax}}{a^2+1}(a\cos x+\sin x)\right]_0^t$

$= \lim\limits_{t\to\infty}\left[\dfrac{e^{at}}{a^2+1}(a\cos t+\sin t) - \dfrac{1}{a^2+1}(a)\right] = \dfrac{1}{a^2+1}\lim\limits_{t\to\infty}\left[e^{at}(a\cos t+\sin t)-a\right]$.

For $a \geq 0$, the limit does not exist due to oscillation. For $a < 0$, $\lim\limits_{t\to\infty}\left[e^{at}(a\cos t+\sin t)\right] = 0$ by the Squeeze Theorem, because $\left|e^{at}(a\cos t+\sin t)\right| \leq e^{at}(|a|+1)$, so $I = \dfrac{1}{a^2+1}(-a) = -\dfrac{a}{a^2+1}$.

63. $f(x) = \dfrac{1}{\ln x}$, $\Delta x = \dfrac{b-a}{n} = \dfrac{4-2}{10} = \dfrac{1}{5}$

(a) $T_{10} = \frac{1}{5\cdot 2}\{f(2) + 2[f(2.2) + f(2.4) + \cdots + f(3.8)] + f(4)\} \approx 1.925444$

(b) $M_{10} = \frac{1}{5}[f(2.1) + f(2.3) + f(2.5) + \cdots + f(3.9)] \approx 1.920915$

(c) $S_{10} = \frac{1}{5\cdot 3}[f(2) + 4f(2.2) + 2f(2.4) + \cdots + 2f(3.6) + 4f(3.8) + f(4)] \approx 1.922470$

64. $f(x) = \sqrt{x}\cos x$, $\Delta x = \dfrac{b-a}{n} = \dfrac{4-1}{10} = \dfrac{3}{10}$

(a) $T_{10} = \frac{3}{10\cdot 2}\{f(1) + 2[f(1.3) + f(1.6) + \cdots + f(3.7)] + f(4)\} \approx -2.835151$

(b) $M_{10} = \frac{3}{10}[f(1.15) + f(1.45) + f(1.75) + \cdots + f(3.85)] \approx -2.856809$

(c) $S_{10} = \frac{3}{10\cdot 3}[f(1) + 4f(1.3) + 2f(1.6) + \cdots + 2f(3.4) + 4f(3.7) + f(4)] \approx -2.849672$

65. $f(x) = \dfrac{1}{\ln x} \quad\Rightarrow\quad f'(x) = -\dfrac{1}{x(\ln x)^2} \quad\Rightarrow\quad f''(x) = \dfrac{2+\ln x}{x^2(\ln x)^3} = \dfrac{2}{x^2(\ln x)^3} + \dfrac{1}{x^2(\ln x)^2}$. Note that each term of $f''(x)$ decreases on $[2,4]$, so we'll take $K = f''(2) \approx 2.022$. $|E_T| \leq \dfrac{K(b-a)^3}{12n^2} \approx \dfrac{2.022(4-2)^3}{12(10)^2} = 0.01348$ and $|E_M| \leq \dfrac{K(b-a)^3}{24n^2} = 0.00674$. $|E_T| \leq 0.00001 \quad\Leftrightarrow\quad \dfrac{2.022(8)}{12n^2} \leq \dfrac{1}{10^5} \quad\Leftrightarrow\quad n^2 \geq \dfrac{10^5(2.022)(8)}{12} \quad\Rightarrow\quad n \geq 367.2$. Take $n = 368$ for T_n. $|E_M| \leq 0.00001 \quad\Leftrightarrow\quad n^2 \geq \dfrac{10^5(2.022)(8)}{24} \quad\Rightarrow\quad n \geq 259.6$. Take $n = 260$ for M_n.

66. $\displaystyle\int_1^4 \frac{e^x}{x}\,dx \approx S_6 = \frac{(4-1)/6}{3}\,[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4)] \approx 17.739438$

67. $\Delta t = \left(\frac{10}{60} - 0\right)/\,10 = \frac{1}{60}$.

Distance traveled $= \int_0^{10} v\,dt \approx S_{10}$

$= \frac{1}{60\cdot 3}[40 + 4(42) + 2(45) + 4(49) + 2(52) + 4(54) + 2(56) + 4(57) + 2(57) + 4(55) + 56]$

$= \frac{1}{180}(1544) = 8.5\overline{7}$ mi

68. We use Simpson's Rule with $n = 6$ and $\Delta t = \frac{24-0}{6} = 4$:

Increase in bee population $= \int_0^{24} r(t)\,dt \approx S_6$

$= \frac{4}{3}[r(0) + 4r(4) + 2r(8) + 4r(12) + 2r(16) + 4r(20) + r(24)]$

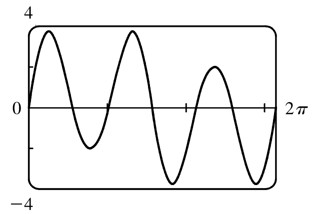
$= \frac{4}{3}[0 + 4(300) + 2(3000) + 4(11{,}000) + 2(4000) + 4(400) + 0]$

$= \frac{4}{3}(60{,}800) \approx 81{,}067$ bees

69. (a) $f(x) = \sin(\sin x)$. A CAS gives

$$f^{(4)}(x) = \sin(\sin x)[\cos^4 x + 7\cos^2 x - 3] + \cos(\sin x)\left[6\cos^2 x \sin x + \sin x\right]$$

From the graph, we see that $\left|f^{(4)}(x)\right| < 3.8$ for $x \in [0, \pi]$.



(b) We use Simpson's Rule with $f(x) = \sin(\sin x)$ and $\Delta x = \frac{\pi}{10}$:

$$\int_0^\pi f(x)\,dx \approx \frac{\pi}{10 \cdot 3}\left[f(0) + 4f\left(\frac{\pi}{10}\right) + 2f\left(\frac{2\pi}{10}\right) + \cdots + 4f\left(\frac{9\pi}{10}\right) + f(\pi)\right] \approx 1.786721$$

From part (a), we know that $\left|f^{(4)}(x)\right| < 3.8$ on $[0, \pi]$, so we use Theorem 7.7.4 with $K = 3.8$, and estimate the error as $|E_S| \le \dfrac{3.8(\pi - 0)^5}{180(10)^4} \approx 0.000646$.

(c) If we want the error to be less than 0.00001, we must have $|E_S| \le \dfrac{3.8\pi^5}{180n^4} \le 0.00001$, so $n^4 \ge \dfrac{3.8\pi^5}{180(0.00001)} \approx 646{,}041.6 \quad \Rightarrow \quad n \ge 28.35$. Since n must be even for Simpson's Rule, we must have $n \ge 30$ to ensure the desired accuracy.

70. With an x-axis in the normal position, at $x = 7$ we have $C = 2\pi r = 45 \quad \Rightarrow \quad r(7) = \frac{2\pi}{45}$.

Using Simpson's Rule with $n = 4$ and $\Delta x = 7$, we have

$$V = \int_0^{28} \pi[r(x)]^2\,dx \approx S_4 = \frac{7}{3}\left[0 + 4\pi\left(\frac{45}{2\pi}\right)^2 + 2\pi\left(\frac{53}{2\pi}\right)^2 + 4\pi\left(\frac{45}{2\pi}\right)^2 + 0\right] = \frac{7}{3}\left(\frac{21{,}818}{4\pi}\right) \approx 4051 \text{ cm}^3.$$

71. (a) $\dfrac{2 + \sin x}{\sqrt{x}} \ge \dfrac{1}{\sqrt{x}}$ for x in $[1, \infty)$. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is divergent by (7.8.2) with $p = \dfrac{1}{2} \le 1$. Therefore, $\displaystyle\int_1^\infty \frac{2 + \sin x}{\sqrt{x}}\,dx$ is divergent by the Comparison Theorem.

(b) $\dfrac{1}{\sqrt{1 + x^4}} < \dfrac{1}{\sqrt{x^4}} = \dfrac{1}{x^2}$ for x in $[1, \infty)$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by (7.8.2) with $p = 2 > 1$. Therefore, $\displaystyle\int_1^\infty \frac{1}{\sqrt{1 + x^4}}\,dx$ is convergent by the Comparison Theorem.

72. The line $y = 3$ intersects the hyperbola $y^2 - x^2 = 1$ at two points on its upper branch, namely $\left(-2\sqrt{2}, 3\right)$ and $\left(2\sqrt{2}, 3\right)$. The desired area is

$$A = \int_{-2\sqrt{2}}^{2\sqrt{2}} \left(3 - \sqrt{x^2 + 1}\right) dx = 2\int_0^{2\sqrt{2}} \left(3 - \sqrt{x^2 + 1}\right) dx \overset{21}{=} 2\left[3x - \tfrac{1}{2}x\sqrt{x^2 + 1} - \tfrac{1}{2}\ln\left(x + \sqrt{x^2 + 1}\right)\right]_0^{2\sqrt{2}}$$

$$= \left[6x - x\sqrt{x^2 + 1} - \ln\left(x + \sqrt{x^2 + 1}\right)\right]_0^{2\sqrt{2}} = 12\sqrt{2} - 2\sqrt{2} \cdot 3 - \ln\left(2\sqrt{2} + 3\right) = 6\sqrt{2} - \ln\left(3 + 2\sqrt{2}\right)$$

Another method: $A = 2\int_1^3 \sqrt{y^2 - 1}\,dy$ and use Formula 39.

73. For x in $\left[0, \frac{\pi}{2}\right]$, $0 \le \cos^2 x \le \cos x$. For x in $\left[\frac{\pi}{2}, \pi\right]$, $\cos x \le 0 \le \cos^2 x$. Thus,

$$\text{area} = \int_0^{\pi/2} (\cos x - \cos^2 x)\,dx + \int_{\pi/2}^{\pi} (\cos^2 x - \cos x)\,dx$$

$$= \left[\sin x - \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^{\pi/2} + \left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x - \sin x\right]_{\pi/2}^{\pi} = \left[\left(1 - \tfrac{\pi}{4}\right) - 0\right] + \left[\tfrac{\pi}{2} - \left(\tfrac{\pi}{4} - 1\right)\right] = 2$$

74. The curves $y = \dfrac{1}{2 \pm \sqrt{x}}$ are defined for $x \geq 0$. For $x > 0$, $\dfrac{1}{2 - \sqrt{x}} > \dfrac{1}{2 + \sqrt{x}}$. Thus, the required area is

$$\int_0^1 \left(\frac{1}{2 - \sqrt{x}} - \frac{1}{2 + \sqrt{x}} \right) dx = \int_0^1 \left(\frac{1}{2 - u} - \frac{1}{2 + u} \right) 2u\, du \quad \left[u = \sqrt{x} \right] \quad = 2 \int_0^1 \left(-\frac{u}{u - 2} - \frac{u}{u + 2} \right) du$$

$$= 2 \int_0^1 \left(-1 - \frac{2}{u - 2} - 1 + \frac{2}{u + 2} \right) du = 2 \left[2 \ln \left| \frac{u + 2}{u - 2} \right| - 2u \right]_0^1 = 4 \ln 3 - 4.$$

75. Using the formula for disks, the volume is

$$\begin{aligned} V &= \textstyle\int_0^{\pi/2} \pi \left[f(x) \right]^2 \, dx = \pi \int_0^{\pi/2} (\cos^2 x)^2 \, dx = \pi \int_0^{\pi/2} \left[\frac{1}{2}(1 + \cos 2x) \right]^2 \, dx \\ &= \textstyle\frac{\pi}{4} \int_0^{\pi/2} (1 + \cos^2 2x + 2 \cos 2x) \, dx = \frac{\pi}{4} \int_0^{\pi/2} \left[1 + \frac{1}{2}(1 + \cos 4x) + 2 \cos 2x \right] dx \\ &= \textstyle\frac{\pi}{4} \left[\frac{3}{2} x + \frac{1}{2} \left(\frac{1}{4} \sin 4x \right) + 2 \left(\frac{1}{2} \sin 2x \right) \right]_0^{\pi/2} = \frac{\pi}{4} \left[\left(\frac{3\pi}{4} + \frac{1}{8} \cdot 0 + 0 \right) - 0 \right] = \frac{3}{16} \pi^2 \end{aligned}$$

76. Using the formula for cylindrical shells, the volume is

$$\begin{aligned} V &= \textstyle\int_0^{\pi/2} 2\pi x f(x) \, dx = 2\pi \int_0^{\pi/2} x \cos^2 x \, dx = 2\pi \int_0^{\pi/2} x \left[\frac{1}{2}(1 + \cos 2x) \right] dx = 2 \left(\frac{1}{2} \right) \pi \int_0^{\pi/2} (x + x \cos 2x) \, dx \\ &= \textstyle\pi \left(\left[\frac{1}{2} x^2 \right]_0^{\pi/2} + \left[x \left(\frac{1}{2} \sin 2x \right) \right]_0^{\pi/2} - \int_0^{\pi/2} \frac{1}{2} \sin 2x \, dx \right) \quad \begin{bmatrix} \text{parts with } u = x, \\ dv = \cos 2x \, dx \end{bmatrix} \\ &= \textstyle\pi \left[\frac{1}{2} \left(\frac{\pi}{2} \right)^2 + 0 - \frac{1}{2} \left[-\frac{1}{2} \cos 2x \right]_0^{\pi/2} \right] = \frac{\pi^3}{8} + \frac{\pi}{4}(-1 - 1) = \frac{1}{8}(\pi^3 - 4\pi) \end{aligned}$$

77. By the Fundamental Theorem of Calculus,

$$\textstyle\int_0^\infty f'(x) \, dx = \lim\limits_{t \to \infty} \int_0^t f'(x) \, dx = \lim\limits_{t \to \infty} \left[f(t) - f(0) \right] = \lim\limits_{t \to \infty} f(t) - f(0) = 0 - f(0) = -f(0).$$

78. (a) $$(\tan^{-1} x)_{\text{ave}} = \lim_{t \to \infty} \frac{1}{t - 0} \int_0^t \tan^{-1} x \, dx \overset{89}{=} \lim_{t \to \infty} \left\{ \frac{1}{t} \left[x \tan^{-1} x - \tfrac{1}{2} \ln(1 + x^2) \right]_0^t \right\}$$

$$= \lim_{t \to \infty} \left[\frac{1}{t} \left(t \tan^{-1} t - \tfrac{1}{2} \ln(1 + t^2) \right) \right] = \lim_{t \to \infty} \left[\tan^{-1} t - \frac{\ln(1 + t^2)}{2t} \right]$$

$$\overset{\text{H}}{=} \frac{\pi}{2} - \lim_{t \to \infty} \frac{2t/(1 + t^2)}{2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

(b) $f(x) \geq 0$ and $\int_a^\infty f(x) \, dx$ is divergent $\Rightarrow$ $\lim\limits_{t \to \infty} \int_a^t f(x) \, dx = \infty$.

$$f_{\text{ave}} = \lim_{t \to \infty} \frac{\int_a^t f(x) \, dx}{t - a} \, dx \overset{\text{H}}{=} \lim_{t \to \infty} \frac{f(t)}{1} \quad \text{[by FTC1]} \quad = \lim_{x \to \infty} f(x), \text{ if this limit exists.}$$

(c) Suppose $\int_a^\infty f(x) \, dx$ converges; that is, $\lim\limits_{t \to \infty} \int_a^t f(x) \, dx = L < \infty$. Then

$$f_{\text{ave}} = \lim_{t \to \infty} \left[\frac{1}{t - a} \int_a^t f(x) \, dx \right] = \lim_{t \to \infty} \frac{1}{t - a} \cdot \lim_{t \to \infty} \int_a^t f(x) \, dx = 0 \cdot L = 0.$$

(d) $$(\sin x)_{\text{ave}} = \lim_{t \to \infty} \frac{1}{t} \int_0^t \sin x \, dx = \lim_{t \to \infty} \left(\frac{1}{t} \left[-\cos x \right]_0^t \right) = \lim_{t \to \infty} \left(-\frac{\cos t}{t} + \frac{1}{t} \right) = \lim_{t \to \infty} \frac{1 - \cos t}{t} = 0$$

79. Let $u = 1/x \quad\Rightarrow\quad x = 1/u \quad\Rightarrow\quad dx = -(1/u^2)\,du$.

$$\int_0^\infty \frac{\ln x}{1+x^2}\,dx = \int_\infty^0 \frac{\ln(1/u)}{1+1/u^2}\left(-\frac{du}{u^2}\right) = \int_\infty^0 \frac{-\ln u}{u^2+1}(-du) = \int_\infty^0 \frac{\ln u}{1+u^2}\,du = -\int_0^\infty \frac{\ln u}{1+u^2}\,du$$

Therefore, $\displaystyle\int_0^\infty \frac{\ln x}{1+x^2}\,dx = -\int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0$.

80. If the distance between P and the point charge is d, then the potential V at P is

$$V = W = \int_\infty^d F\,dr = \int_\infty^d \frac{q}{4\pi\varepsilon_0 r^2}\,dr = \lim_{t\to\infty} \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_t^d = \frac{q}{4\pi\varepsilon_0}\lim_{t\to\infty}\left(-\frac{1}{d}+\frac{1}{t}\right) = -\frac{q}{4\pi\varepsilon_0 d}.$$