Fall 2016 Math 2B - Midterm II

Name :

Student ID # :

Seat :

I certify that this exam was taken by the person named and done without any form of assistance including books, notes, calculators and other people.

Signature :

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- *This exam consists of 4 problems.*
- *Read directions for each problem carefully.*
- *Please show all work needed to arrive at your solutions.*
- *Justify all your answers.*

Problem 1 : Evaluate the following integrals.

(a) $\int_0^2 (x^2-4)e^{-x}\,dx$ 2 Integration by parts **[10 pts.]**

Let $u = x^2 - 4$ $v = -e^{-x}$

$du = 2x\,dx$ $dv = e^{-x}\,dx$

$$\int_0^2 (x^2-4)e^{-x}\,dx = \left[-(x^2-4)e^{-x}\right]_0^2 + 2\int_0^2 x e^{-x}\,dx$$

let $u = x$ $v = e^{-x}$

$du = dx$ $dv = -e^{-x}\,dx$

$$= -4 + 2\left[\left[-xe^{-x}\right]_0^2 + \int_0^2 e^{-x}\,dx\right]$$

$$= -4 + 2\left[-2e^{-2} + \left[-e^{-x}\right]_0^2\right]$$

$$= -4 - 4e^{-2} - 2e^{-2} + 2 = \boxed{-2 - 6e^{-2}}$$

(b) $\displaystyle\int \frac{\tan^3(\sqrt{t})\sec(\sqrt{t})}{\sqrt{t}}\,dt$ [10 pts.]

We do the substitution $u = \sqrt{t}$ $\quad du = \dfrac{dt}{2\sqrt{t}}$

$$\int \frac{\tan^3(\sqrt{t})\sec(\sqrt{t})}{\sqrt{t}}\,dt = 2\int \tan^3(u)\sec(u)\,du$$

$$= 2\int (\sec^2(u)-1)\tan u \sec u\,du$$

We do the substitution

$v = \sec u \quad dv = \tan u \sec u\,du$

$$= 2\int (v^2-1)\,dv$$

$$= 2\left(\frac{v^3}{3} - v\right) + C$$

$$= 2\left(\frac{\sec^3 u}{3} - \sec u\right) + C$$

$$\boxed{= 2\left(\frac{\sec^3\sqrt{t}}{3} - \sec\sqrt{t}\right) + C}$$

(c) $\int \frac{\sqrt{x^2-9}}{x^4}\,dx$ [10 pts.]

We do the trigonometric substitution

$$x = 3\sec\theta \qquad dx = 3\sec\theta\tan\theta\,d\theta$$

$$\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta$$

$$\int \frac{\sqrt{x^2-9}}{x^4}\,dx = \int \frac{3\tan\theta}{(3\sec\theta)^4}\,3\sec\theta\tan\theta\,d\theta$$

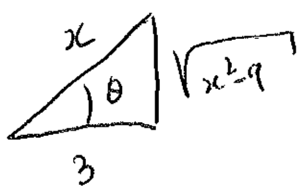
$$= \frac{1}{9}\int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta$$

$$= \frac{1}{9}\int \sin^2\theta\cos\theta\,d\theta$$

We make the substitution $u = \sin\theta \qquad du = \cos\theta\,d\theta$

$$= \frac{1}{9}\int u^2\,du = \frac{1}{27}u^3 + C$$

$$= \frac{1}{27}\sin^3\theta + C$$



$x = 3\sec\theta$

$\sec\theta = \frac{x}{3}$

$\cos\theta = \frac{3}{x}$

Thus $\sin\theta = \frac{\sqrt{x^2-9}}{x}$

$$\int \frac{\sqrt{x^2-9}}{x^4}\,dx \boxed{= \frac{1}{27}\left(\frac{\sqrt{x^2-9}}{x}\right)^3 + C}$$

(d) $\displaystyle\int \frac{x^3+1}{x^3+2x}\,dx$ [10 pts.]

We perform a long division because the rational function is improper. We obtain

$$\int \frac{x^3+1}{x^3+2x}\,dx = \int 1 + \frac{-2x+1}{x^3+2x}\,dx = x + \int \frac{-2x+1}{x^3+2x}\,dx$$

$$x^3+2x = x(x^2+2)$$

$$\frac{-2x+1}{x^3+2x} = \frac{A}{x} + \frac{Bx+C}{x^2+2} = \frac{(A+B)x^2 + Cx + 2A}{x(x^2+2)}$$

$$\begin{cases} A+B=0 \\ C=-2 \\ 2A=1 \end{cases} \qquad \begin{cases} A=\frac{1}{2} \\ B=-\frac{1}{2} \\ C=-2 \end{cases}$$

$$= x + \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{x\,dx}{x^2+2} - 2\int \frac{dx}{x^2+2}$$

In the second integral we do $u = x^2+2$
$du = 2x$

$$= x + \frac{1}{2}\ln|x| - \frac{1}{4}\int \frac{du}{u} - \frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)$$

$$\boxed{= x + \frac{1}{2}\ln|x| - \frac{1}{4}\ln|x^2+2| - \sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C}$$

Problem 2 : Determine whether each improper integral below is convergent or divergent. Evaluate those that are convergent.

(a) $\displaystyle\int_1^{+\infty} \frac{dy}{y^{\sqrt{2}}}$ **[5 pts.]**

$\sqrt{2} > 1$ Thus $\displaystyle\int_1^{+\infty} \frac{dy}{y^{\sqrt{2}}}$ is $\boxed{\text{convergent}}$

and $\displaystyle\int_1^{+\infty} \frac{dy}{y^{\sqrt{2}}} \boxed{= \frac{1}{\sqrt{2}-1}}$

(b) $\displaystyle\int_0^1 x^3 \ln x \, dx$ **[10 pts.]**

Discontinuity at 0 Let $0 < t \leq 1$.

We integrate by parts

Let $u = \ln x$ $\qquad v = \frac{x^4}{4}$

$du = \frac{dx}{x}$ $\qquad dv = x^3 \, dx$

$$\int_t^1 x^3 \ln x \, dx = \left[\frac{x^4}{4} \ln x\right]_t^1 - \int_t^1 \frac{x^4}{4} \frac{1}{x} \, dx$$

$$= -\frac{t^4}{4} \ln t - \frac{1}{4} \int_t^1 x^3 dx = -\frac{t^4}{4} \ln t - \frac{1}{4} \left[\frac{x^4}{4}\right]_t^1$$

$$= -\frac{t^4}{4} \ln t - \frac{1}{16} + \frac{t^4}{16} \xrightarrow[t \to 0^+]{} \boxed{-\frac{1}{16}}$$

$\boxed{\text{convergent}}$

(c) $\int_{-\infty}^{+\infty} e^{2x+1}\,dx$ [10 pts.]

We study first $\int_0^{+\infty} e^{2x+1}\,dx$.

Let $t \geq 0$.

$$\int_0^t e^{2x+1}\,dx = \left[\frac{1}{2} e^{2x+1}\right]_0^t = \frac{1}{2}\left(e^{2t+1} - e\right) \xrightarrow[t \to +\infty]{} +\infty$$

$\int_0^{+\infty} e^{2x+1}\,dx$ is divergent

Thus $\int_{-\infty}^{+\infty} e^{2x+1}\,dx$ is $\boxed{\text{divergent too.}}$

Problem 3 : Find the length of the curve

$$y = \frac{x^2}{4} - \frac{1}{2}\ln x$$

for $\frac{1}{2} \leq x \leq 1$. **[10 pts.]**

Let $f(x) = \frac{x^2}{4} - \frac{1}{2}\ln x$ for $\frac{1}{2} \leq x \leq 1$.

$$f'(x) = \frac{x}{2} - \frac{1}{2x}$$

$$L = \int_{1/2}^{1} \sqrt{1 + f'(x)^2}\, dx$$

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2}$$

$$= \sqrt{1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}} = \sqrt{\frac{1}{2} + \frac{x^2}{4} + \frac{1}{4x^2}}$$

$$= \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2} = \frac{x}{2} + \frac{1}{2x}$$

$$L = \int_{1/2}^{1} \frac{x}{2} + \frac{1}{2x}\, dx = \left[\frac{x^2}{4} + \frac{1}{2}\ln x\right]_{\frac{1}{2}}^{1}$$

$$= \frac{1}{4} - \frac{1}{16} + \frac{1}{2}\ln 2 = \boxed{\frac{3}{16} + \frac{1}{2}\ln 2}$$

Problem 4 : Determine whether each of the following sequences is convergent or divergent. Justify your answer and specify the limit when you can.

(a) $a_n = e^{\frac{2n^2-1}{3n^2+n}}$ **[3 pts.]**

$$\frac{2n^2-1}{3n^2+n} \xrightarrow[n\to+\infty]{} \frac{2}{3}$$

exp is continuous at $\frac{2}{3}$.

Thus $\boxed{a_n \xrightarrow[n\to+\infty]{} e^{\frac{2}{3}}}$ $\boxed{\text{convergent}}$

(b) $b_n = \left(-\frac{5}{2}\right)^n + 2$ **[3 pts.]**

$\left(-\frac{5}{2}\right)^n$ is a geometric sequence. $-\frac{5}{2} < -1$

Thus it has no limit.

(b_n) is $\boxed{\text{divergent}}$ and has no limit.

(c) $c_n = (3n-5)e^{-n}$ **[3 pts.]**

$c_n = \frac{3n-5}{e^n} = f(n)$ with $f(x) = \frac{3x-5}{e^x}$

By l'Hospital rule $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \frac{3}{e^x} = 0$

$\boxed{\text{Thus } (c_n) \text{ is convergent and } c_n \xrightarrow[n\to+\infty]{} 0}$

(d) $d_n = \frac{(-1)^n}{n}$ **[3 pts.]**

$|d_n| = \frac{1}{n} \xrightarrow[n\to+\infty]{} 0$

Thus $\boxed{d_n \xrightarrow[n\to+\infty]{} 0 .}$ (d_n) is $\boxed{\text{convergent.}}$