This sample 2J midterm exam is meant for practice only. Length and content of the actual midterm may vary, at the discretion of the instructor.

Try to complete the test in 50 minutes, without consulting any book or notes, and without using a calculator.
Problem 1

Use Gaussian elimination to solve the linear system:

\[
\begin{align*}
    x_1 + x_2 - x_3 + x_4 &= 1 \\
    2x_1 + x_2 + 2x_4 &= 3 \\
    -x_1 - 2x_2 + 3x_3 - x_4 &= 0.
\end{align*}
\]
Problem 2

a Find the determinant of the matrix \( A = \begin{pmatrix} 1 & -1 & 3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & 2 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix} \).

b If \( B \) is a \( 5 \times 5 \) matrix with determinant 3, find:

b1. the determinant of the matrix \((-2B)\);

b2. the determinant of the matrix obtained by \( B \) by first switching the second and fourth row, and then diving the third column by 2.
Problem 3

a Give an example of augmented matrix for a system of 3 equations in 4 unknowns with no solutions.

b Give an example of augmented matrix for a system of 3 equations in 4 unknowns with infinitely many solutions.

c Is there any system of 3 equations in 4 unknowns with a unique solution? If so, write down an example.
Problem 4

Find the $LU$ factorization of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix}$. 
Problem 5

The matrix \( A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \) is diagonalizable.

a) Find the eigenvalues of \( A \).

b) Find the corresponding eigenvectors.

c) Find a diagonal matrix \( D \) and a nonsingular matrix \( X \) such that
\[
A = XDX^{-1}.
\]
Just give \( X \) and \( D \). Do not compute \( X^{-1} \) and do not evaluate the product!
Problem 6

Decide whether each of the following matrices is diagonalizable.

You should not need to find the eigenvectors!

a) The matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.

b) The matrix $B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

c) The matrix $C = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$.
Problem 7

a Find the inverse of the matrix

\[
A = \begin{pmatrix}
1 & -2 & 3 \\
0 & 1 & -4 \\
0 & 0 & 1
\end{pmatrix}.
\]

Use any method you like!

b Use your answer from part a to find the (unique) solution of the system

\[
Ax = b, \quad \text{with}
\]

\[
b = \begin{pmatrix}
1 \\
0 \\
2
\end{pmatrix}.
\]
Problem 8

**True or False Questions.** Mark the correct answer. (*No explanations required.*)

a. A homogeneous system $Ax = 0$ is always consistent.
   
   true  false

b. If a square matrix is diagonalizable, its eigenvalues are distinct.
   
   true  false

c. Every two similar matrices have the same trace.
   
   true  false

d. Every two row equivalent matrices have the same determinant.
   
   true  false

e. If $A$ and $B$ are invertible, $(AB)^{-1} = A^{-1}B^{-1}$.
   
   true  false
Problem 9

*Recall that a matrix $B$ is called “symmetric” if it is equal to its transpose.*

Show that, for any $n \times n$ matrix $A$, the matrix $B = AA^T$ is symmetric.
Additional problems for practice.
Problem A  [Brief explanation required]

Let $A$ be a $4 \times 4$ matrix with characteristic polynomial

$$p(\lambda) = -\lambda(\lambda + 1)(\lambda - 3)(-\lambda + 1).$$

a. Is $A$ diagonalizable?

b. Is $A$ invertible?
Problem B [Brief explanation required]

Recall that two matrices $A$ and $B$ are similar if there exists an invertible matrix $S$ such that $A = S^{-1}BS$.

Show that if $A$ is similar to $B$, and $B$ is similar to $C$, then $A$ is similar to $C$. 
Problem C [Brief explanation required]

Let $A$ be a $5 \times 5$ matrix such that

$$A^T = -A.$$

Show that $A$ is singular.

*Hint:* compute the determinant of $A$. 