Prime numbers and Diophantine equations

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1. Warm-up

a. Given a 3 gallon jug and a 5 gallon jug (without any markings), is it possible to get exactly 1 gallon of water from a well? If so, how? If not, why not?

b. In how many ways can a debt of \$69 be paid using only \$5 and \$2 bills?

2. Diophantine equations

For each of the following exercises, find as many integer solutions as you can.

Observations:

We notice that if a+b=c, with a and b even numbers, then c must be _____.

d. Are there any solutions to $5\Box +10 = 11$?

Observations:

e. If $6\Box +12 = k$, what can be said about the number k?

f. If 30□+25 -= m, what can be said about the number m?

Make your own theorem:

If ? is _____, then 30□+? ≥ =20 has no solutions.

3. Prime Numbers

Definition: A natural number larger than 1 is called <u>prime</u> if its only divisors are 1 and itself.

Examples of prime numbers: 2, 3, 5, 7, ...

- a. Can you name the next 5 prime numbers?
- b. What is the largest prime number you know?
- c. How many prime numbers have been found?
- d. How many prime numbers are there?

Try out the Sieve of Eratosthenes:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Unique Factorization Theorem:

Every positive integer other than 1 can be factored into prime factors in exactly one way (except possibly for the order of the factors).

Example: **140** = $14.10 = 2.7.2.5 = 2^2.5.7$

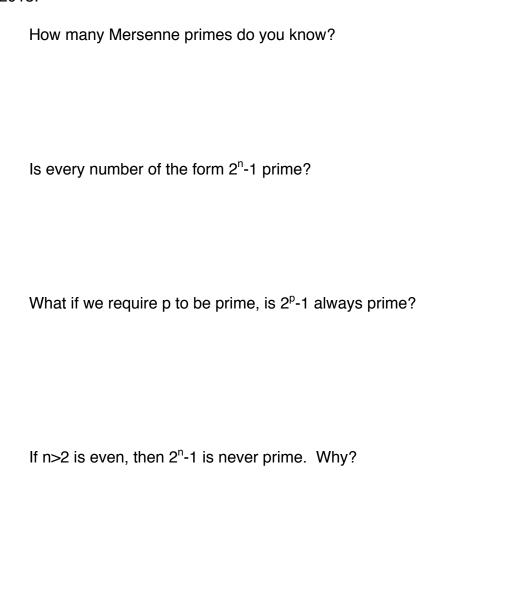
- 4. Factor each of the following numbers into a product of their prime factors.
 - a. 60
 - b. 7280
 - c. 107
 - d. 48944
 - e. 7900200

Theorem: There are infinitely many prime numbers.

Why?

Types of Primes:

A **Mersenne prime** is a prime number of the form 2ⁿ-1. The Great Internet Mersenne Prime Search is a computer program that uses the computing power of thousands of volunteers to search for prime numbers. The largest known prime number is 2^{43,112,609}-1. It has 12,978,189 digits and was found in January of 2013.



What about 2ⁿ-1 for n a multiple of 3?