Problem 0: Warm-Up Problems

1) Find the quotient and remainder when:

(a) 108 is divided by 3.
(b) 129 is divided by 7.

2) What is the remainder of:

(a) 16×2 after division by 5.
(b) $12039102394 + 12340984921423$ after division by 5.

3) Which of the following are true?

$7 \equiv 1 \pmod 3$
$7 \equiv 147 \pmod 3$
$7 \equiv 3 \pmod 3$
$76 \equiv -152 \pmod 3$

4) If $x \equiv 5 \pmod 8$ and $y \equiv 6 \pmod 8$, what number is $x + y$ congruent to modulo 8? What number is xy congruent to modulo 8.

5) A *remainder class modulo* n is the collection of integers which give the same remainder when divided by n. Dividing by n gives us n possible remainders, $0, 1, 2, \ldots n-1$ and so there are n remainder classes. For example, if $n = 2$ the two remainder classes are:

$$\overline{0} = \{\ldots, -9, -6, -3, 0, 3, 6, \ldots\}$$
$$\overline{1} = \{\ldots, -8, -5, -2, 1, 4, 7, \ldots\}$$
$$\overline{2} = \{\ldots, -7, -4, -1, 2, 5, 8, \ldots\}$$

What are the remainder classes modulo 4?

Remainders when Dividing

Given two integers a and n, we may divide a by n. If n does not divide a evenly, then there is a *remainder* r, which will be less than n. For example,

$$
\begin{array}{rl}
 & 2500(\mathbf{r}17) \\
25 \,\big) & \overline{62517} \\
 & \underline{50} \\
 & 125 \\
 & \underline{125} \\
 & \quad 17
\end{array}
$$

For any integer a and divisor n, there are two integers q and r called the *quotient* and *remainder* such that

$$a = q \times n + r \quad \text{and} \quad 0 \leq r < n$$

So from our example above: $62517 = 2500 \cdot 25 + 17$.

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MODULAR ARITHMETIC CHEAT SHEET

We say that a is *congruent to b modulo n* and write:

$$a \equiv b \pmod{n}$$

if a and b have the same remainder when divided by n.

If $a \equiv c \pmod{n}$ and $b \equiv d \pmod{n}$ then,

$$a + b \equiv c + d \pmod{n}$$

$$a - b \equiv c - d \pmod{n}$$

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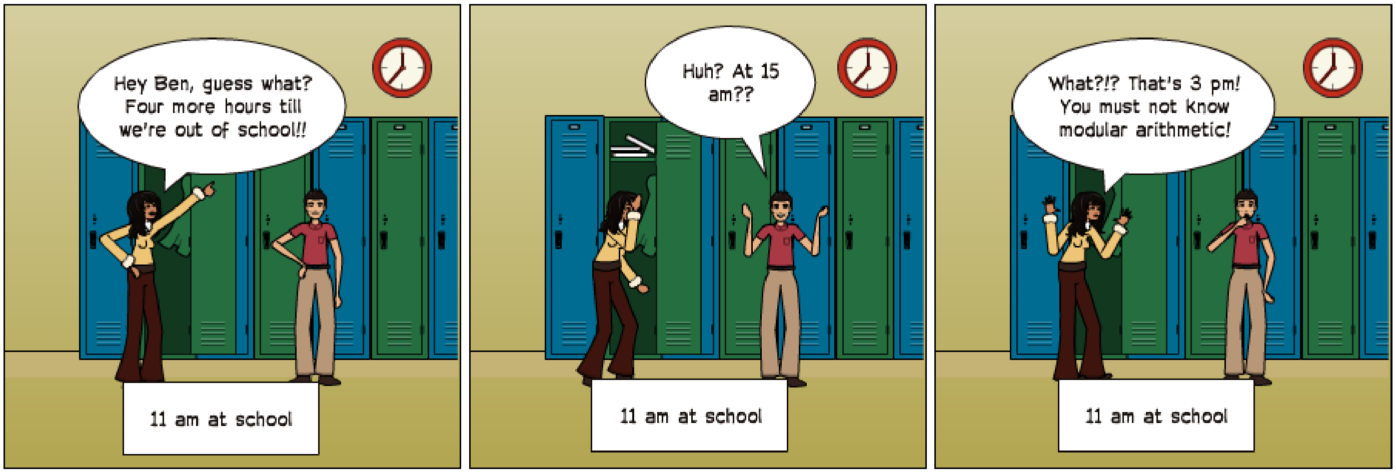
Problem 1: A Clock Arithmetic



1) Marty the marathon runner has a unique method of resting after his marathons: he goes into hibernation. If Marty enters his cryo-static sleep chamber at 6 pm, and sets the timer for 76 hours, what time will he wake up?

2) On Monday morning, Sam broke his stubborn ways and decided to try green eggs and ham. Much to his suprise, he loved them. He loved them so much, he decided he would eat green eggs and ham for the next 200 days. What day of the week will it be when Sam can stop eating green eggs and ham?

3) In the year 2340, Joanna, who is a spectacular snow-hoverboard rider, charters a flight to the moon to ride its killer slopes. She is going to try and break the world-record and complete a 7110° rotation. If she is facing due west when she enters the air off the jump, and she rotates counter-clockwise what direction should she be facing if she is to break the record?

Problem 2: A Modular Game

This is a two player game between player A and player B.

Modular Addition Game: The game is played as follows:

* To start, player A picks a positive integer n
* Player B decides who will go first
* A running total T is set to 0
* At each player's turn, they pick a number m and add it to T
* The player loses if T is congruent to a previous total modulo n

Here is an example play through:

Alice and Bob play the Modular Addition Game. Alice chooses the modulus $n = 5$ and Bob decides to go first. The turns are as follows:

Round #	1	2	3	4	5
Alice		-88		-23	
Bob	7		102		13
Running total T	7	-81	21	-2	11

On the last play, Bob loses, because he makes the running total 11 and $11 \equiv 21 \pmod 5$. What number should Bob have picked instead?

Try the game for yourself!

Player A:
Player B:
Modulus n:

Round #	1	2	3	4	5	6	7	8	9	10
Player A										
Player B										
Total T										

How many moves can a game have? What strategy should player A follow? What about player B?

Player A:
Player B:
Modulus n:

Round #	1	2	3	4	5	6	7	8	9	10
Player A										
Player B										
Total T										

Player A:
Player B:
Modulus n:

Round #	1	2	3	4	5	6	7	8	9	10
Player A										
Player B										
Total T										

Player A:
Player B:
Modulus n:

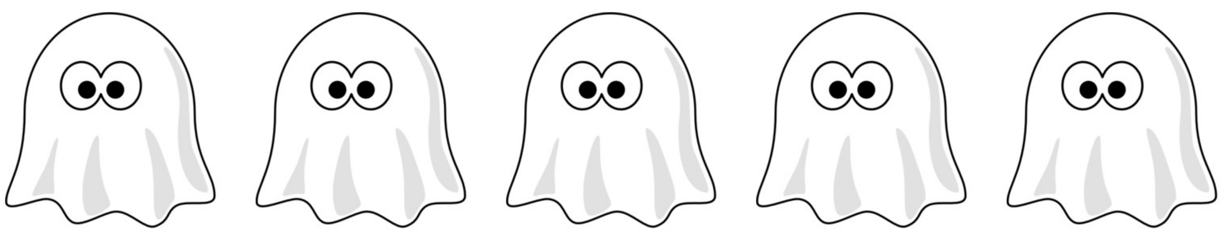
Round #	1	2	3	4	5	6	7	8	9	10
Player A										
Player B										
Total T										

Problem 3: A Frightful Halloween

Todd, Eileen, Jane, and Eric decide to go to a haunted house for Halloween. Todd, who is deathly scared of ghosts, asks Eileen, Jane, and Eric to go through the haunted house before him and count all the ghosts they see. Eileen, Jane, and Eric agree to this, but they decide to play a trick on Todd. They come up with this plan: Eileen will count the ghosts by 3's, Jane will count the ghosts by 5's, and Eric will count the ghosts by 19's. After getting through the haunted house they come back to Todd:

Eileen says: "I counted the ghosts by 3's and there were 2 left over. "
Jane says: "I counted the ghosts by 5's and there were 3 left over."
Eric says: "I counted the ghosts by 19's and there were 7 left over."

What is the smallest number of ghosts Todd should be prepared for? What if Eileen, Jane, and Eric are really mischievous and say: "There are at least 500 ghosts and at most 1500 ghosts" (this is a big haunted house!). How many ghosts should Todd watch out for? (There is more than one number!)



Problem 4: Even, Threeven, and Throdd Oh My!

0, 1, 2, 3, 4, 5, 6, 7...

E, O, E, O, E, O, E, O...

In Miss Cardinality's fascinating Math class, Jack and Danielle discuss the properties of even and odd numbers. "Jack, did you know that an even number times and odd number is even? Or that an even number times an even number is even?" says Danielle. Jack responds, "No, I didn't! But did you know that an even plus an odd is odd? Or that an even plus an even is even?" To which Danielle responds, "No, I didn't realize that! How cool!"

Terry, who fancies himself as an amateur philosopher, overhears this conversation and can't help himself: "But how do you guys *know* that an even times an odd is even? Or that an even plus an odd is odd? If I give you a one-hundred digit even and a two-thousand digit odd could you prove to me that if we multiply them we will get an even number and if we add them, we will get an odd number? How can you say that you know 21031240128492141254184912840129021912484146 times 1129048219047821940759073518273589235728357 is odd??"

Jack and Danielle, who are human, and hence smarter than calculators, decide they do not want to compute

$$21031240128492141254184912840129021912484146 \times 1129048219047821940759073518273589235728357$$

Instead, they want to find a clever way to prove to Terry that they are indeed correct. Can you help them out? In addition, figure out (and prove!) the following rules:

a) even + even = ?	**b)** even + odd = ?	**c)** odd + odd = ?
d) even × even = ?	**e)** even × odd = ?	**f)** odd × odd = ?

After disproving Terry's doubts so handedly, Jack and Danielle are having so much fun, they decide to call a number

* *threeven* if it is divisbile by 3
* *throdd1* if it leaves a remainder of 1 when divided by 3
* *throdd2* if it leaves a remainder of 2 when divided by 3

Using these definitons, they figure out rules for adding and multiplying like above. Can you figure them out also?

a) threeven + throdd1 = ?	**b)** threeven + throdd2 = ?	**c)** throdd1 − throdd2 = ?
d) throdd2 − throdd1 =?	**e)** threeven × throdd1 = ?	**f)** throdd1 × throdd2 = ?

Problem 5: Rules Are Made for . . . Division?

Cole Sear claims he has a sixth-sense and can tell if a number is divisble by 3. “I see divisors of three” he claims. Johnathan is highly suspect of this and so challenges Cole to a divisbility-of-3-duel.

Johnathan: “84” Cole: “Yes”

Johnathan, goes and checks and sees that $84 = 3 \cdot 28$. “Well that was an easy one, let’s try something harder” thinks Johnathan.

Johnathan: “189” Cole: “Yes”

Again, Johnathan checks and sees that $189 = 3 \cdot 63$. “Ok, let me give him one that’s not divisible by 3” Johnathan says to himself.

Johnathan: “973” Cole: “No”

“Wow! He actually is pretty good” thinks Johnathan.

Johnathan: “1539”	Cole: “Yes”
Johnathan: “2762”	Cole: “No”
Johnathan: “1011457”	Cole: “No”
Johnathan: “333699817245”	Cole: “Yes”

Johnathan checks and all these are correct too! How is Cole doing this?

There is a rule for divsion by 9 and 11 as well. Can you think of what the rule for 9 would be? What about the rule for division by 11?

Problem 6: A Calendar for the Ages

The Gregorian calendar, which most nations use today, is based on a year containing 365 days. Every fourth year is a leap year (containing 366 days),and years divisible by 100 if and only if they are also divisible by 400. For example, 1900 was not a leap year, but 2000 was.

a) Show that the calendar repeats itself every 28 years that do not include a turn of the century which is not a leap year.

b) What day of the week was January $1, 1901$? (January $1, 2013$ was a Tuesday).

c) In what years of the 20th century does February have five Sundays?

d) What day of the week, if any, can never be February 29?

2013

January

MON	TUE	WED	THU	FRI	SAT	SUN
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

February

MON	TUE	WED	THU	FRI	SAT	SUN
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28			

March

MON	TUE	WED	THU	FRI	SAT	SUN
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

April

MON	TUE	WED	THU	FRI	SAT	SUN
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30					

May

MON	TUE	WED	THU	FRI	SAT	SUN
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

June

MON	TUE	WED	THU	FRI	SAT	SUN
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30

July

MON	TUE	WED	THU	FRI	SAT	SUN
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

August

MON	TUE	WED	THU	FRI	SAT	SUN
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

September

MON	TUE	WED	THU	FRI	SAT	SUN
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30						

October

MON	TUE	WED	THU	FRI	SAT	SUN
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

November

MON	TUE	WED	THU	FRI	SAT	SUN
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	

December

MON	TUE	WED	THU	FRI	SAT	SUN
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30	31					

Problem 7: Bonus

Some More Useful Facts About Modular Arithmetic

* Let p be a prime number, and n be an integer not divisible by p. If for some integers a, b, we have $na \equiv nb \pmod p$ then $a \equiv b \pmod p$. (In essence, we can "divide" by n).
* Let p be a prime number and a an integer not divisible by p. Then $a^{p-1} \equiv 1 \pmod p$.
* If p is a prime number then for any integer a, $a^p \equiv a \pmod p$.
* $a \equiv b \pmod n$ if and only if $a - b$ is divisible by n if and only if $a = kn + b$ for some integer k
* If p is a prime and p divides ab then p divides a or p divides b

Use these facts, and some ingenuity to solve the following problems:

1) Find the remainder when 2^{100} is divided by 101.

2) Find the remainder when 3^{102} is divided by 101.

3) Prove that $300^{3000} - 1$ is divisible by 1001.

4) Prove that $7^{120} - 1$ is divisible by 143.

5) The sum of the numbers a, b, and c is divisible by 30. Prove that $a^5 + b^5 + c^5$ is divisible by 30.

6) Let p be a prime number, and a, b be two integers. Prove that $(a+b)^p \equiv a^p + b^p \pmod p$. (Notice how much easier this is! $(3a+2b)^5 = 243a^5+810a^4b+1080a^3b^2+720a^2b^3+240ab^4+32b^5$ but $(3a+2b)^5 \equiv 3a + 2b \pmod 5$)

7) Show that if $a \equiv b \pmod 3$ then $\frac{2}{3}(a^2+ab+b^2)$ can be written as the sum of three non-negative squares.

8) Let n be a natural number not divisible by 17. Prove that either $n^8 + 1$ or $n^8 - 1$ is divisible by 17.

9) Let p be a prime not equal to 3. Prove that the number $111\ldots1$ (p ones) is not divisible by p.

10) Let $p > 5$ be a prime. Prove that the number $111\ldots1$ ($p - 1$ ones) is divisible by p.

11) Let p be a prime. Prove that $1 \cdot 2 \cdot 3 \cdots (p-1) \equiv -1 \pmod p$.