HIGH SCHOOL MATH CIRCLE - UCI
February 2 2014

Total: 27 problems. http://www.math.uci.edu/~mathcircle/

Part 1 of 3 (problems 1 to 8)

(**1**) **Chessboard v.1**: Divide the chessboard into two connected parts so that the first part is 4 squares more than the second one, but the second part contains 4 black squares more than the first one. A part is considered connected if it stays in one piece; connection must be by a segment.

(**2**) **Painting a fence**: From Monday to Friday a worker painted a fence in 8 hours' shifts. On Monday he worked twice as slow as during midweek (Tuesday, Wednesday, Thursday). On Friday he worked twice as fast as during midweek and finished his job after 6 hours. Assuming that he painted 300 meters more on Friday than on Monday, how long was the fence?

(**3**) **4-digit numbers**: Determine the number of 4-digit numbers each composed of distinct digits with the first digit divisible by 2 and the sum of the first and the last digit divisible by 3.

(4) Birthdays: The Simpsons family celebrates only those birthdays when one's age equals to the sum of the digits of his/her birth year. Adam's celebration was in 2013 and Betty's celebration was in 2014. Who is older, and by how many years?

(**5**) **Crepes and Honey**: Karlsson bought in a cafeteria several crepes (25 rubles per piece) and several jars with honey (340 rubles per jar). When he told Smidge the total amount he spent, Smidge was able to determine the number of crepes and the number of jars with honey. Can it happen that Karlsson spent more than 2000 rubles?

(**6**) **Treasure v.1**: Brothers found a treasure of gold and silver. They divided it so that each share was 100 kg. The oldest brother got 30 kg (more than anyone else) of gold and one fifth of all the silver. How much gold was there in the treasure?

(**7**) **Chessboard v.2**: Divide the chessboard into two connected parts so that the first part is 6 squares more than the second one, but the second part contains 6 black squares more than the first one. A part is considered connected if it stays in one piece; connection must be by a segment.

(**8**) **5-digit numbers**: Determine the number of 5-digit numbers each composed of distinct digits with the first digit divisible by 2 and the sum of the first and the last digits divisible by 3.

Part 2 of 3 (problems 9 to 18)

(**9**) **Exchange rates**: At the beginning of year an exchange rate of US dollar to euro was 0.8. An expert predicted that during this year an exchange rate euro to ruble would increase by 8% while a rate US dollar to ruble would drop by 10%. If his prediction is correct, what would be an exchange rate of US dollar to euro by the end of the year?

(**10**) **Divisors v.1**: Ann and Betty thought of a number (each of her own). Then each girl wrote all the divisors of her number: Ann wrote 10 numbers and Betty wrote 9 numbers. How many distinct numbers were written on the board if the greatest number written twice was 50?

(**11**) **Maximizing area v.1:** A closed broken line is built along the lines of a grid, with its total length equal to 36 cell sides. What is the maximal area bounded by this line?

(**12**) **Three chords**: Consider a circle and three equal chords passing through one point. Prove that each chord is a diameter.

(**13**) **Treasure v.2**: Brothers found a treasure of gold and silver. They divided it so that each share was 100 kg. The oldest brother got 25 kg (more than anyone else) of gold and one eighth of all the silver. How much gold was there in the treasure?

(**14**) **Villages v.1**: The distance between two villages A and B is 45 km. Three friends have two bicycles, the speed of a cyclist is 15 km/h and the speed of a hiker is 5 km/h. What is the minimal time needed for them to go from A to B? Two people cannot ride the same bike simultaneously and they cannot leave the bike on the road unattended.

(**15**) **Sum plus product**: Lev took two natural numbers and added their sum to their product, getting 1000. Which numbers might those be? Find all possible pairs.

(**16**) **Divisors v.2**: Alex and Ben thought of a number (each of her own). Then each girl wrote all the divisors of her number, Ann wrote 10 numbers and Betty wrote 9 numbers. How many distinct numbers were written if both students wrote number 6?

(**17**) **Treasure v.3**: Brothers found treasure of gold and silver. They split it so that each weighed 100 kg. The oldest brother got 1/5 of all gold, and 1/7 of all silver. The youngest brother got 1/7 of all gold. What part of all silver did the youngest brother get?

(**18**) **Maximizing area v.2**: A closed broken line is constructed along the lines of a grid, with its total length equal to 2014 cell sides. What is the maximal area bounded by this line?

Part 3 of 3 (problems 19 to 27)

(**19**) **Villages v.2**: The distance between two villages A and B is 45 km. Three friends have two bicycles, the speed of a cyclist is 15 km/h and the speed of a hiker is 5 km/h. What is the minimal time needed for them to go from A to B? Two people cannot ride the same bike simultaneously but they can leave the bike on the road unattended.

(**20**) **A pentagon**: Consider a convex pentagon. For each pair of its diagonals intersecting inside, consider the smallest angle between them. Find all possible values of the sum of all these five angles.

(**21**) **Disk, v.1**: A disc of radius 1 is given. Prove that one can find 3 non-overlapping pieces of the disc which could be rearranged into a rectangle with dimensions 1 × 2.4. One can rotate and turn over the pieces.

(**22**) **2014-digit numbers**: Let a and n be natural numbers. Given that a^n is a 2014-digit number, find the smallest positive integer k such that a cannot be a k-digit integer.

(**23**) **Pavel-sum v.1**: Pavel invented a new way to add numbers. For two numbers a and b, the pavel-sum is given by: $a \# b \ := \ (a+b)/(1-ab)$ (if it is defined). He gave three numbers a, b and c to Boris and Michael and asked Boris to pavel-add a and b and then pavel-add c to the result, while Michael was asked to pavel-add b and c, and then pavel-add a to the result. Could Boris and Michael get different results?

(**24**) **Solving for** x: Let $f(x) = x^3 + 9x^2 + 27x + 24$. Solve the equation $f(f(f(f(x))))=0$.

(**25**) **Disk, v.2**: A disc of radius 1 is given. Prove that one can find four non-overlapping pieces of the disc which could be rearranged into a rectangle of dimensions 1 × 2.5. One can rotate and turn over the pieces.

(**26**) **Inner squares**: 100000 squares were drawn inside a given square with the side 100. Diagonals of distinct inner squares do not intersect. Prove that at least one of the inner squares has a side length less than 1.

(**27**) **Pavel-sum v.2**: Pavel invented a new way to add numbers. For two numbers a and b the pavel-sum is given by $a \# b \ := \ (a+b)/(1-ab)$ (if it is defined). He gave four numbers a, b, c and d to Boris and Michael and asked Boris to pavel-add a and b, then pavel-add c, and finally pavel-add d to the result, while Michael was asked to pavel-add c and d, and then pavel-add b, and finally pavel-add a. Could Boris and Michael get different results?