

## CHAPTER 3

### Pick's Theorem

Not a great deal is known about **Georg Alexander Pick** (Austrian Mathematician; 1859 - 1942), but it seems he did play some nontrivial role in the early career of Albert Einstein:

Pick was the driving force behind the appointment and Einstein was appointed to a chair of mathematical physics at the German University of Prague in 1911. He held this post until 1913 and during these years the two were close friends. Not only did they share scientific interests, but they also shared a passionate interest in music. Pick, who played in a quartet, introduced Einstein into the scientific and musical societies of Prague. In fact Pick's quartet consisted of four professors from the university including Camillo Krner, the professor of mechanical engineering.<sup>1</sup>

Suggestions have been made that he played a direct role in the development of Einstein's *general relativity theory* as Pick introduced him to some of the essential work in differential geometry of the time.<sup>2</sup> While Einstein was able to emigrate to the United States through a position at Princeton University, Pick could not avoid the Nazis. He was sent to the Theresienstadt concentration camp where he perished on 26 July 1942.

Pick is largely remembered for a beautiful geometric result he discovered in 1899, but was not widely known until it was publicized in the book Mathematical Snapshots by Steinhaus three-quarters of a century later.

Big Question - We would like to determine the areas and perimeters of GeoBoard polygons. Is there a way to do this simply by counting pegs?

1. Suppose you happened upon a real, two-dimensional object and you needed to determine its perimeter. What would you do?
2. Suppose now that you had to determine the area of the object. What would you do?
3. Which is easier to determine, the perimeter or the area? Does it matter what the polygon is or is there a general rule? Explain.

A **Geoboard** is a board of pegs arranged in a regular, rectangular array around which rubber-bands can be placed to create geometric polygons. Several images of Geoboards in action are shown in Figure 2. If you have access to one, it may be helpful below. Alternatively, you can use an online version (e.g. [http://nlvm.usu.edu/en/nav/frames\\_asid\\_277\\_g\\_1\\_t\\_3.html?open=activities&from=topic\\_t\\_3.html](http://nlvm.usu.edu/en/nav/frames_asid_277_g_1_t_3.html?open=activities&from=topic_t_3.html) or <http://www.mathplayground.com/geoboard.html>) or simply use dot-paper or graph-paper.

On a Geoboard the unit length is the distance between adjacent vertical (or horizontal) pegs. The unit area is the area of a square all of whose boundary pegs are pairwise adjacent.

4. Make a dozen or so rectangles on a Geoboard. For each determine the area and the perimeter.
5. Is there anything common about the area measurements or the perimeter measurements? Is this surprising?

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<sup>1</sup><http://www-history.mcs.st-and.ac.uk/Biographies/Pick.html>

<sup>2</sup>See e.g. [http://en.wikipedia.org/wiki/Georg\\_Alexander\\_Pick](http://en.wikipedia.org/wiki/Georg_Alexander_Pick).

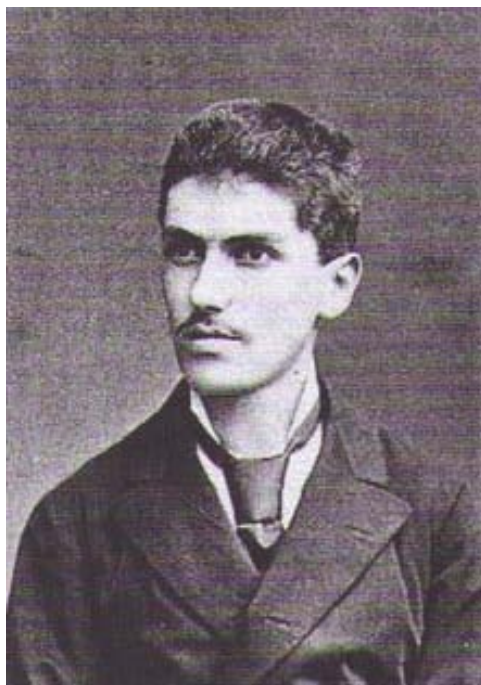


FIGURE 1. Georg Alexander Pick.

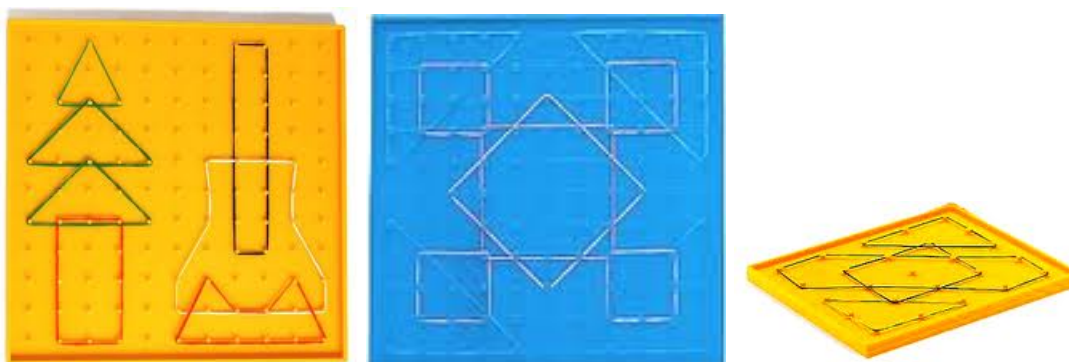


FIGURE 2. Geoboards.

6. For each of these same rectangles, record the number of *boundary pegs*, i.e. those that are touched by the rubber-band, and the number of *interior pegs*, i.e. those that are within the confines of the polygon created by the rubber-band.
7. Do you see a relationship which allows you to predict the area of a rectangle by knowing the number of boundary and interior points? Explain.
8. Do you see a relationship which allows you to predict the perimeter of a rectangle by knowing the number of boundary and interior points? Explain.
9. Do you think that any pattern you found above will hold for all rectangles? Either prove your result or find a counter-example to your conjecture.

10. If you have found a positive result, can you extend this result to other non-rectangular GeoBoard polygons? If not, explain. If so, describe carefully the class of objects for which the result holds and any limitations.
11. Now make a dozen or so triangles. For each, determine the area. Carefully describe your method and how you know that it is correct. (If you use a formula, you must justify the formula *and* each application of it.)
12. For each of your triangles, determine the perimeter. (Hint: You may need to remind yourself of the *Pythagorean theorem*.)
13. Is there anything common about the area measurements measurements of the triangles? Is this surprising?
14. Is there anything common about the perimeter measurements? Is this surprising?
15. How do these answers compare to what you reported in Investigation 5?
16. Is there a way to predict the perimeter of the triangles by knowing the number of boundary and interior points? Explain.
17. Is there a way to predict the area of the triangles by knowing the number of boundary and interior points? Explain.
18. For at least one of perimeter and area you should have found a positive result. Describe this result precisely with an algebraic formula.
19. Do you think that this result will hold for other GeoBoard polygons than rectangles and triangles? Explain.

A *heuristic argument* is one that uses analogy, insight or organized intuition to explain a result.

20. For a heuristic argument why the positive result may hold, which is more important to the measure in question (area or perimeter), boundary or interior points? I.e. if you wanted to increase the measure in question (area or perimeter), do boundary or interior points make a greater contribution? How much greater?

If you relied on the area formula for triangles that you learned in high school it may have limited your choice of triangles above. In fact, to find the area of a random triangle theoretically is fairly complicated unless the triangle is set up so the base and height are nearly obvious. Given the Cartesian (i.e.  $(x, y)$ ) coordinates of each of the vertices  $A, B, C$  of the triangle the area can be computed using the *surveyor's formula*

$$A = \frac{1}{2} [x_A y_B + x_B y_C + x_C y_A - x_B y_A - x_C y_B - x_A y_C],$$

which is also known as the *shoelace formula* or *Gauss' area formula* after **Carl Freidrich Gauss** (German mathematician; - ). If the lengths of each side of the triangle are known, the area can be computed using *Heron's formula*

$$A = \sqrt{\frac{l_A + l_B + l_C}{2} \left( \frac{l_A + l_B + l_C}{2} - l_A \right) \left( \frac{l_A + l_B + l_C}{2} - l_B \right) \left( \frac{l_A + l_B + l_C}{2} - l_C \right)},$$

where  $l_A, l_B, l_C$  are the lengths of the triangle's sides, which was discovered by **of Alexandria Heron** (Greek mathematician and engineer; 10 AD - 70AD).

Triangular area doesn't seem quite as easy as the formulas makes it out to seem, does it?

21. Determine the area of the area of the Geoboard polygon on the left of Figure 3. Explain your reasoning in detail.
22. Determine the area of the area of the Geoboard polygon in the center of Figure 3. Explain your reasoning in detail.
23. Determine the area of the area of the Geoboard polygon on the right Figure 3. Explain your reasoning in detail.

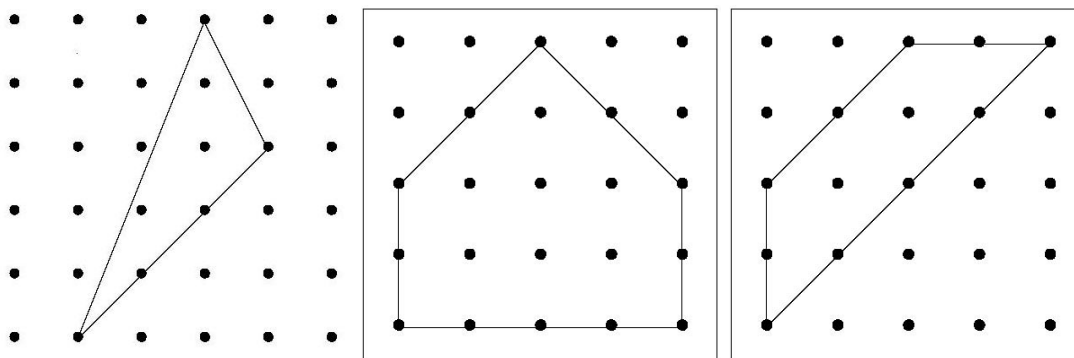


FIGURE 3. Geoboard polygons.

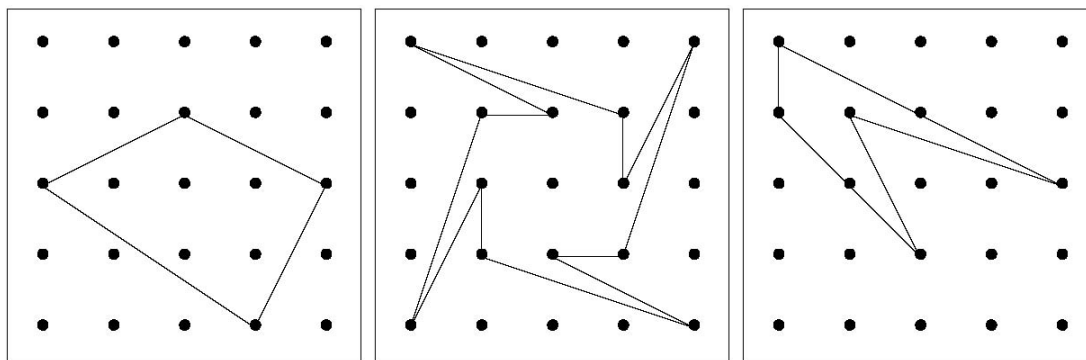


FIGURE 4. Geoboard polygons.

24. Determine the area of the area of the Geoboard polygon on the left of Figure 4. Explain your reasoning in detail.
25. Determine the area of the area of the Geoboard polygon in the center of Figure 4. Explain your reasoning in detail.
26. Determine the area of the area of the Geoboard polygon on the right Figure 4. Explain your reasoning in detail.

The positive result that you found in Investigation 18 is called *Pick's theorem*.

27. Does Pick's theorem hold for the polygons in Figure 3? Explain.
28. Does Pick's theorem hold for the polygons in Figure 4? Explain.
29. Use Pick's theorem to explain why the observation in Investigation 13 is valid.
30. Do you think that Pick's theorem will hold for all Geoboard polygons? Explain.

### 1. Applications of Pick's Theorem

Whether we and our politicians know it or not, Nature is party to all our deals and decisions, and she has more votes, a longer memory, and a sterner sense of justice than we do.

Wendell Berry (American Poet; - )

Documenting the amount of ice at the Earth's poles is of fundamental importance in our understanding of the causes and impact of global warming. Figure ?? shows the extent of the polar sea ice in 1979 and 2012.

## 2. Generalizing Pick's Theorem

In May, 1991 the National Council of Teachers of Mathematics published an article in its journal *Mathematics Teacher*, a journal for grade 6 - 14 mathematics teachers. The title of the article is "Pick's Theorem Extended and Generalized."

Many Geoboards have a square grid on the front and a different grid on the back. Typical alternatives are equilateral triangle lattices and hexagonal (like honeycomb) lattices. The article shows how Pick's formula can be adapted to hold for these other shapes as well.

The abstract of the article reads:

Our mathematics teacher set us an assignment to investigate Pick's theorem for square lattices. When I finished early, he suggested that I extend the theorem to cover triangular lattices, which I was able to do without too much trouble. The next step was obvious - could I extend the theorem to all lattices? The hexagonal lattice proved difficult, but I worked on it that night and the next day. While I was walking home from school, I realized that if I introduced a third variable for the type of lattice, I had a formula to cover all lattices. I arrived at the formula through a process of seeing patterns and guessing and checking, and I am not, at this stage, able to give a more mathematical proof.

Typically mathematical articles are not published without containing "a more mathematical proof." But this was not a typical article. The author here, Christopher Polis, was *an eighth grade student* when he discovered how to extend Pick's theorem to general Geoboard lattices of different shapes!

## 3. Proving Pick's Theorem

In mathematics, if a pattern occurs, we can go on to ask, Why does it occur? What does it signify? And we can find answers to these questions. In fact, for every pattern that appears, a mathematician feels [s]he ought to know why it appears.

W. W. Sawyer (; -)

Pick's theorem is surprising in its simplicity. But why does it work? Mathematicians are not satisfied that a result is true until there is a proof. Not compelled that a proof is really necessary after all of the evidence supporting Pick's theorem above?

31. For the Geoboard polygon in Figure 6, determine the area, number of interior points and number of boundary points.
32. Does Pick's theorem hold for this figure?
33. For the Geoboard polygon in Figure ??, determine the area, number of interior points and number of boundary points.
34. Does Pick's theorem hold for this figure?
35. Can you explain why these *counterexamples* arise, at least intuitively?
36. Do these examples bother you? Explain.

In Investigation 20 you were asked to think about how adding boundary or interior points contributed to an increase in area. Let's try to apply this heuristic to Pick's formula more formally.

37. On graph paper draw a Geoboard rectangle. Find the area, number of boundary points and number of interior points.
38. Draw vertical and horizontal dotted/colored lines exactly between each of the lines on your graph paper.

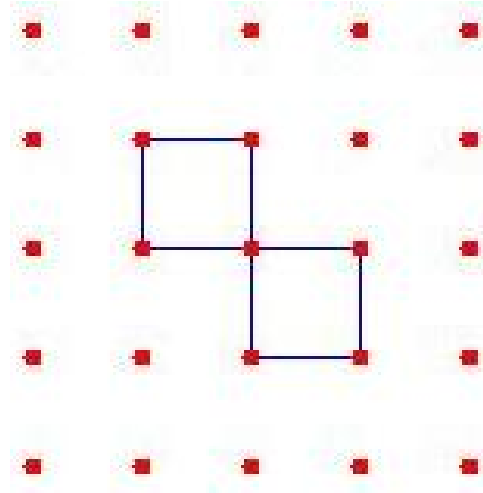


FIGURE 5. Does Pick's theorem hold?

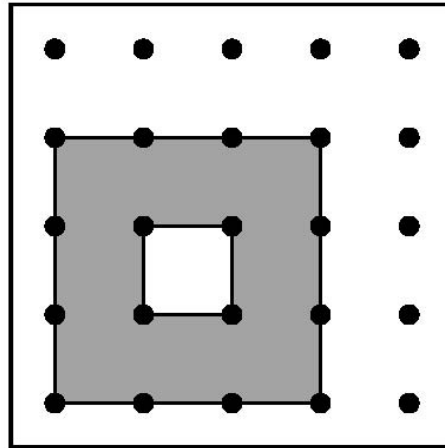


FIGURE 6. Does Pick's theorem hold?

39. Explain why your new figure shows that each interior point contributes exactly one unit of area to the total.
40. Does your figure show that each boundary point contributes exactly one-half unit of area? Explain.
41. Use your figure to prove Pick's theorem for rectangles.

So here's the question - Can we extend this to general Geoboard polygons? It turns out we can.<sup>3</sup>

For each point, boundary and interior, of a Geoboard figure, define the **visibility measure** at that point to be the relative percentage you can see into the interior of the Geoboard figure from that point.

<sup>3</sup>This approach is from "Pick's Theorem Revisited" by Dale E. Varberg, *American Mathematical Monthly*, vol. 92, no. 8, October 1985, pp. 584-7.

For example, for the corner of a rectangle the visibility measure is  $\frac{1}{4}$ . For any non-corner peg on the boundary of a rectangle the visibility measure is  $\frac{1}{2}$ .

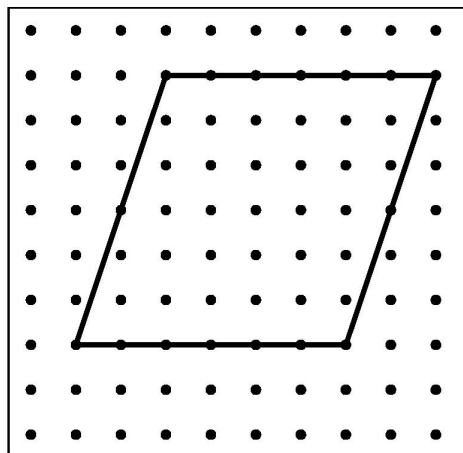


FIGURE 7. Geoboard quadrilateral.

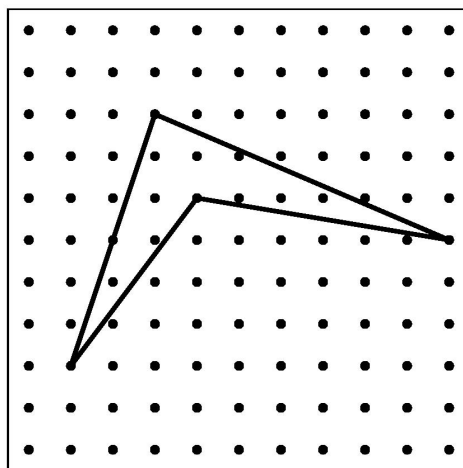


FIGURE 8. Geoboard quadrilateral.

42. Find the visibility measure of each interior and boundary point in Figure 3.
43. For each Geoboard figure, add up all of the visibility measures. What do you notice?
44. Find the visibility measure of each interior and boundary point in Figure 4.
45. For each Geoboard figure, add up all of the visibility measures. What do you notice?
46. Find the visibility measure of each interior and boundary point in Figure 7.
47. Find the area of the Geoboard polygon in Figure 7. How does the area compare to the sum of the visibility measures?
48. Find the visibility measure of each interior and boundary point in Figure 8.

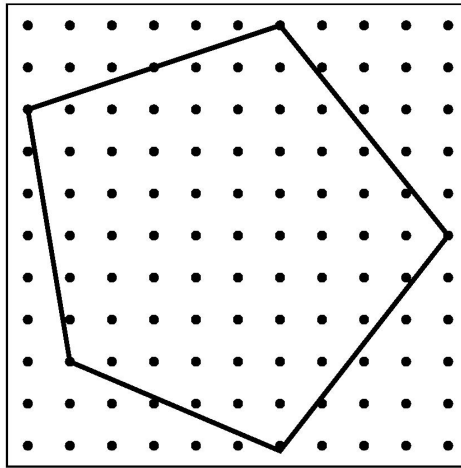


FIGURE 9. Geoboard pentagon.

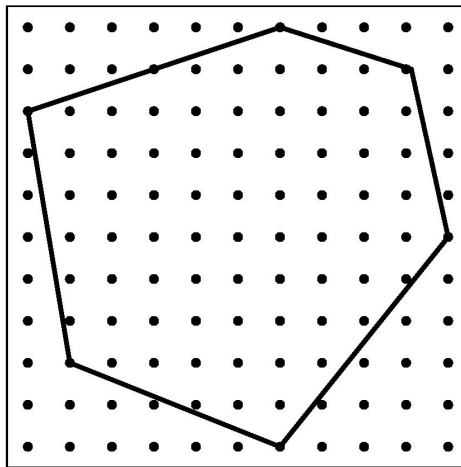


FIGURE 10. Geoboard hexagon.

49. Find the area of the Geoboard polygon in Figure 8. How does the area compare to the sum of the visibility measures?
50. Find the visibility measure of each interior and boundary point in Figure 9.
51. Find the area of the Geoboard polygon in Figure 9. How does the area compare to the sum of the visibility measures?
52. Find the visibility measure of each interior and boundary point in Figure 10.
53. Find the area of the Geoboard polygon in Figure 10. How does the area compare to the sum of the visibility measures?
54. Find the visibility measure of each interior and boundary point in Figure 11.
55. Find the area of the Geoboard polygon in Figure 11. How does the area compare to the sum of the visibility measures?



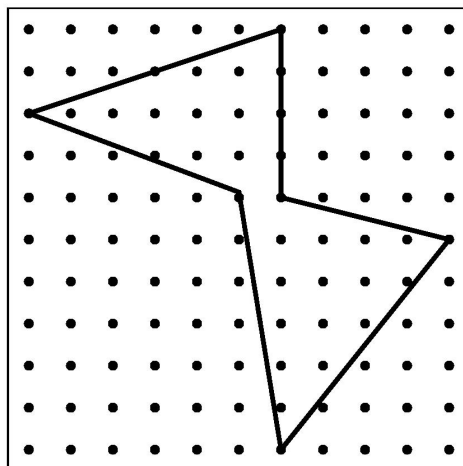


FIGURE 11. Geoboard hexagon.

56. You should notice that certain types of interior and boundary points have predictable visibility measures. What are they?
57. Do you think that the relationship between the area and the sum of the visibility measures will hold for all Geoboard figures? Why?

For any Geoboard polygon,  $S$ , define the **measure**  $M(S)$  to be the sum of the visibility measures of the interior and boundary visibility measures.

58. You've already proven that  $M(S) = A(S)$  whenever  $S$  is a rectangle. Explain.
59. Suppose the Geoboard polygon  $S$  is composed of two other Geoboard polygons  $T$  and  $R$  who may share boundaries but are otherwise non-overlapping. Prove that  $M(S) = M(T) + M(R)$ .<sup>4</sup>
60. Explain why, in the context of the previous problem,  $A(S) = A(T) + A(R)$ .
61. Explain why  $M(S) = A(S)$  must be true for right, Geoboard triangles.
62. Explain why it now follows that  $M(S) = A(S)$  for all Geoboard triangles.
63. Can all Geoboard polygons be decomposed into Geoboard triangles? Explain.
64. Combine these results to explain why it follows that  $M(S) = A(S)$  for all Geoboard polygons  $S$ .

We've now found an alternative way to measure areas of Geoboard polygons. How's this help us prove Pick's theorem? With rectangles and in Investigation ?? the visibility measures of points were very different depending on the type of point that we had.

65. For a given Geoboard figure, if you add together all of the visibility measures that equal 1, what is the sum equal to? (Equal to in relation to terms that are essential to Pick's formula.)
66. For a given Geoboard figure, if you add together all of the visibility measures that equal  $\frac{1}{2}$ , what is the sum equal to? (Equal to in relation to terms that are essential to Pick's formula.)

The angle formed on the inside of a Geoboard polygon where two straight line segments come together is called the **interior angle** of the polygon.

67. For several triangles you have already considered, determine each of their interior angles and the sum of their interior angles. What do you notice?
68. For several quadrilaterals you have already considered, determine each of their interior angles and the sum of their interior angles. What do you notice?

<sup>4</sup>This property is called *subadditivity*.

- 69. For several pentagons you have already considered, determine each of their interior angles and the sum of their interior angles. What do you notice?
- 70. For several hexagons you have already considered, determine each of their interior angles and the sum of their interior angles. What do you notice?
- 71. Make a conjecture about the sum of the interior angles of a polygon with  $n$  sides.

A *simple polygon* is a polygon whose sides are connected to each other one after another until closed without *intersecting*.

- 72. Figure ?? provides a “Proof Without Words” that the sum of the exterior angles in a simple polygon with  $n$  sides agrees with your answer above. Explain this proof. Illustrate how it works with several other Geoboard polygons you have considered above.
- 73. Do you find this proof to be compelling? Explain.
- 74. Does your formula for the sum of the interior angles apply to the Geoboard polygon in Figure 6? So is this a counterexample to our angle sum conjecture?
- 75. Does your formula for the sum of the interior angles apply to the Geoboard polygon in Figure ??? So is this a counterexample to our angle sum conjecture?
- 76. Complete the proof of Pick’s theorem by determining the sum of the visibility measures of the points whose visibility measures differ from 1 and  $\frac{1}{2}$ .

Story about the kid. Amateur mathematicians.

Lorenze and weather!!! Is all of this in the Reasoning book? It should be. Along with the Want of a Nail stuff.

#### 4. Teacher's Guide

The result that  $\text{Boundary} = \text{Perimeter}$  in 9 does indeed hold for all rectangles. There are several proofs, all of which are of interest.

One approach is simply to disconnect the boundary at a single peg. The boundary can be straightened out and it is clear that there is a one-to-one correspondence between each “peg” and unit of perimeter.

An algebraic approach is to notice that along each edge the number of pegs is one more than the length of that edge. So if  $p_{\text{base}}$  and  $p_{\text{height}}$  are the number of pegs along the base and the height of the rectangle, then  $p_{\text{base}} = b + 1$  and  $p_{\text{height}} = h + 1$ . The perimeter of the rectangle is  $P = 2b + 2h$ . We can add the pegs along each side of the rectangle, noting that this will count each corner twice, so that the total number of pegs is  $p_{\text{boundary}} = p_{\text{base}} + p_{\text{height}} + p_{\text{base}} + p_{\text{height}} - 4 = (b+1) + (h+1) + (b+1) + (h+1) - 4 = 2b + 2h + 4 - 4 = 2b + 2h = P$ .

An inductive approach is also very interesting. The result is clearly true for a unit square. To make a  $1 \times 2$  rectangle from the unit square, move the right side over one unit to the right. To complete the rectangle, two units of perimeter will need to be added and two more boundary pegs will need to be added. So we still have  $P = \text{Boundary}$ . In fact, if we move *any* edge of a rectangle one unit in a direction perpendicular to its length to make a larger rectangle, we will have to add exactly two new units of perimeter and two new boundary pegs. This enables us to create any rectangle from the initial square while insuring at each stage that  $P = \text{Boundary}$ .

Polygons like triangles certainly do not have  $P = \text{Boundary}$  and students are quick to assume that the result only holds for rectangles in . However, this result can naturally be extended to figures such as the one pictured in Figure 12. Caution must be used in generalizing too far however. Pick's theorem is equivalent to Euler's theorem for polyhedra (put ref here??), so all of the counterexamples and refutations that occur there also arise here. For example, the polygon pictured in Figure 13 does not satisfy  $P = \text{Boundary}$ , nor does it satisfy Pick's theorem.

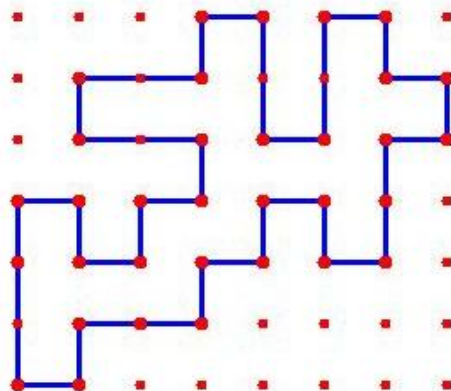


FIGURE 12. A non-rectangular Geoboard polygon for which  $P = \text{Boundary}$ .

As soon as triangles are introduced there will be questions of whether certain pegs are boundary pegs or not. For example in Figure 14 it is clear that there are, respectively, none and one boundary points on the hypotenuses (not including the vertices). It is *not* so clear on an actual Geoboard. The thickness of the rubber bands and the thickness of the pegs make several more points seem candidates for boundary points. This is a wonderful opportunity for students to think back about properties of

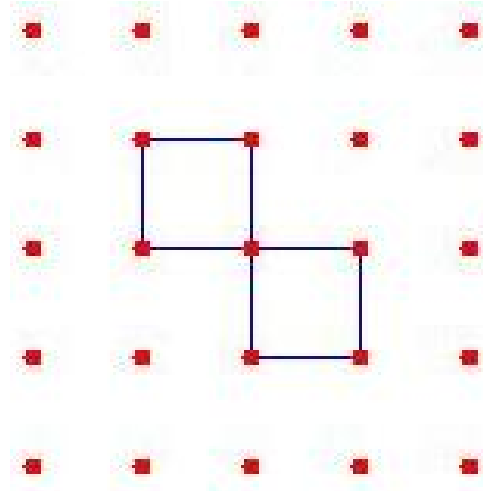


FIGURE 13. A counterexample to  $P = \text{Boundary}$  for *rectilinear Geoboard polygons* and Pick's theorem.

lines, including slope. (For example, see the video <http://artofmathematics.westfield.ma.edu/maa-focus/posters/BoundaryPointDiscussion.mov>).

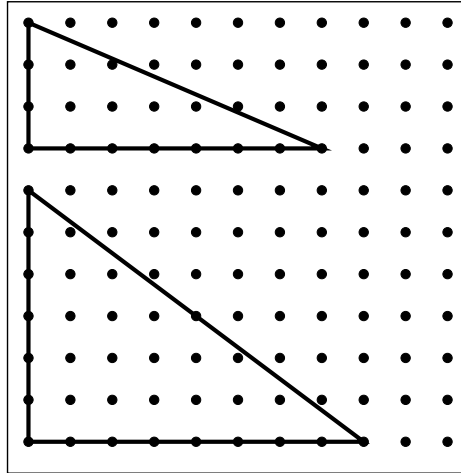


FIGURE 14. What constitutes a boundary peg on a Geoboard.

In trying to discover Pick's theorem the students often make very interesting geometric patterns (e.g. rectangles that are 3 by 4, then 3 by 5, then 3 by 6, etc., and then find interesting patterns. But not the more general relationship discovered by Pick. Often it helps to ask them how they are organizing their data. If it is just in the order they created the geometric polygons, other orders can help them. For example, consider all of the geometric polygons that have the same number of boundary pegs. How does the area vary then? Or consider all of the geometric polygons that have the same number of interior pegs. How does the area vary then?