Payoff Matrices

A payoff matrix is a way to express the result of players' choices in a game. A payoff matrix does not express the structure of a game, such as if players take turns taking actions or a player has to make a choice without knowing what choice the other will make.

Parts of a Payoff Matrix

Here is a payoff matrix for a game between two players. Each player has two actions they can take: Player 1 can choose either Right or Left, and Player 2 can choose Heads or Tails. More complicated games could have more rows or columns. For example, if Player 2 chose a color from the rainbow, there'd be seven columns for them instead of just two. Similarly, if a game has multiple choices per player, each possible combination of choices shows up on the table; if Player 1 had to choose a direction twice, they would have four rows: Right Right, Right Left, Left Right, and Left Left.

		Player 2	
		Heads	Tails
Player 1	Right	4 3	2 1
	Left	1 2	3 2

Every combination of choices from Player 1 and Player 2 produces an outcome in the grid. The number in the bottom left corresponds to the value for Player 1; the other is the value for Player 2. Bigger numbers are of course better. For example, in the first game, Player 2 prefers the outcome from Right and Heads over all others, while Player 1 is indifferent between Heads and Tails if they choose Left on their action.

		Player 2	
		Heads	Tails
Player 1	Right	9001 1	20 -10
	Left	1 -3	500 -3

For our purposes today, only the relative values matter. That is, we only care about each player's preference between each pair of options, not *how much* better a particular option is.

1. What are each player's relative preferences in each of the two games above?

2. If one choice is better than another for a player, no matter how that player's opponent chooses, the better choice is said to *dominate* the worse choice, and the worse choice is said to be *dominated* by the better choice. If a strategy is said to be *dominant* if it dominates all other strategies. Does either player have a dominant choice in either game?

Nash Equilibriums

A Nash equilibrium is a pair of strategies where neither player can benefit from being the only one to change their strategy. In the game to the right, the pair (Right, Tails) is not a Nash equilibrium, since Player 2 gets a better outcome by switching to Heads ($2 \to 4$) and Player 1 gets a better outcome by switching to Left ($1 \to 2$). This game has two Nash equilibriums. Can you find them?

		Player 2	
		Heads	Tails
Player 1	Right	4 3	2 1
	Left	1 2	3 2

For each of the following games, circle the Nash equilibriums and dominant strategies, if any exist.

		Player 2	
		Heads	Tails
Player 1	Right	9001 1	20 -10
	Left	1 -3	500 -3

		Player 2	
		Red	Blue
Player 1	Red	0 0	3 2
	Blue	2 3	0 0

		Player 2	
		Call	Fold
Player 1	Check	0 0	-6 6
	Fold	6 -6	6 -6

		Player 2	
		Drive	Walk
Player 1	Drive	-1 -1	2 1
	Walk	1 2	1 1

In the two bottom games, notice that in some cases, a player changing their strategy neither improves nor lessens their result. To deal with this case, we can say that domination is either *strong* or *weak*. In strong domination, the dominant strategy is strictly better than the other; in weak domination, the dominant strategy is never any worse than the other.

1. Is there a relationship between strong dominance and Nash equilibriums? How about weak dominance and Nash equilibriums?

2. Can there be more than one Nash equilibrium in a row or column? How?

Coordination Games

A coordination game is one where both players are better off cooporating.

Driving

The simplest type of coordination game is one where players are rewarded entirely for how much they cooperate. A good example of this is two oncoming drivers deciding what side of the road to drive on so that they don't crash. Fill out the payoff matrix for this situation.

What are the Nash equilibrium(s)? Are there any dominant strategies?

		Driver 2	
		Right	Left
Driver 1	Right		
	Left		

If two drivers on a narrow road had to choose at moment's notice which side to swerve to, how often do you think they'd avoid crashing? How does this related to laws about which side of the road people should drive on?

Pure Coordination

		Sibling 2	
		Beach	Home
Sibling 1	Beach		
	Home		

Two siblings want to meet up after school at either the beach or at home. They both would hate to not have a chance to hang out, and both prefer the beach. Fill out the payoff matrix for this situation.

What are the Nash equilibrium(s)? Are there any dominant strategies?

Do you think this will have a better outcome than the simple coordination case? Why or why not?

Why do you think this sort of coordination game is called a "pure coordination" game?

Battle of the Sexes

A husband and wife have agreed to go out for the night, but as they are leaving work, they realize that they forgot to decide whether to go to the opera or the football game. Unable to contact each other, they must choose where to go. If they each get a value 1 for going to their preferred activity (football for the husband, opera for the wife) and a value 2 for being in the same place as their spouse, fill out the payoff matrix. *Hint: Each spouse should have each value between 0 and 3 show up once as a payoff.*

		Wife	
		Opera	Football
Husband	Opera		
	Football		

What are the Nash equilibrium(s)? Are there any dominant strategies?

How do you think this will turn out compared to other the other coordination games? (Which activity is each spouse more likely to go to?)

The Stag Hunt

		Hunter 2	
		Stag	Hare
Hunter 1	Stag		
	Hare		

Two hunters go into the wood prepared to hunt either a stag or hares. A stag has at least four times as much meat on it as a day's worth of hares, but both hunters must work together to take the stag down. On the other hand, each hunter can reliably trap hares. Fill out the payoff matrix based on how much meat each hunter obtains.

What are the Nash equilibrium(s)? Are there any dominant strategies?

One of the choices is risky, but rewarding, while the other is reliable. Which one do you think is better?

Questions

1. In general, how many Nash equilibriums do coordination games have? Is there also a pattern for dominant strategies?

2. Do you think that the players in the coordination games will do better if they are able to communicate before making their choices? What does this say about the value of cell phones?

The Prisoners’ Dilemma

Two prisoners have been arrested for the same crime and are being held in separate cells, unable to communicate. There is not evidence to pin the worse crime on either of them, so both face a year in prison for a lesser crime. They are each offered a bargain: testify against your accomplice and you’ll be set free, but they’ll face the three-year sentence for the worse crime. However, if both rat the other out, they split the blame and serve two years each. Fill out the payoff matrix. *Hint: You can value each year in prison as -1.*

		Prisoner 2	
Prisoner 1			

1. Circle each Nash equilibrium and dominant strategy.
2. Are there any outcomes that are strictly better for both players than the Nash equilibrium(s)?
3. Does rational, selfish action always produce the best outcome? Why or why not?

During the Cold War, both NATO and the Warsaw Pact had the option to either make more nuclear weapons or reduce their nuclear stockpile. Not having an arsenal to match the other side would be a severe disadvantage, but if both sides ended up evenly matched, they’d save money by making fewer weapons.

		Warsaw Pact	
		More	Less
NATO	More		
	Less		

Rank the four outcomes from 4 (best) to 1 (worst) for each side and use that information to fill out the payoff matrix to the right. Historically, both sides kept making more weapons. Do you think this was the rational choice?

Anticoordination Games

An anticoordination game is one where both players are better off choosing opposite strategies.

Chicken

In the game of chicken, two people drive straight at each other, and the first one to swerve loses. If both swerve at the same time, the result is a tie, and if neither swerves, then the result is a disastrous crash. Use this information to fill out the payoff matrix, assuming that a crash is by far the worst option for both players.

		Driver 2	
		Swerve	Stay
Driver 1	Swerve		
	Stay		

What are the Nash equilibrium(s)? Are there any dominant strategies?

Given the high cost of crashing, would you swerve or stick to the game to the bitter end?

Bigger Questions

1. Do you think you could get a better outcome by signalling your intentions to your opponent before you start driving? This is called a *threat*. Which action would you threaten to do?

2. Do you think that a promise to take a particular action is enough, given that you could be lying to scare the other person off?

3. A threat is called *credible* if the person making it won't have an incentive to go back on it later. Is a verbal threat credible? How about disabling your car's steering?

4. Will a credible threat work if the other player isn't aware of it? (e.g. If you did something to your car in secret.)

Challenge Problems

1. Construct a game without any Nash equilibriums.

		Player 2	
Player 1			

2. Construct a game where one player has a dominant strategy, but that strategy always produces a better outcome for the other player.

		Player 2	
Player 1			

3. Construct a game where each player is indiferent between their own choices, but not the other player's choices. Do you think this would be an interesting game to play?

		Player 2	
Player 1			

4. Construct a game where a pair of irrational players can do better than a pair of rational players. *Hint: You've already seen at least one game like this.*

		Player 2	
Player 1			

5. Construct a game where both players have a strictly dominant strategy, but the resulting Nash equilibrium is not the best outcome for either player. (This game is called Deadlock.)

		Player 2	
Player 1			

6. Construct a game where both players have a weakly dominant strategy.

		Player 2	
Player 1			

Mixed Strategies

A mixed strategy is one in which each strategy is played with fixed probability. (A pure strategy can be seen as a mixed strategy where one of the probabilities is 1 and the others are all 0.) A mixed strategy equilibirum is one in which both players use mixed strategies.

With two players, the mixed strategy equilibrium will occur when each player's choice makes the other player indifferent between the two options. Let's use the first game as an example and find Player 1's equilibrium mixed strategy. Let p be the probability that Player 1 picks Right. It immediately follows that Player 1 picks Left at probability $1-p$. To solve for p, find the expected value for Heads and Tails, and set them equal.

		Player 2	
		Heads	Tails
Player 1	Right	4 3	2 1
	Left	1 2	3 2

$$E(Heads) = 4p + 1(1-p) = 1 + 3p$$

$$E(Tails) = 2p + 3(1-p) = 3 - p$$

so

$$1 + 3p = 3 - p$$

Thus $p = \frac{1}{2}$. Hence Player 1 will reach the mixed strategy equilibrium when they play Right and Left with equal probability. A similar computation shows that Player 2 should also be equally inclined to play Heads and Tails at the equilibrium. By plugging these values back into the formulas for the expected values, we expect Player 1 to average a value of 2 and Player 2 to average a value of 2.5.

1. Find the mixed strategy equilibrium for each game you've already solved. How do the expect values of the mixed strategy line up with the pure strategy equilibriums (if any)?

2. Do you think there will always be a Nash equilibrium (pure or mixed) for every 2-player game? Why or why not?

Ten-Penny Game #1: One-time Offer

Two Players

Never play this game with the same person more than once.

1. Start with either ten pennies or a nickel and five pennies. Flip a coin to determine who will be Player 1 and who will be Player 2. Player 1 starts with with all ten cents.

2. Player 1 offers Player 2 a portion of the ten cents.

   - If Player 2 accepts the offer, split the money according to the offer.
   - If Player 2 rejects the offer, return all ten cents to the pot.

3. Fill out the table with the results.

Player 1	Player 2	Offer	Your Winnings

1. What was the smallest offer that was made? What was the smallest offer that was accepted? How about the largest?

2. Assuming you are trying to get as much money as possible, how small of an offer should you take as Player 2?

3. Did you ever make an offer like that as Player 1? Did anyone take it?

4. If you refused such an offer (or an ever better offer), why did you do it? Is there more at stake than the money?