Random walks and even more peregrinations

Geoff Galgon, Garrett Ervin

# 1 Counting and walking preliminaries

For this math circle, it will help to know how to count the number of ways to pick $k$ objects from a group of $n$ objects. This number is denoted $\binom{n}{k}$. For example, the number of ways to choose a group of two people (say, a president and a vice president) from a group of seven people is $\binom{7}{2}$. The number of ways that exactly three heads can appear in a sequence of five coin flips is $\binom{5}{3}$. The number of ways to select a team of five people from a group of eleven is $\binom{11}{5}$. And so on. The formula for calculating $\binom{n}{k}$ is $\frac{n!}{(n-k)!k!}$. We will not prove this formula, but you may take it for granted. For example, we have $\binom{5}{2} = 10$ and $\binom{11}{8} = 165$.

**Question 1.1.** A tipsy man stands on a number line at the origin. Every time he takes a step, it is to the right with probability 1/2 and to the left with probability 1/2. So, after one step, it is equally likely that he stands at $-1$ and at $+1$. After two steps, what are his possible positions? Are they equally likely? If not, give the probability that the man stands at each of the possible positions.

**Question 1.2.** By a *path* we will mean a particular sequence of steps. So, if the man takes two steps, the possible paths are right-right, right-left, left-right, left-left. We will denote these paths as $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$ respectively. Each path *is* equally likely to occur. In the case of two steps, each path occurs with probability 1/4.

If the man walks three steps, how many possible paths are there, and what is the probability that each one occurs? In general, if the man walks $N$ steps, what is the probability that each particular path occurs?

**Question 1.3.** Let us denote a path of $N$ steps taken by the man as $(e_0, e_1, \dots, e_N)$, where every $e_i$ is either 1 or $-1$ to indicate either a step to the right or to the left, respectively. For such a path, the mans final position is given by $e_0 + e_1 + \dots + e_N$. If $N$ is an even number, it is possible that the man ends up back at his starting place (that is, at 0). For $N = 2$, we have seen that there are exactly two paths ending at 0. If $N = 4$, what is the number of paths ending at 0? What about for $N = 6$? In general, for $N$ an even number, what is the number of paths ending at 0? Let $T_n$ denote the number of paths that end at 0 after $N = 2n$ steps.

**Question 1.4.** Combine your answers from questions 2 and 3 to calculate, for a walk of even length $N$, the probability that the man ends up at 0.

**Question 1.5.** For a walk of length 3, find the probability that the man ends up at 1. What about for a walk of length 4? Of length 5? In general, for a walk of odd length $N$, calculate the probability that the man finishes his walk at $+1$.

**Question 1.6.** Generalize your calculations: For a walk of even length $N$, calculate the probability of ending at $m$ (for $m$ an even number between $N$ and $-N$). What about for odd $N, m$?

**Question 1.7.** Suppose now that the probability that our merry drunkard moves to the right at every step is 3/4 and the probability that he moves to the left is 1/4. Repeat the above with these new probabilities.

## 2 Further walking, three sheets to the wind

**Question 2.1.** A man who's had too much to drink stands at the edge of a cliff. If he walks forward from here he falls and meets his end. At each step, he walks forward with probability 1/3, and backwards with probability 2/3. What is the probability that the man eventually dies by walking off the cliff (assuming he is capable of taking infinitely many steps)?

Hint: Let $x_0$ denote the probability that he dies, given that he starts at the edge of the cliff. Let $x_1$ denote the probability that he dies given that he starts one step behind the edge of the cliff. See if you can find two equations which relate $x_0$ and $x_1$ and allow you to solve for $x_0$, the quantity in question.

**Question 2.2.** Solve the question above more generally: A man who's had too much to drink stands at the edge of a cliff. If if walks forward from the edge he falls and meets his end. He walks forward with probability $p$, and backwards with probability $1 - p$. What is the probability that the man eventually dies by walking off the cliff (assuming he is capable of taking infinitely many steps)? Draw a graph of these probabilities with the $y$-axis probability of death and the $x$-axis probability of moving forward with one step.

**Question 2.3.** In order to fall of the cliff, the man has to appear (back) at the edge of the cliff, and then take a step forward. Another method of computing the probabilities above first involves counting the number of paths that he can take which lead him back to the edge at step $N$ (even), as in the section above, *but only those in which he doesn't at any point step over the cliff.* How many paths are there with two steps allowed? How many paths are there with four steps?

**Question 2.4.** How many such paths (where he never falls off the cliff but returns to the edge at the end) are there with 6 steps? What about with 8 steps? Let $C_n$ denote the number of such paths for an even $N = 2n$.

**Question 2.5.** What proportion of the total number of paths that end up at the starting position do those which never cross over to one side ($C_n$) fill in these examples? That is, what is $\frac{C_n}{T_n}$?

**Question 2.6.** The "$C$" in the $C_n$ above stands for "Catalan," and these $C_n$ are called the Catalan numbers. There are many different equivalent ways of defining them. We have seen one here, as the number of paths starting and ending at 0 in $2n$ steps never crossing to the other side of 0. As another way of viewing these numbers, consider an $n \times n$ grid, and count the number of possible paths which travel from one corner to the diagonally opposite corner, never crossing the diagonal.

For example, for $n = 1$, the grid includes the vertices $\{(0,0),(0,1),(1,0),(1,1)\}$. If we have to travel from the vertex $(0,0)$ to the vertex $(1,1)$ without crossing the diagonal, there is only one path, $\langle(0,0),(0,1),(1,1)\rangle$. For $n = 2$, the grid includes the 9 vertices, $\{(0,0),\ldots,(2,2)\}$, and there are two paths that don't cross the diagonal. Explain why this characterization for $C_n$ works. That is, why is it equivalent to our previous formulation?

**Discussion Question 2.7.** Using the grid characterization given above, can you prove the formula you found for $C_n$ in Question 2.5? This is difficult, we will help you.

**Question 2.8.** Instead of the traditionally macabre framing adopted above, let's suppose that our man is walking on a number line, starting at the origin. Additionally, suppose that he as a $1/2$ chance of moving to the left or to the right. What is the probability that he returns to the origin eventually?

Hint: As done previously, it may be helpful to calculate the probability that he is at the origin after $N$ ($N$ is even) steps. Then the expectation of the number of times he hits the origin will be the infinite sum over all even $N$ of these probabilities (why?). What does this quantity tell you?

## 3 Discussion problems

**Question 3.1.** Suppose now that our man moves up to two dimensions, and on the Cartesian grid he can move up, down, left, or right with probabilities $1/4$, respectively. What is the probability that he returns to the origin?

**Question 3.2.** Suppose now that our man steps into three dimensions, and can move any of forward, backwards, left, right, up, or down, with probabilities $1/6$, respectively. What is the probability that he returns to the origin?

Note: This requires a computer to get an accurate number.