Game Theory

UCI Math Circle, Fall 2015

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(1) (Prisoner's Dilemma) Two members of a criminal gang are arrested and imprisoned. Each prisoner is in solitary confinement with no means of communicating with the other. The prosecutors lack sufficient evidence to convict the pair on the principal charge. They hope to get both sentenced to a year in prison on a lesser charge. Simultaneously, the prosecutors offer each prisoner a bargain. Each prisoner is given the opportunity either to: betray the other by testifying that the other committed the crime, or to cooperate with the other by remaining silent. The offer is:

- If A and B each betray the other, each of them serves 2 years in prison.
- If A betrays B but B remains silent, A will be set free and B will serve 3 years in prison (and vice versa).
- If A and B both remain silent, both of them will only serve 1 year in prison (on the lesser charge).

What kind of option would a rational criminal choose? If both criminals are assumed to be rational, what can we expect out of the situation? What situations in life are similar to this scenario.

(2) (Sequential Games) Sony and Toshiba are trying to decide which standard system they want to adopt: HD vs Blue Ray. Sony prefers Blue Ray and Toshiba prefers HD, but they both prefer that they are using the same system. Is there an outcome that is satisfactory to both of them? What if Sony has to decide which system they'll adopt first?

(3) (Mixed Strategies) Rock-Paper-Scissors can be modeled as a game played by two people who each have three options. Is there a winning strategy? If strategies are allowed to be any random combination of the three options, is there a strategy that can be considered "the best"? In what way?

(4) (Dollar Auction) Suppose that I auction off a dollar, starting at any price you want (say 1 cent). The only catch is that the second highest bidder must pay me the amount of money they bid as well, despite the fact that they get no dollar. How much should the dollar sell for?

Normal Form

A two-person game is a game played by two individuals, A and B, who choose from a collection of strategies. Each pair of strategies (A_i, B_j) is associated with a pair of utilities (a_i, b_j), so that if A takes on the strategy A_i and B takes on the strategy B_j then A is awarded with a_i utility and B is awarded with b_j utility. Players choose their strategies with the goal of maximizing their utility.

Let's say that A and B are playing the Prisoner's dilemma. Their first options, $A1$ and $B1$, are to stay silent. Their second options, $A2$ and $B2$ are to betray one another. Their choices and outcomes can be described in a 2×2 matrix, called the normal form of the game.

Prisoner's Dilemma

	B1	B2
A1	$(-1, -1)$	$(-3, 0)$
A2	$(0, -3)$	$(-2, -2)$

Keep in mind that the number on the left in the pairing represents the utility for A, and the number on the right represents the utility for B.

A zero-sum game is a game in which one player's gain is necessarily another players loss. Suppose players A and B play a game where A hides a button in one of their two hands, and B guesses where the button is. If B guesses correctly then A pays them \$2, otherwise B pays A \$1. To make things interesting, they agree that if the button was in A's left hand all along, they will double the payoffs. The Normal form of the game can be written as follows:

Button Button (Zero-Sum)

	B1	B2
A1 (Left)	$(-4, 4)$	$(2, -2)$
A2 (Right)	$(1, -1)$	$(-2, 2)$

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	B1	B2
A1	-4	2
A2	1	-2

Zero-sum games are typically written in normal form with just the payoffs for the first player A, because the payoff for B can be inferred.

- In the Prisoner's dilemma, suppose that person A is considering strategy $A2$. Assuming that B has already picked her strategy (which could be either one), should A ever considering changing her strategy to $A1$?

- Suppose you play a game of rock-paper-scissors against a friend, where the winner gives the loser a dollar, making it a zero-sum game. Write out the normal form of the game.

Rock, Paper, Scissors

	B1	B2	B3
A1			
A2			
A3			

- Suppose you play a game of chicken using cars with your friend, where each of you can take a strategy of swerve or stay. If one of you swerves and the other stays then the person who swerves is embarrassed, resulting in some loss (negative utility). If you both swerve then no one is embarrassed. If neither person swerves then your cars collide and are badly damaged, the worst situation for both of your. Write out the normal form of this game.

Chicken

	B1	B2
A1 (Stay)		
A2 (Swerve)		

- In the second example (2) on the first page, we talked about Sony and Toshiba trying to decide on whether to adopt HD DVD technology or Blue Ray. If the companies have to make the decision simultaneously, they each have two strategies: HD or Blue Ray. The company prefers their proprietary format (Sony likes Blue Ray and Toshiba likes HD), their next preference is for them to use their favorite format and their competitors to use the other, and their third choice is for both companies to use the same format (i.e. Sony's third choice is both of them using HD). Write out the normal form of the game.

	B1 (HD)	B2 (BR)
A1 (HD)		
A2 (BR)		

- What if in the last problem Sony chooses first. This is now a different kind of game, because Toshiba doesn't just have two strategies: it can use information about Sony's decision to make it's own. Find all of Toshiba's strategies and write down the normal form for this game.

	B1 (HD/HD)	B2 (HD/BR)	B3 (BR/HD)	B4 (BR/BR)
A1 (HD)				
A2 (BR)				

Dominant Strategies

Consider the following game in normal form:

	B1	B2	B3
A1	(4,3)	(5,1)	(6,2)
A2	(2,1)	(8,4)	(3,6)
A3	(3,0)	(9,6)	(2,8)

How can we tell what strategies the two players might consider playing?

We say that some strategy A_i for player A strictly dominates another strategy A_j if $U_A(A_i, B_k) > U_A(A_j, B_k)$ for all strategies B_k that B can choose. Here we use U_A to denote the utility for A, which is the left number in the pairing. This simply means that the choice A_i is always better for A than choosing A_j, regardless of what B does.

If a strategy dominates another, written $A_j > A_i$, then A should never have a reason to choose A_i. In the above game, $B3 > B2$ because $2 > 1$, $6 > 4$, and $8 > 6$. We can thus eliminate $B2$ as an option for B and construct a new game.

	B1	B3
A1	(4,3)	(6,2)
A2	(2,1)	(3,6)
A3	(3,0)	(2,8)

- Continued eliminating dominated strategies for B and A until you narrow down how the game is actually played. What strategies should A and B choose? How are these the "best" strategies?
- This is called the iterated elimination of strictly dominated strategies. What happens if we apply this method to the Prisoner's dilemma?
- The following game is called a stag hunt:

Stag Hunt

	B1	B2
A1	$(2, 2)$	$(0, 1)$
A2	$(1, 0)$	$(1, 1)$

Are there strictly dominating strategies in this game?

- Suppose that two crooks are meant to be playing a Prisoner's dilemma, but they are very prideful. In particular, not betraying is worth 4 years in prison to each of them. How does the game change? Write out the normal form.

	B1	B2
A1		
A2		

  Does this game feature strictly dominating strategies?

- Recall the conflict between Sony and Toshiba. If Sony chooses first, are there strictly dominating strategies? What are they, and what is the outcome of the game if both players are perfectly rational.

- The more realistic situation is that Sony was in a better position than Toshiba. Let's suppose that Sony is confident that if they use Blue Ray and Toshiba uses HD, Blue Ray will eventually catch on. We suppose Sony feels this way because of their use of Blue Ray in the PS3 and their successful marketing campaign. Sony's preferences and Toshiba's are different, so that the situation (if they made their decisions simultaneously) would be as follows.

	B1 (HD)	B2 (BR)
A1 (HD)	(2,4)	(1,1)
A2 (BR)	(3,2)	(4,3)

  Now suppose that Sony picks first, so Toshiba has four strategies again. Write out the normal form. Eliminate weak strategies and determine what should be the outcome of the game.

	B1 (HD/HD)	B2 (HD/BR)	B3 (BR/HD)	B4 (BR/BR)
A1 (HD)				
A2 (BR)				