Level 1 Math Circles
Beginning Probability

Lesson Plan by Alex Sutherland

October 3rd, 2016

Pass out coins. Have everyone flip their coin 5 times and record their results. Go around the class and get results. Record on the board with following tables:

0 Heads:
1 Heads:
2 Heads:
3 Heads:
4 Heads:
5 Heads:

Total Heads:
Total Tails:

Depending on time and materials, repeat with unfair coins.

Flipping Coins

Emma and Freddy are hanging out after school and flipping quarters. Freddy is about to flip another quarter.

1) How likely is it that the quarter lands heads up?

Answer: $\frac{1}{2}$

2) How likely is it that the quarter lands tails up?

Answer: $\frac{1}{2}$

3) What is the sum of these likelihoods?

Answer: $\frac{1}{2} + \frac{1}{2} = 1$

Imagine that Emma flips a coin and it lands heads up. She then flips another coin and it also lands heads up. Emma then flips a third coin and it lands heads up. She is about to flip a fourth coin.

4) What is the likelihood that the fourth coin lands heads up?

Answer: $\frac{1}{2}$

5) What is the likelihood that the fourth coin lands tails up?

Answer: $\frac{1}{2}$

6) What is the sum of these likelihoods?

Answer: $\frac{1}{2} + \frac{1}{2} = 1$

After Emma is finished flipping the fourth coin, Freddy collects the four coins and prepares to flip them.

7) How likely is it that all four quarters land heads up?

Answer: $\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{16}$

8) How likely is it that exactly three quarters land heads up?

Answer: There are 4C3=4 ways of getting exactly 3 heads. There are 2^4 possible sequences of coin flips. So, the probability is $\frac{4}{16} = \frac{1}{4}$.

9) How likely is it that exactly two quarters land heads up?

Answer: There are 4C2=6 ways of getting exactly 2 heads. There are 2^4 possible sequences of coin flips. So, the probability is $\frac{6}{16} = \frac{3}{8}$.

10) How likely is it that exactly one quarter lands heads up?

Answer: There are 4C1=4 ways of getting exactly 1 head. There are 2^4 possible sequences of coin flips. So, the probability is $\frac{4}{16} = \frac{1}{4}$.

11) How likely is it that no quarters land heads up?

Answer: There are 4C0=1 ways of getting exactly 1 heads. There are 2^4 possible sequences of coin flips. So, the probability is $\frac{1}{16}$.

12) What is the sum of these likelihoods?

Answer: $\frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16} = 1$

Emma and Freddy now have access to as many quarters as they like. Suppose Emma wants to flip n quarters.

13) What is the likelihood that all n quarters land heads up?

Answer: $\left(\frac{1}{2}\right)^n$

14) What is the likelihood that i of the n quarters land heads up?

Answer: There are nCi ways of getting exactly i heads up. There are 2^n possible sequences of coin flips. So, the probability of getting exactly i heads up is $\frac{nCi}{2^n}$.

15) What is the likelihood that the exact number of heads up quarters is less than or equal to i?

Answer: $\sum\limits_{k=1}^{i} \frac{nCk}{2^n} = \frac{1}{2^n} \sum\limits_{k=1}^{i} nCk$

Flipping Unfair Coins

Emma and Freddy find a stash of n unfair coins. Assume that each coin has probability P of landing heads up.

1) If Freddy flips a coin, what is the chance that coin i lands tails up?

Answer: $1 - P$

2) If Freddy flips all n coins, what is the chance that all coins land heads up?

Answer: P^n

3) If Freddy flips all n coins, what is the chance that exactly i of them land heads up?

Answer: $(nCi)P^i(1-P)^i$

Now, assume each coin has it's own probability of landing heads up (i.e that coin i has probability P_i of landing heads up).

4) If Emma flips a coin, what is the chance that coin i lands tails up?

Answer: $1 - P_i$

5) If Emma flips all n coins, what is the chance that all coins land heads up?

Answer: $\prod_{k=1}^{n} P_k$

6) If Emma flips all n coins, what is the likelihood that i consecutive coins land heads up?

Answer: $\sum_{k=1}^{n-i}(1-P_1)\dots(1-P_{k-1})P_k\dots P_{k+i}(1-P_{k+i-1})\dots(1-P_n)$

<u>Candy Time</u>

Gwendolyn has a bowl of red, yellow, orange and green candies. There are 5 red candies, 3 yellow candies, 6 orange candies, and 2 green candies. Gwendolyn is about to randomly pick a candy out of the bowl.

1) How likely is it that she will pick a yellow candy?

Answer: $\frac{3}{16}$

2) What is the chance that the candy she will pick is red or green?

Answer: $\frac{5}{16}+\frac{2}{16}=\frac{7}{16}$

Suppose Gwendolyn puts the first candy back in the bowl and then decides she will randomly pick four candies out of the bowl.

3) What is the likelihood that all four candies will be orange?

Answer: $\frac{6}{16}\left(\frac{5}{15}\right)\left(\frac{4}{14}\right)\left(\frac{3}{13}\right)=\frac{6(5)(4)(3)}{16(15)(14)(13)}=\frac{6}{4(14)(13)}=\frac{3}{364}$

4) What is the likelihood that all four candies will be green?

Answer: There are not 4 green candies, so 0.

5) What is the chance that the second and fourth candy will be red?

Answer: The probability that only candy 2 and 4 are red is
$\frac{11}{16}\left(\frac{5}{15}\right)\left(\frac{10}{14}\right)\left(\frac{4}{13}\right)=\frac{55}{1092}$

6) How likely is it that exactly two of the four candies will be red?

The probability that only candy 1 and 2 are red is
$\frac{5}{16}\left(\frac{4}{15}\right)\left(\frac{11}{14}\right)\left(\frac{10}{13}\right)=\frac{55}{1092}$

Observe that the positions of which exact spaces are red is irrelevant to the probability. As there are 6 pairs of positions ((1,2), (1,3), (1,4), (2,3), (2,4), (3,4)), the probability that exactly two of the four candies are red is $6\left(\frac{55}{1092}\right)=\frac{55}{182}$.

8) How likely is it that none of the four candies will be neither yellow nor green?

Answer: $\frac{11}{16}\left(\frac{10}{15}\right)\left(\frac{9}{14}\right)\left(\frac{8}{13}\right) = \frac{7920}{43680} = \frac{33}{182}$

<u>Other Problems</u>

1) Ingrid is showing Jennifer her three cards. One card is red on both sides, one card is green on both sides, and one card is red on one side and green on the other. Ingrid then puts the three cards behind her back. After shuffling the cards, Ingrid randomly picks a card and shows Jennifer one side of the card she picked. If the side shown to Jennifer is green, what is the likelihood that the other side of the card is green?

Answer: Observe that three are green sides. Two of the green sides have an opposite side that is green. Only one of the green sides have an opposite side that is red. As such, there is a $\frac{2}{3}$ chance that the other side of the card shown to Jennifer is green.

2) The USA is struck with a new deadly disease called Disease X. 1% of the population is infected with Disease X. However, the only way to know if someone has Disease X is to take a special medical test. However, this test is not perfect. If a sick person takes the test, there is a 98% chance it will return positive and and a 2% chance it will return negative. If a healthy person takes the test, there is a 97% chance it will return negative and a 3% chance that it will return positive. Kendrick lives in the USA and wants to know if he is sick. As a result, Kendrick takes the test and it comes back positive. What is the chance that Kendrick is sick?

Answer: There are two possible ways that Kendrick could have gotten a positive result, either he is sick and got a true positive or he is healthy and got a false positive. The chance of the first case is $0.01(0.98) = 0.0098$. The chance of the second case $0.99(0.03) = 0.0297$. Therefore, he is sick with probability $\frac{0.0098}{0.0098+0.0297} = \frac{0.0098}{0.0395} = 0.248$.

3) Consider the game where you are given 50 white marbles and 50 black marbles. The player puts all 100 marbles in two separate bowls, but can split up the marbles however they choose. They are then blindfolded and the position of the bowls are randomized. While still blindfolded, the player picks a bowl and then picks a marble from the bowl. If the player picks a white marble, they win. If they do not pick a white marble, they lose. Lisa gets to play this game. How should Lisa distribute the marbles to maximize her chances of winning?

Answer: Let w represent the number of white marbles in bowl one and b represent the number of black marbles in bowl one. Without loss of generality, $0 \leq w \leq 25$ and $0 \leq b \leq 25$.

The winning probability is $\frac{1}{2}\left(\frac{w}{w+b}\right)+\frac{1}{2}\left(\frac{50-w}{100-w-b}\right)$. Consider $w=1$, $b=0$.

$$\frac{1}{2}(1)+\frac{1}{2}\left(\frac{49}{99}\right)=\frac{99}{198}+\frac{49}{198}=\frac{148}{198}$$

Also, observe

$$\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{2}\left(\frac{49}{98}\right)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$
$$\frac{1}{2}(0)+\frac{1}{2}\left(\frac{50}{100}\right)=0+\frac{1}{2}=\frac{1}{2}$$
$$\frac{1}{2}\left(\frac{2}{2}\right)+\frac{1}{2}\left(\frac{48}{98}\right)=\frac{98}{196}+\frac{48}{196}=\frac{146}{196}$$

$$\frac{1}{2}\left(\frac{25}{50}\right)+\frac{1}{2}\left(\frac{25}{50}\right)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$