MATH CIRCLE ISOPERIMETRIC PROBLEMS

HUNG TRAN

1. Queen Dido and the City of Carthage

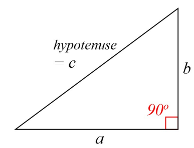
- From a story described in Virgil's *Aeneid*.
- Queen Dido escapes from her evil brothers and flees to North Africa.
- She purchases land from the local ruler, King Jarvas of Numidia. After negotiations, they agree that she could only have as much land as *she could enclose by an ox's hide.*
- Queen Dido has the hide cut into small strips and stitches them together.
- Obviously, she would like as much land as possible. And since she understands geometry, she outlines the shape with the biggest area possible.
- That's the beginning of the City of Carthage (now in Tunisia).
- What is that shape? In other words, find the shape with a given perimeter that encloses the largest area (**isoperimetric problem**) !!!

Figure 1. Queen Dido (From "Shape and Form of the Nature Universe" by Hilderbrandt-Troma)

2. Problems

Recall Pythagorean theorem for a right triangle:

$$a^2 + b^2 = c^2.$$



$c^2 = a^2 + b^2$

Also the trigonometric identity:

$$\sin\theta = \frac{a}{c}.$$

Warming Up:

1. Using graph papers, construct an isosceles triangle (side 5 and base 6), an equilateral triangle (side 16/3), a rectangle (sides 3 and 5), a square (side 4). Find the areas of those figures and find the one with most area.

2. Show that $\sin\theta \leq 1$. When does equality happen?

3. Suppose $a + b = 8$, find the maximum value of ab.

4. Along all rectangles of perimeter 16, find the one with the largest area.

5. Show that Area$(\triangle ABC) = \frac{1}{2}AB.AC.\sin\angle BAC$.

Accelerating: General triangles and quadrilaterals.

6. Let Δ be the triangle that maximizes area among all triangles with two sides $a > 0$ and $b > 0$. What is the shape of Δ? What is its area?

7. Let Δ be the triangle that maximizes area among all triangles with a base $a = 6$ and perimeter $p = 16$. What is the shape of Δ? What is its area? *Hint: Fix side AB of length a and draw the set of all points C such that the perimeter is p. What is the shape of this set? At what point on this set the distance is furthest away from AB?.*

8. Let Δ be the triangle that maximizes area among all triangles of perimeter $p = 16$. What is the shape of Δ? What is the area of Δ?

9. Let $\square$ be the quadrilateral that maximizes area among all quadrilaterals of perimeter $p = 16$. What is the shape of $\square$? What is the area of $\square$?

Finishing:[1] Suppose Ω is a region of perimeter $p = 16$ that maximizes the area.

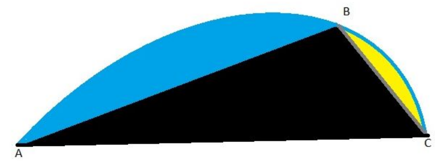
10. A region is convex if, given any 2 points A, B inside, the segment AB must be within the region. Intuitively, a region is convex if it has no holes and the boundary has no dents.

Intuitively and graphically, explain whether Ω is convex.

11. Let ℓ be a straight line that partitions Ω into two pieces of equal perimeter. Do the two pieces have the same area and why?

[1]Some problems here are inspired by "A historical review of the isoperimetric theorem in 2-D, and its place in elementary plane geometry" by Alan Siegel

12. Let the boundary of Ω intersects ℓ at point A and C. Let H be one of those two pieces and B a point on the boundary of H and not on AC. What is the value of $\angle ABC$ and why?



13. What is the shape of H and Ω? What is Area(Ω)?