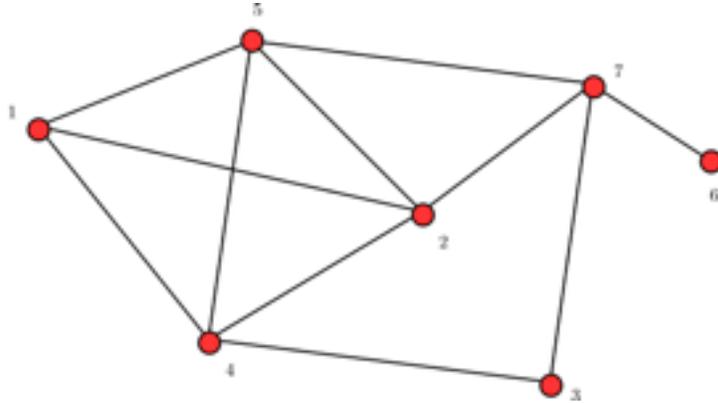


Graph theory worksheet — UCI Math Circle

A graph is something that looks like this.



It has vertices, and edges. Each edge connects two vertices. It is used to model various things where there are ‘connections’. For example, it could be cities and roads between them, or it could be the graph of friendship between people: each vertex is a person and two people are connected by an edge if they are friends. There is no interaction between edges that intersect in the middle.

1. How many vertices are there in the above graph? How many edges?
2. Can you list the vertices of the graph? Can you list the connections of each edge? (this is like giving the ‘friend list’ for each person)

The degree of a vertex is the number of edges that connect to it. (e.g. the number of friends of a person)

4. In the above graph, how many vertices of degree 1, 2, 3, 4, 5 are there?
5. Make a graph with 5 vertices where each vertex has degree 2.

6. If you had to do the same but with degree 3 instead of degree 2, i.e. make a graph with 5 vertices, where each vertex has degree 3, you could never do it. Why? (hint: If you add the degrees of every vertex in a graph, it is always an even number. Why?)

A clique is a group of vertices that are all connected to each other (e.g. a group of people who are all friends with each other). A k -clique in a graph is a clique with k people in it.

7. How many 2-cliques are there in the above graph? How many 3-cliques? How many 4-cliques?

8. What is the number of paths that go from vertex 1 to vertex 6?

9. What is the length of the shortest path from vertex 1 to vertex 6?

10. How many graphs are there that have 4 vertices. (assuming at most one connection is allowed between two vertices and no vertex can connect to itself via an edge)

11. How many graphs are there that have n vertices.

A graph with n vertices where every two vertices are connected to each other is called a fully connected graph. Also called a K_n (everyone's friends, in fact best friends)

12. Can you draw a K_5 , a K_6 ?

13. In K_n , how many edges are there?

14. How many k -cliques are there in K_n ?

15. If you had to describe a graph to a friend of yours as a text message without any pictures, how would you do it? (there is more than one way to do this!)

16. Dr Wazzaa thought about problem 14 and decided that, to describe a graph, it is enough to give the degree of each vertex. Prove Dr Wazzaa wrong by showing that there can be two graphs that are different, but the degrees of their vertices are the same.

17. Seven people in a room have shaken hands. Six of them have shaken exactly two people's hands. How many people's hands has the 7th person shaken?

18. Six soccer teams are competing in a tournament. Every team is to play three games, each against a different team. You are in charge of pairing up the teams to create a schedule of games that will be played. Ignoring the order and times of the games, how many different schedules are possible?

19. 20 soccer teams take part in a tournament. On the first day all the teams play one match. On the second day all the teams play a further match. Prove that after the second day it is possible to select 10 teams, so that no two of them have yet played each other.

20. There are 16 cities in a kingdom. The king wants to have a system of roads constructed so that one can go along those roads from any city to any other one without going through more than one intermediate city and so that no more than 5 roads go out of any city.

(a) Prove that this is possible.

(b) Prove that if we replace 5 by 4 in the problem, the king's desire will become unrealizable.