International Mathematics

TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2007.**

1. Black and white checkers are placed on an 8×8 chessboard, with at most one checker on each cell. What is the maximum number of checkers that can be placed such that each row and each column contains twice as many white checkers as black ones?

2. Initially, the number 1 and a non-integral number x are written on a blackboard. In each step, we can choose two numbers on the blackboard, not necessarily different, and write their sum or their difference on the blackboard. We can also choose a non-zero number of the blackboard and write its reciprocal on the blackboard. Is it possible to write x^2 on the blackboard in a finite number of moves?

3. D is the midpoint of the side BC of triangle ABC. E and F are points on CA and AB respectively, such that BE is perpendicular to CA and CF is perpendicular to AB. If DEF is an equilateral triangle, does it follow that ABC is also equilateral?

4. Each cell of a 29×29 table contains one of the integers 1, 2, 3, ..., 29, and each of these integers appears 29 times. The sum of all the numbers above the main diagonal is equal to three times the sum of all the numbers below this diagonal. Determine the number in the central cell of the table.

5. The audience chooses two of five cards, numbered from 1 to 5 respectively. The assistant of a magician chooses two of the remaining three cards, and asks a member of the audience to take them to the magician, who is in another room. The two cards are presented to the magician in arbitrary order. By an arrangement with the assistant beforehand, the magician is able to deduce which two cards the audience has chosen only from the two cards he receives. Explain how this may be done.

Note: The problems are worth 3, 4, 4, 5 and 5 points respectively.

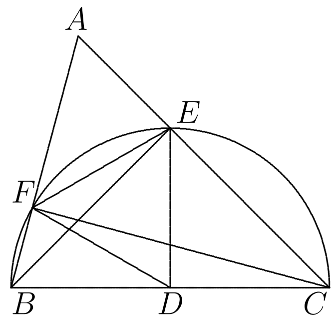
Solution to Junior O-Level Fall 2007

1. The number of checkers in each row must be a multiple of 3, since there are twice as many white chckers as black ones. Since there are only 8 cells in each row, the maximum number of checkers in each row in 6. Since there are 8 rows, the maximum number of checkers overall is 48. This can be attained by the placement shown below.

●	●	○	○	○	○		
●	○	○	○	○			●
○	○	○	○			●	●
○	○	○			●	●	○
○	○			●	●	○	○
○			●	●	○	○	○
		●	●	○	○	○	○
	●	●	○	○	○	○	

2. We first write down $x+1$ and $x-1$. Then we can write down $\frac{1}{x+1}$, $\frac{1}{x-1}$ and their sum $\frac{2}{x^2-1}$. The reciprocal of this number is $\frac{x^2-1}{2}$. Adding this number to itself yields x^2-1, and adding 1 to it yields x^2.

3. Let DEF be an equilateral triangle. Construct a semicircle with centre D and radius DE. The diameter BC of the semicircle is perpendicular to DE, with F closer to B than to C. Since $DF = DE$, F also lies on this semicircle. Extend BF and CE to meet at A. Since $\angle BEC = 90° = \angle BFC$, BE and CF are indeed altitudes of triangle ABC. Since A lies on the extension of CE and DE is the perpendicular bisector of BC, $AB < AC$. Hence ABC is not equilateral.



4. There are 29 numbers on the main diagonal, 29×14 numbers above it and 29×14 numbers below it. The sum of the largest 29×14 numbers is $29(16+17+\cdots+29) = 29 \times 7 \times 45$ while the the sum of the smallest 29×14 numbers is $29(1+2+\cdots+14) = 29 \times 7 \times 15$. Since the former is exactly three times as large as the latter, the largest 29×14 numbers are all above the main diagonal and the smallest 29×14 numbers are all below the main diagonal. In other words, every number on the main diagonal, including the central cell, is 15.

5. The magician and her assistant can agree beforehand to arrange the numbers 1 to 5 in order on a circle, so that 1 follows 5. If the audience chooses two adjacent cards, say 3 and 4, the assistant will choose the two cards after them, which are 5 and 1. If the audience chooses two non-adjacent cards, say 3 and 5, the assistant will choose the cards after them, namely, 4 and 1. If the magician receives two adjacent cards, say 2 and 3, she will know that the audience

must have chosen 5 and 1. If the magician receives two non-adjacent cards, say 2 and 5, she will know that the audience must have chosen 1 and 4.