Functional equations — 1.

What is a function?

Technically speaking, it would be rather impossible to give a proper notion of a function. We will need to use other words, such as 'relation', 'map' or other, which are just synonyms of a word function and need to be properly defined.

Question: What we can do? We will get understanding of not what the object is, but *how we can work with it.*

Assume, we have two sets, call them A and B, and the aim is to describe a function, which maps A on B. We will just say, that for exery x from set A there is a choice of an element y from set B. This element y we will denote as $f(x)$, it's called the image of x.

Notation: $f : A \longrightarrow B$; $y = f(x)$ or $f : x \mapsto y$

Composition of functions.

Definition. Let us have two functions $y = f(x)$, $g : A \longrightarrow B$ and $z = g(y)$, $f : B \longrightarrow C$. Then a function $z = F(x) = g(f(x))$, $F : A \longrightarrow C$ is called a **composition** of functions g and f. Notation: $F = g \circ f$.

Remark 1. There can be a composition of more than two functions, i. e. :

$$z = F(x) = f_1 \circ f_2 \circ \ldots \circ f_n = f_1(f_2(f_3 \ldots (f_n(x)) \ldots)).$$

Remark 2. Note, that $g \circ f$ and $f \circ g$ are different functions.

Example 1. *Let* $g(x) = x^2$, $f(y) = \cos y$. *Then* $F(x) = (f \circ g)(x) = \cos x^2$.

Example 2. *Let* $\varphi(x) = x^3$. *Then* $(\varphi \circ \varphi)(x) = (x^3)^3 = x^9$.

Example 3. *Let* $\psi(x) = \dfrac{2x+3}{3x+1}$. *Then* $(\psi \circ \psi)(x) = \dfrac{2\,\frac{2x+3}{3x+1} + 3}{3\,\frac{2x+3}{3x+1} + 1} = \dfrac{13x+9}{9x+10}$.

1. Let $f(x) = \dfrac{x^2+1}{x-1}$, $g(x) = \dfrac{x+1}{x-1}$. Find $f \circ g$.

2. Let $\varphi(x) = \sqrt{16 - (x+5)^2} - 5$, $\quad x \in [-5; -1]$. Find $\varphi \circ \varphi$.

3. Let $f(x) = \dfrac{x+1}{1-x}$. Find:

a) $f \circ f$; b) $f \circ f \circ f$; c) $f \circ f \circ f \circ f$; d) $f \circ f \circ f \circ f \circ f$.

Homework

4. Find $(\varphi \circ \varphi)(x)$, if

a) $\varphi(x) = -x$; b) $\varphi(x) = \dfrac{1}{x}$; c) $\varphi(x) = -\dfrac{1}{x}$; d) $\varphi(x) = \dfrac{x}{x-1}$.

5. Let $\varphi(x) = \dfrac{1}{1-x}$. Find $(\varphi \circ \varphi \circ \varphi)(x)$.

6. Let $\varphi(x) = \dfrac{x}{2}$. Find $\underbrace{\varphi \circ \varphi \circ \ldots \circ \varphi}_{n \text{ times}}$.

The simplest functional equations

Functional equations are equations where unknown is a function. Generally, those functions are connected with well-known functions using composition.

Examples of functional equations: $f(x+5) - 2\,f(x+3) + f(3x) = 0$ or $f(x-y) = f(x) + f(2y)$, where $f(x)$ is unknown, and the aim is to find $f(x)$.

Definition. Function $f(x)$ is called **a soultion of a functional equation** on set M, if it satisfies this equation for all values of variable x in the set M.

Problems

1. Prove that a function $f(x) = A \cdot 2^x + B \cdot 3^x$, where A and B some constant numbers, is a solution of $f(x+2) - 5\,f(x+1) + 6\,f(x) = 0$.
2. Proof, that a function $f(x) = Cx$, where C is some constant, is a solution of $f(x+y) = f(x) + f(y)$.
3. Prove, that $f(x) = x^\alpha$ for any α is a solution of $f(x \cdot y) = f(x) \cdot f(y)$.
4. Prove, that $f(x) = \operatorname{sgn}(x)$ is a solution of $f(x \cdot y) = f(x) \cdot f(y)$.

Remark. Last two tasks show, that different functions may be solutions of the same equation. To solve a functional equation means to find all of the solutions or prove that they do not exist.

5. Prove, that $f(x) = g\left(\dfrac{x^2+1}{x-1}\right)$, where g is any function, is a solution of
6. Find all functions $f(x)$ defined on the set of real numbers, such that $3\,f(x) - f^2(1) + 5x - 1 = 0$ for any real x.
7. Find all functions $f(x)$ and $g(x)$ satisfying next condition for all $x \neq 0$

$$\begin{cases} f(2x) + 2g(2x) = \dfrac{2x^2+x+1}{x}, \\ f\left(\dfrac{1}{x}\right) + g\left(\dfrac{1}{x}\right) = \dfrac{x^2+x+1}{x}. \end{cases}$$

8. Find all functions f defined on the set of all real numbers, such that for all real x and y this function satisfies

$$\sin x + \cos y = f(x) + f(y) + g(x) - g(y).$$

Homework

9. Prove, that a function $f(x) = a^x$ for any $a > 0$ is a solution of

$$f(x+y) = f(x) \cdot f(y).$$

10. Find all functions $f(x)$ defined on real numberssatisfying for any real x

$$f(x) = f(0) \cdot \sin x - f\left(\frac{\pi}{2}\right) \cdot \cos x + x.$$

11. Find all functions $f(x)$ and $g(x)$ defined on real numbers, such that for any real numbers x and y the following holds

$$f(x) + f(y) + g(x) - g(y) = x^3 + \sqrt[3]{y}.$$

12. A function $f(n)$ from integers to integers satisfies the condition:

$$f(n) = \begin{cases} n - 10, & \text{if } n > 100, \\ f\left(f\left(n+11\right)\right), & \text{if } n \leq 100. \end{cases}$$

Prove that $f(n) = 91$ for all $n \leq 100$.

Functional equations-3. Substituion in equations without free variables

1. Find all functions $f(x)$, such that for every $x \neq \pm 1$ it satisfies
$$f\left(\frac{x}{x+1}\right) = x^2.$$
2. Find all functions $f(x)$, such that for every $x \neq 0$ it satisfies
$$f(x) - 2f\left(\frac{1}{x}\right) = 2^x.$$
3. Find all functions $f(x)$, such that for every $x \neq 1$, $x \neq 0$ it satisfies
$$f(x) + f\left(\frac{1}{1-x}\right) = x.$$
4. Find all functions $f(x)$, such that for every $x \neq -1$ it satisfies
$$f\left(\frac{x}{x+1}\right) + 2f(x+1) = x+1.$$
5. Let a, b, c be some constant numbers, $a^2 \neq b^2$. Find all functions $f(x)$ satisfying
$$af(x-1) + bf(1-x) = cx.$$
6. Find all functions $f(x)$, such that for every $x \neq 0$ it satisfies
$$3f(-x) + f\left(\frac{1}{x}\right) + f(x) = x.$$
7. Find all functions $f(x)$, such that for every $x \neq 0$, $x \neq -1$ it satisfies
$$f\left(\frac{x-1}{x+1}\right) + f\left(-\frac{1}{x}\right) + f(x) = \frac{x^3+2x^2-2x-1}{x\,(x+1)}.$$

Homework

8. Find all functions $f(x)$, satisfying $f\,(2x+1) = x^2$.
9. Let a, c, k be some constant numbers and $a \neq c$. Find all functions $f(x)$, such that for every $x \neq -c$, $x \neq 1$ it satisfies $f\left(\dfrac{a+x}{c+x}\right) = kx$.
10. Let $a \neq \pm 1$. Find all functions $f(x)$, such that for every $x \neq 0$ it satisfies
$$af(x) + f\left(\frac{1}{x}\right) = ax.$$
11. Find all functions $f(x)$, such that for every $x \neq 0$ it satisfies
$$(x+1)\,f(x) + f\left(\frac{1}{x}\right) = 1.$$
12. Let $a \neq \pm 1$ and let $g(x)$ be some function. Find all functions $f(x)$, such that for every $x \neq 1$ it satisfies
$$f\left(\frac{x}{x-1}\right) - af(x) = g(x).$$
13. Find all functions $f(x)$, such that for every $-5 \leq x \leq -1$ it satisfies
$$f\left(\sqrt{16-(x+5)^2} - 5\right) + xf(x) = x.$$
14. Let $a \neq 0$ be some constant number. Find all functions $f(x)$, such that for every $x \neq 0$, $x \neq a$ it satisfies
$$f(x) + f\left(\frac{a^2}{a-x}\right) = x.$$

15. Find all functions $f(x)$, such that for every $x \neq \pm\dfrac{1}{3}$ it satisfies

$$f\left(\frac{x+1}{1-3x}\right) + f(x) = x.$$

16. Let a and b be some constant numbers, $a \neq 0$, $a^2 \neq b^2$. Find all functions $f(x)$, such that for every $x \neq \pm 1$ it satisfies

$$af(x+2) + bf\left(\frac{x+2}{x+1}\right) = x^2.$$

17. Let a and b be some constant real numbers, $a \neq \pm 1$, and let n be a natural number. Find all functions $f(x)$, satisfying

$$af(x^n) + f(-x^n) = bx^n.$$

18. Let a be some constant number, $a \neq \pm 1$. Find all functions $f(x)$, such that for every $x \neq 0$ и $x \neq \pm 1$ it satisfies

$$f\left(\frac{x}{x+1}\right) + af\left(\frac{x+1}{x}\right) = x + 2.$$

Rules

Setup

1. The board consists of a grid of 8×8 squares.
2. Squares are adjacent horizontally, vertically or diagonally.
3. The game is played by two or four players.
4. Each of the player has his own camp: a 3×3 square adjoint to a corner of the whole board.
5. Each player has a set of pieces in a distinct color, of the same number as squares in each camp.
6. The board starts with all the squares of each player's camp occupied by a piece of that player's color.

Objective

The winning objective is to be the first player to race all one's pieces into the opposing camp — the camp diagonally opposite one's own.

Play sequence

1. Players randomly determine who will move first.
2. Pieces can move in eight possible directions (up, down, left, right and diagonally).
3. Each player's turn consists of moving a single piece of one's own color in one of the following plays:

    3a. One move to an empty square:

    Place the piece in an empty adjacent square. This move ends the play.

    3b. One or more jumps over adjacent pieces:

    An adjacent piece of any color can be jumped if there is an adjacent empty square on the directly opposite side of that piece. Place the piece in the empty square on the opposite side of the jumped piece. The piece which was jumped over is unaffected and remains on the board. After any jump, one may make further jumps using the same piece, or end the play.

4. Then the other player makes his step in the same way.

First set of tasks

0. Play as many as possible games, to understand game mechanics and simplest rules of the game. Try to see logical connections, and try to distinguish somehow good moves and bad moves. The more you play, the deeper you understand how to play this game.
1. To finish the rules you need to write down when the game ends.
2. To make the game proper you need to add up extra rules, so the game will always end up somehow (one of the players winning or draw). Also you need to express properly the winning conditions.
3. Find ways to measure moves of each player to describe which move (sequence of moves) is better. Can you do it numerically?

Symmetric polynomials (2 variables case)

Theorem (Vieta). *Let x_1, x_2 be roots of polynomial x^2+px+q. Then*
$$\begin{cases} x_1+x_2=-p; \\ x_1x_2=q. \end{cases}$$

Both x_1+x_2 and x_1x_2 are polynomials over variables x_1 and x_2 and moreover those polynomials do not change, if one replace x_1 with x_2 and x_2 with x_1. We will study such polynomials.

Definition. Polynomial $f(x,y)$, which does not change if one exchange variables x and y are called **symmetric**

Example. Polynomial $f_1(x,y)=x^2y+xy^2$ is symmetric, while $f_2(x,y)=x^3-3y^2$ is not.

Definition. Polynomials $\sigma_1=x+y$ and $\sigma_2=xy$ are called **elementary symmetric polynomials** over two variables.

Aside of σ_1 and σ_2 we will be using polynomials x^2+y^2, x^3+y^3,..., x^n+y^n, which are called **power sums** and we will denote them as S_n:
$$S_n=x^n+y^n.$$
We will try to express first several of S_n using σ_1 and σ_2:

$$\begin{aligned} S_2&=x^2+y^2=(x+y)^2-2xy=\sigma_1^2-2\sigma_2;\\ S_3&=x^3+y^3=(x+y)\left(x^2-xy+y^2\right)=\sigma_1^3-3\sigma_1\sigma_2;\\ S_4&=x^4+y^4=\left(x^2+y^2\right)^2-2x^2y^2=\sigma_1^4-4\sigma_1^2\sigma_2+2\sigma_2^2. \end{aligned}$$

From that you may assume, that it will hold for any n, which is your *first task:*

Lemma. *Every power sum $S_n=x^n+y^n$ can be represented as a polynomial over σ_1 and σ_2.*

I hope your soultion for the first task was constructive (so it gives you way to construct that polynomial over $sigma_1$ and $sigma_2$), so you will easily show us such representation for S_5, S_6 and S_7.

Extra task: There also direct formulas for S_n, which you may try to prove (or at least to show that those formulas are equal to each other):

$$\boxed{S_n=x^n+y^n=\left(\frac{\sigma_1+\sqrt{\sigma_1^2-4\sigma_2}}{2}\right)^n+\left(\frac{\sigma_1-\sqrt{\sigma_1^2-4\sigma_2}}{2}\right)^n} \tag{1}$$

$$\boxed{S_n=\frac{1}{2^{n-1}}\sum_{m=0}^{\left[\frac{n}{2}\right]}\binom{n}{2m}\sigma_1^{n-2m}\left(\sigma_1^2-4\sigma_2\right)^m.} \tag{2}$$

$$\boxed{S_n=n\cdot\sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^m(n-m-1)!}{m!\,(n-2m)!}\sigma_1^{n-2m}\sigma_2^m,} \tag{3}$$

where $\left[\frac{n}{2}\right]$ means the biggest integer less than $\frac{n}{2}$.

The third formula was constructed by Edward Waring in 1779.

Using the proven lemma, one can extend it to the proof of the main theorem about symmetric polynomials

Theorem (Main)**.** *Any symmetric plynomial over two variables can be represented as a polynomial over* σ_1, σ_2.

This the main task to prove the main theorem. Aso one can show that such representaion is unique (it does not depend on the way we obtain this representation, we will always end up with the same thing).